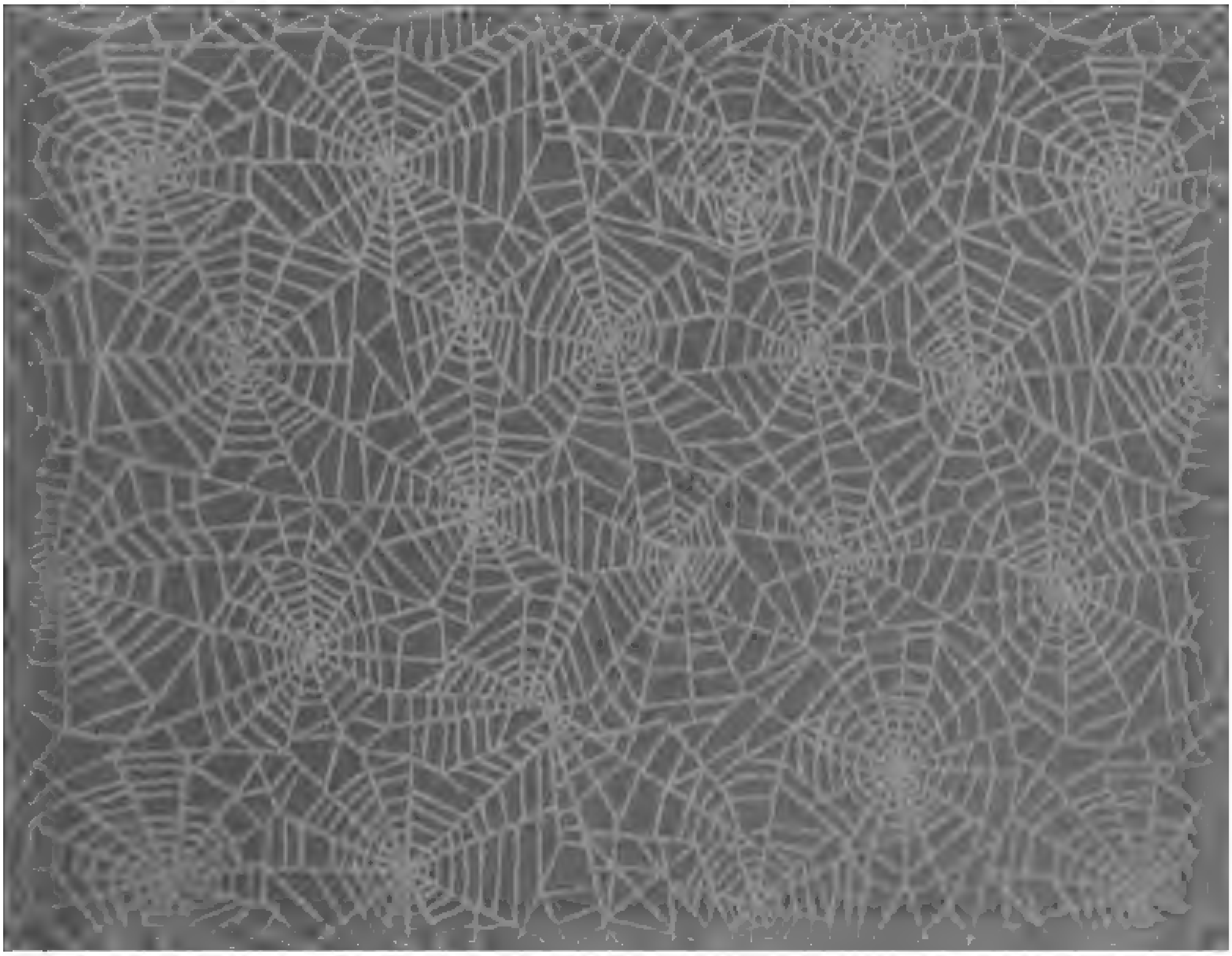

A

COURSE

OF

MATHEMATICS,

DESIGNED FOR THE USE

OF THE

OFFICERS AND CADETS,

OF THE

ROYAL MILITARY COLLEGE.

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VOL. I.

2v. in 1

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PREFACE.

THIS Volume contains *Arithmetic*, *Geometry*, *Plane Trigonometry*, and *Mensuration*.

As the Arithmetic is principally designed for those who are acquainted with the first rules, we have entered upon Fractions immediately after the division of whole numbers: this seems the order which naturally presents itself, because fractions result from the division of integers. The examples therefore in all the subsequent branches, are indiscriminately in whole numbers and fractions.

A thorough knowledge of Fractions, with the proper management of the Rules of Proportion, will enable the student very readily to comprehend nearly all that is necessary to be acquired in Arithmetic: for most of the other branches, as Single Position, Fellowship, Barter, Rules of Exchange, Discount, and Interest, are only applications of the Rule of Three. We therefore abridge the usual number of heads, and give a greater variety of examples under that of Proportion. Simple and Compound Interest however, are made separate articles. But Permutations, Combinations, and Alligation, with the exception of an example or two, are omitted; because nothing more than a partial and imperfect knowledge of those rules can be attained without the help of Algebra.

It will be perceived that the rules in general are not systematically detached from the demonstrations; this, the student whose object is real knowledge, will not consider as a defect in method, because it may frequently prove the means of enforcing the study of *principles*. A more commodious arrangement might therefore have been adopted for those who wish to acquire the *practice* of arithmetic only. That examples however, may not be wanting, we have added a great variety in the different rules, beginning with Vulgar Fractions. See from *p.* 125 to *p.* 159.

Euclid's Elements of Geometry, in the most concise form, generally make a separate work, and are therefore too extensive to be admitted at length in a volume of this kind. But we have endeavoured to give all the theorems necessary for the two most useful practical branches, *Trigonometry* and *Mensuration*: the latter however, is supposed to include such figures only as depend on right-lines and the circle. And with a view to facilitate the transition from *theory* to *practice*, when ratios or proportions are concerned, we have sometimes abridged the demonstrations by referring to analogous operations in the arithmetic. This may be deemed ungeometrical: but it ought to be remembered, that many who study Euclid do not wholly comprehend the doctrine of proportion as it is laid down in the fifth Book, without tracing the methods of demonstration by means of an arithmetical, or algebraic process.

Under Surveying the reader is not to expect the methods of plotting and measuring estates; but only such trigonometrical problems as are generally applicable to surveying. This part however, with the articles on Heights and Distances, are principally intended as introductory to the construction of military maps and plans. And to complete, or rather to render the Trigonometry independent, a table of logarithms sufficiently extensive for common practice is subjoined.

The subjects which compose this volume have so frequently been handled at full length in separate publications, that *new principles* cannot be expected in a work which may be considered as an abridgement, or compilation. What originality it is therefore entitled to, must principally consist in the arrangement. Most of the examples however, in the application of Trigonometry were selected from actual operations during the summer months in the field. And the practical questions and problems in the other parts of the volume, which are adapted to military concerns, have been furnished from the author's manuscript papers that from time to time were drawn up for the use and instruction of the Officers in the Senior Department of the College.

This edition is much more correct than the former: and several improvements and additions will be found in both the Arithmetic and Geometry.

High Wycombe,
May, 1807.

CONTENTS OF VOL. I.

ARITHMETIC.

1. ARITHMETIC is the science of numbers, or the art of computing by means of the ten numeral digits, or figures; 0 *cipher*, 1 *one*, 2 *two*, 3 *three*, 4 *four*, 5 *five*, 6 *six*, 7 *seven*, 8 *eight*, 9 *nine*.

All numbers may be denoted by those figures variously combined. And the rule which teaches their different values according to their different places, is called NOTATION, or

NUMERATION.

LET the number 444444444444 be proposed: then the different values of the same figure 4 will be as follows:

Hundreds of thousands of millions	Tens of thousands of millions	Thousands of millions	Hundreds of millions	Tens of millions	Millions	Hundreds of thousands	Tens of thousands	Thousands	Hundreds	Tens	Units
4	4	4	4	4	4	4	4	4	4	4	4

The first figure on the right stands for *four units*, being its simple value; the next for *four tens*, or *forty*, or *ten times* its simple value; the third for *four hundreds*, or a *hundred times*

its simple value; the fourth for *four thousands*, or a *thousand times* its simple value, &c. and the four together or 4444 denote *four thousand four hundred and forty four.*

Hence it appears that the values increase from the right to the left in a decuple proportion, each figure standing for ten times the value of the preceding one.

It is also evident that in reading of numbers there is a constant repetition of *hundreds, tens,* and *units*, at every three figures: thus, the three first on the right denote *four hundred and forty-four;* the next three, *four hundred and forty four thousands;* the next three, *four hundred and forty-four millions;* and the next three, *four hundred and forty-four thousands of millions,* &c.

Therefore in reading of large numbers, if we divide them into periods of six figures each, the first period to the right will be *units, tens, hundreds, and thousands;* the next period will be *millions;* the next *millions of millions, or bi-millions,* or *billions;* the next *tri-millions* or *trillions,* &c. &c.

For example, let 12802410007815104906709 be a proposed number:

Trillions	Billions	Millions	Thousands	Hundreds	Tens	Units
12802	410007	815104	906	7	0	9

Then dividing it into periods as above, it will be read thus: *twelve thousand eight hundred and two trillions, four hundred ten thousand and seven billions, eight hundred and fifteen thousand one hundred and four millions, nine hundred and six thousand, seven hundred and nine.*

3. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9, are called *significant* figures, because each has a value by itself, but the *cipher* or *zero* 0 stands for nothing if alone; when annexed however, on the right hand to other figures, or any number, it increases the value ten times: thus 7 denotes only *seven*, but 70 is *seven tens*

or *seventy;* and 700 *seventy tens* or *seven hundred;* also 11, signifies only *eleven*, but 110 *eleven tens*, or *one hundred and ten;* 1100 *eleven hundreds*, or *one thousand one hundred;* &c.

And therefore in setting down a proposed number, the places of significant figures must be supplied by ciphers when the former are wanting, as in the following example:

Nine hundred and seventy six	976
Nine hundred and seventy	970
Nine hundred and six	906
Seven thousand nine hundred and six	7906
Seven thousand	7000
Seventeen thousand and six	17006
Ten thousand	10000
One hundred ten thousand and six	110006
One hundred thousand one hundred	100100
One hundred thousand	100000
One million and one	1000001
One million	1000000

OF THE ROMAN NUMERALS OR NOTATION.

4. THE Romans made use of seven capital letters to express numbers,

Namely	I.	V.	X.	L.	C.	D.	M.
Value	1.	5.	10.	50.	100.	500.	1000.

The intermediate and other numbers are denoted by two or more of those letters joined or repeated till the sum of the whole make up the proposed number, the characters of the greatest value being set to the left; thus, VI is 6; VII, 7; VIII, 8; and MDCLXVI, 1666. Sometimes a less character is put to the left of a greater, and then it represents their difference as IV, 4; IX, 9; XL, 40; XC, 90; CD, 400. Also IↃ stands for D or 500; and CIↃ for M or 1000. Every C and Ↄ annexed on each side increases the value ten times; thus

CCIↃↃ is 10000. A bar or stroke over a letter increases the value 1000 times, as $\overline{X}$ is 10000, and $\overline{C}$ 100000, &c.

This notation is frequently used for the dates, numbering the chapters or sections of books, &c.

SIMPLE ADDITION.

5. SIMPLE ADDITION consists in finding the sum of two or more numbers of the same denomination. This is done in the following manner;

Place the numbers under each other, so that units are exactly under units, tens under tens, hundreds under hundreds, &c. and draw a line under them. Then add the row of units together, and find how many tens are in the sum.—Set down exactly under the units what remains more than those tens, or when nothing remains, a cipher, and carry one for every ten to the second row.—Next, add up the second row, together with the number carried, then proceed with the sum as before. And in this manner continue the operation till the whole is finished.

Examp. 1. Let the sum of 543 and 246 be required?

	543
	246
Sum.	789

Ex. 2. Required the sum of 57854, 480, and 769?

	57854
	480
	769
Sum	59103

In this addition I proceed thus:—9 and 0 and 4 make 13 which is 1 ten and 3 over, therefore I put down the 3 and carry 1 to the rank of tens; next, 6 and 8 are 14 and 5 make 19 and 1 I carried make 20, which is 2 tens and 0 over, therefore I put down a cipher and carry 2; again, 7 and 4 make 11 and 8 are 19 and 2 that were carried make 21, which is 1 to put

down and 2 to be carried; next, the 2 carried and 7 make 9; lastly, as there is nothing carried to the 5 it becomes the last figure in the sum.

The reason for placing units under units, tens under tens, hundreds under hundreds, &c. and carrying the tens to the left, is manifest from Notation. But because the whole must be equal to the sum of all its parts, if we add together the units in one sum, the tens in another, the hundreds in a third, &c. and add the several sums together, it will prove the addition; and perhaps the reason for carrying the tens will appear more obvious.

The sum of the units....................	13	
Of the tens..................	190	
Of the hundreds..............	1900	
Of the thousands.............	7000	
Of the tens of thousands....	50000	
Sum	59103	as before.

6. Another method of proving addition, is to cut off the upper line, then having added all the other lines together, add the upper line to the sum.

Ex. 3.	98764	Proof.	
	51238	98764	
	72045	51238	
	76958	72045	
	1039	76958	
	8460	1039	
Sum	308504	8460	
		209740	sum without the upper line.
		98764	upper line.
	Sum	308504	as before.

7. When the numbers to be added are large, and consist of many ranks, divide them into two or more parts, and find the sum of each part separately, then add the several sums together.

Ex. 4.		Proof.	
	987654321	987654321	
	123456780	123456780	
	592763184	592763184	
	790041376	1703874285	sum
	598472867	790041376	
	984799999	598472867	
	624875932	984799999	
	100926793	2373314242	sum
	994876823	624875932	
Sum	5797868075	100926793	
		994876823	
		1720679548	sum

	1703874285	the three sums.
	2373314242	
	1720679548	
Sum	5797868075	as before.

But the usual method of proving Addition is to begin at the upper line and add downwards, in the same manner as it was added upwards, then if the sums agree, we may conclude the work is right.

SIMPLE SUBTRACTION.

8. Simple Subtraction is the operation of taking a less number from a greater, or finding the difference of two proposed numbers: thus, 1 subtracted from 7 leaves 6, which is the difference of 1 and 7; 8 subtracted from 10 leaves 2, the difference of 8 and 10; 22 subtracted from 33 leaves 11 the difference; for 2 units taken from 3 units leaves 1 unit; and 2 tens taken from 3 tens leaves 1 ten; therefore 1 ten and 1 unit, or 11 is the difference. And hence it is evident that in placing numbers for subtraction, units must stand under units, tens under tens, hundreds under hundreds, &c. as in addition.

Ex. 1.	From	33
	Take	22
Difference or remainder		11

9. The method of proving subtraction is to add the less number and the difference or remainder together, for their sum must evidently be equal to the greater number if the work is right: thus, let the difference of 4356 and 3213 be required.

Ex. 2.	4356	
	3213	
Difference	1143	
Proof	4356	the sum of 3213 and 1143.

10. When the figure to be subtracted is greater than that directly above it, the method of operating is easily derived thus:

Let the difference of 41 and 18 be required:

41
18

differ. 23; here 8 cannot be subtracted from 1, but if 10 is taken from the 40 and added to the 1 the sum is 11, then 8 from 11 and 3 remains; consequently the 1 which stands under the 4 must be subtracted from 3 (or 4 lessened by 1), and the remainder is 2. In like manner proceed with any other number of figures.

Ex. 4. From 823
Take 636

Rem. 187: here 6 from 13 (10 added to 3) and 7 remains; 1 from 11 (10 added to 2 lessened by 1) and 8 remains; 6 from 7 (8 lessened by 1) and 1 remains. But it evidently comes to the same thing if we augment the lower figures by 1 instead of lessening the upper figures; thus 6 from 13 and 7 remains; 4 from 12 and 8 remains; 7 from 8 and 1 remains.

Ex. 5. From 14040
Take 3051

Rem. 10989; here 1 from 10 and 9 remains; 5 from 13 (10 added to 4 lessened by 1) and 8 remains; 0 from 10 lessened by 1 and 9 remains; 3 from 4 lessened by 1 and 0 remains: lastly as there is nothing to subtract from the 1, it becomes the left-hand figure of the remainder.

If we augment the lower figures by 1 instead of diminishing the upper ones, the process will be thus: 1 from 10 and 9 remains; 6 from 14 and 8 remains; 1 from 10 and 9 remains; 4 from 4 and 0 remains.

Ex. 6.	From	1000001	*Ex.* 7. From	81010215
	Take	1101	Take	901016
	Rem.	998900	Rem.	80109199
	Proof	1000001	Proof	81010215

11. Or subtraction may be performed by setting down such figures for the remainder that when added to the less number shall give the greater.

Thus, from 9875
Take 2301

Rem. 7574; here 1 and 4 make 5, therefore 4 is the remainder; 0 and 7 make 7 for the remainder; 3 and 5 make 8, therefore 5 is the remainder; 2 and 7 make 9, therefore 7 is the remainder.

When the lower figure is greater than that directly above, it is evident that the next lower figure must be augmented by 1.

Thus, from 10126
Take 1357

Rem. 8769; here 7 and 9 make 16, therefore 9 remains; 6 (or 5 augmented by 1) and 6 make 12, therefore 6 remains; 4 (or 3 augmented by 1) and 7 make 11, therefore 7 remains; 2 (or 1 augmented by 2) and 8 make 10, therefore 8 remains.

SIMPLE MULTIPLICATION.

12. Simple Multiplication consists in finding the sum or amount of a proposed number taken or repeated a given number of times, and may be denominated a compendious method of Addition: for example, suppose 6 is to be taken 3 times:

6
6
6

then the addition gives 18, but by multiplication we say 3 times 6 make 18.

The number to be multiplied is called the *multiplicand*; that by which you multiply, the *multiplier*; and the result is

called the *product*. The multiplicand and multiplier are without distinction called the terms or *factors* of the multiplication, because they *make* the product or number sought: thus 3 times 5 *make* 15.

13. But in the first place it will be necessary to learn perfectly the following Table, which contains the products of every two of the 9 digits.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

To find the product of two figures in this table, look for one of them in the left-hand column, and for the other at top, then their product will be found where the vertical column from the top intersects the horizontal one from the left. Let 6 and 7 be proposed, then the columns meet at 42; for 6 times 7, or 7 times 6 make 42.

14. The rule for multiplying by a single figure is derived from addition; thus: Let the sum of 3 times 875, or, which is the same thing, the product of 875 by 3, be required?

	875	Multiplicand	875
	875	Multiplier	3
	875	Product	2625
Sum	2625		

To perform the addition; 5, 5, and 5 make 15, or 5 more than 1 ten; 7, 7, and 7 make 21, and 1 make 22, or 2 more than 2 tens; next 8, 8, and 8 make 24, and 2 make 26. But in the multiplication we say 3 times 5 make 15, or 5 more than 1 ten; 3 times 7 make 21, and 1 make 22, or 2 over 2 tens; lastly, 3 times 8 make 24, and 2 (for the 2 tens) make 26. Therefore in multiplication, 1 must be carried to the left for every 10 in the products, and the overplus set down as in addition.

Examp. 2.

Multiply	987600543210
By	7
Product	6913203802470

Ex. 3.

Multiply	123456789
By	9
	1111111101

15. When the multiplier consists of one figure with ciphers on the right, multiply by that figure, and annex the ciphers to the right of the product.

Ex. 4.

Multiply	11	
By	300	
Product	3300	this is evident from Notation.

When the multiplier consists of several figures.

16. Begin at the right, and multiply by each figure separately, and set down the products so that the units of the second line may stand under the tens of the first, the units of the third line under the tens of the second, and so on: then add all the products together for the amount.

Ex. 5.

Multiply	231
By	323
	693
	462
	693
Product	74613

The reason for setting down the products by the single figures in this manner will be manifest, if we consider that the whole amount must (in the present example) consist of 3 times 231, 20 times 231, and 300 times 231, when added together:

3 times 231	....	693
20 times 231	....	4620
300 times 231	....	69300
	Sum	74613.

Here if the ciphers are cancelled (as having no value in the addition) the first figure of any line, or product by a single figure, must necessarily fall one place to the left of that above it. And hence the rule for multiplying by several figures is deduced.

17. When ciphers are between other figures in the multiplier, neglect them, remembering to set down the lines of products as far to the left as they would be if the ciphers were others figures.

Ex. 6.	Multiply	5772
	By	230045
		28860
		23088
		17316
		11544
	Product	1327819740

18. If ciphers are at the right hand of one or both factors, find the product of the other figures, to which annex all the ciphers on the right.

Ex. 7.	Multiply	6300
	By	7000
		441
		252
	Product	29610000

19. When one of the factors is the product of two or more single figures, the other factor may be multiplied by one of the figures, and the product by another, and so on: then the last result will be the answer.

Ex. 8. Let 4615 be multiplied by 72, or 9 times 8.

	4615
	9
	41535
	8
Product	332280

The reason of this operation is obvious; for 9 times any number repeated 8 times, is evidently that number repeated 72 times.

Methods of proving Multiplication.

I.

20. Make the multiplicand and multiplier change places; then if the products agree, the work is right.

Examp.	Multiply	6847	*Proof.*
	By	7806	7806
		41082	6847
		54776	54642
		47929	31224
	Product	53447682	62448
			46836
			53447682

II.

21. Find what is over the exact number of nines in the sum of the digits of each factor, then multiply the excesses together, and find the excess above nines in the digits of this product, which excess ought to be the same as the excess above nines in the digits of the whole product or answer.

Examp.	Multiply	836	—8, excess above nines
	By	727	—7, excess above nines
		5852	
		1672	
		5852	
	Product	607772	—2, excess above nines,

The product of the two excesses 8 and 7 is 56, which gives 2 for the excess above nines, the same as the excess in the whole product or amount.

This method of proof is founded on the following property of the number 9;—*every number, the sum of whose digits is an exact number of nines, is itself an exact number of nines.* This is easily proved as follows: any number containing an exact number of tens must consist of the same number of nines and of units; thus 1 nine and 1 unit make 1 ten; 2 nines and 2 units make 2 tens; 7 nines and 7 units make 7 tens; 60 nines and 60 units make 60 tens, &c.; consequently, if the nines are taken out of the tens in any number, the remainder will be as many units as there are tens in that number; for example, the nines taken from the tens in 670 will leave 67 units; and the nines taken from the 6 tens in 67 will leave 6 units, which, with the 7 units, make 13 the sum of the digits in 670; therefore if *all* the nines are cast out of 670, the remainder will be 4 (the difference of 13 and 9); and because 4 wants 5 of 9, it is evident that 675, and also the sum of its digits, are each an exact number of nines. And the same method of proof will extend to other numbers.

From hence it follows, that when the nines are cast out of any number, and also out of the sum of its digits, the remainders will be the same.

And in multiplication it is also evident, that when the sum of the digits in one factor is an exact number of nines, the sum of the digits in the product will be an exact number of nines.

In the foregoing example where the excesses above nines in the factors are 8 and 7, the product 607772 is the sum of 836 multiplied by 720, and

436 less by 8 multiplied by 7, and 8 multiplied by 7; the two former parts are exact nines (one of the factors in each being nines) and since the latter part (8 multiplied by 7) is the product of the two excesses in the factors, the truth of the rule is manifest.

This method of proving multiplication by casting out the nines, is probably as ancient as the present system of arithmetic, for we find it in Lucas de Burgo's *Summa de Arithmetica*, &c. printed in 1494. But though a convenient rule, there are circumstances in which it may fail; thus if two figures should be transposed in the product, or the value of one figure too great and another as much too little, or a 9 be set down instead of 0, or the contrary: in all these cases, the excess above nines will evidently be the same as in the true product.

SIMPLE DIVISION.

22. SIMPLE DIVISION consists in finding how often a less number is contained in, or may be taken from a greater number of the same denomination; and is a compendious method of subtraction. Or it is the method of resolving a given number into a proposed number of equal parts. Thus, if 2 and 10 are the numbers, the former is contained 5 times in the latter: or if 10 be divided into 2 equal parts, each part will be five.

23. The number to be divided is called the *dividend*.—— That by which you divide the *divisor*.-----And the number of times the latter is contained in the former is called the *quotient*.

Dividend.
Divisor 2) 10 (5 *Quotient.*

24. *To perform Division.* Find how often the divisor is contained in as many of the left hand figures of the dividend as are just necessary, which will give the first figure in the quotient. Multiply the divisor by this quotient figure and subtract the product from the aforesaid figures of the dividend, then bring down the next figure of the dividend to the right of the remainder. Find

how many times the divisor is contained in the remainder so increased, for the second figure of the quotient, but if it be 0 times, put a cipher, and bring down another figure; then proceed as before till all the figures are brought down.

Examp. 1. Let 83401190 be divided into 2 equal parts.

```
               Dividend
     Divisor 2 ) 83401190 ( 41700595 Quotient
                 8
                 -
                 3                        Proof
                 2                      41700595
                 -                             2
                 14                     --------
                 14                     83401190
                 --                     ========
                  011
                   10
                   --
                    19
                    18
                    --
                     10
                     10
                     --
```

In this example the quotient is half the dividend, therefore if we multiply 41700595 by 2, the product will be 83401190.

25. Hence to prove Division, *multiply the divisor and quotient together, then if the product is the same as the dividend, the work is right.*

26. When there is no remainder after the last subtraction, the quotient will be a whole number, as in the preceding example; but if there be a remainder, place it over the divisor with a line between, on the right of the other figures, and you have the *fractional* part of the quotient.

Ex. 2. Let 101 be divided into 2 equal parts.

```
         2 ) 101 ( 50½ quotient, or the half of 101.
              10
              --
   Remainder   1
              --
```

The fraction $\frac{1}{2}$ denotes half, or 1 divided into 2 equal parts, and is the fractional part of the quotient.

Ex. 3. Divide 713391049 into 7 equal parts.

```
7 ) 713391049 ( 101913007 quotient, or the answer.
    7
    13                              Proof.
     7                            101913007
    63                                    7
    63                            713391049
      9
      7
     21
     21
      049
       49
```

Ex. 4. 9) 8257576 (917508$\frac{4}{9}$ *quotient.*

```
          81
           15                      Proof.
            9                      917508
           67                           9
           63                      8257572
            45                           4 remainder
            45                     8257576
             76
             72
Remainder     4
```

If the integral part of the quotient be multiplied by the divisor 9, and the remainder 4 added to the product, the sum is the dividend, as in the proof.

When the divisor however, is only *one* figure, it is usual to perform the subtraction mentally and set down the quotient under the dividend: thus,

```
9 ) 8257576
    917508 4/9 quotient.
```

In this division I proceed thus:—the nines in 82 are 9, and 1 over; the nines in 15 is 1, and 6 over; the nines in 67 are 7, and 4 over; the nines in 45 are 5; the nines in 7 0 times, and 7 over; the nines in 76 are 8, and 4 over.

Ex. 5. Let 67550595 be divided into 211 equal parts.

```
211 ) 67550595 ( 320145 quotient or answer.
      633
       425
       422
         305
         211
          949
          844
          1055
          1055
```

To find how often the divisor (211) is contained in the numbers of the several steps of the operation, first enquire how many times 2 (the left figure of the divisor) is contained in 6 (the left figure of the dividend); this gives 3 for the first figure in the quotient; next, the 2's in 4 are 2 for the second figure; thirdly, 211 the divisor being greater than 30, a cipher or 0 will be the third figure; fourthly, the 2's in 3 give 1; next, the 2's in 9 give 4; and lastly, the 2's in 10 are 5.

27. But when the dividend is a large number, and the divisor consists of several figures, a table may be formed containing the products of the divisor by the several digits, as in the next example:

Ex. 6. Divide 1447859740478 by 1783.

1783 multiplied by	1	give	1783
	2	....	3566
	3	....	5349
	4	....	7132
	5	....	8915
	6	....	10698
	7	....	12481
	8	....	14264
	9	....	16047

1783) 1447859740478 ($812035749\frac{11}{1783}$ quotient.

```
14264
 2145
 1783
  3629
  3566
   6374
   5349
   10250
    8915
    13354
    12481
      8737
      7132
      16058
      16047
Remainder  11
```

Proof.

```
     812035749
          1783
    2436107247
   6496285992
  5684250243
 812035749
 1447859740167
            11
 1447859740178
```

28. Those who are expert in the practice of division, sometimes omit the products, and set down the remainders only.

Thus, (taking the last example.)

```
1783 ) 1447859740478 (812035749 11/1783
       2145
        3629
          6374
           10250
             13354
               8737
                16058
                   11 remainder.
```

And the division is sometimes performed without bringing down the figures of the dividend.

```
Thus, 1783 ) 1447859740478 ( 812035749 11/1783.
              2142325335 ( 1
               36603370 ( 1
                  11 86
                     1
```

Where the remainders stand under the corresponding figures of the dividend, as before.

In these contracted methods, the remainders are obtained by performing the subtraction while you multiply. Thus to find 214 the first remainder: 8 times 3 make 24, and 4 make 28, therefore 4 is the right-hand figure of the remainder; next 8 times 8 make 64, and 2 (the tens carried) make 66, and 1 make 67, consequently 1 is the next figure; again, 8 times 7 are 56, and 6 (the tens carried) make 62, and 2 make 64, therefore 2 is the other figure of the remainder. And in the same manner the other remainders are found.

29. When the divisor is a number with ciphers on the right, cut them off, and also the like number of figures from the right of the dividend, then divide the remainder of the dividend by that of the divisor in the usual manner, and bring down the figures cut off from the dividend to the right of what remains after this division, if any thing, for the whole remainder; otherwise the figures cut off will be the true remainder.

Ex. 7. Divide 245135 by 2500.

```
25,00 ) 2451,35 ( 98 135/2500 quotient.
        225
         201
         200
   Rem.  135
```

Remainder

96. Those who are expert in the
times omit the products, and set dow

Thus, (taking the last example.)

```
1783 ) 1447859740478 (812035749 11/1783
       2145
        3629
          6374
          10250
            13354
              8737
               16058
                  11  remainder.
```

And the division is sometimes performed without bringing down the figures of the dividend.

```
Thus, 1783 ) 1447859740478 ( 812035749 11/1783.
                2142795535 ( 1
                 36603370 ( 1
                    11 86
                       1
```

Where the remainders stand under the corresponding figures of the dividend, as before.

In these contracted methods, the remainders are obtained by performing the subtraction while you multiply. Thus to find 214 the first remainder: 8 times 3 make 24, and 4 make 28, therefore 4 is the right-hand figure of the remainder; next 8 times 8 make 64, and 2 (the tens carried) make 66, and 1 make 67, consequently 1 is the next figure; again, 8 times 7 are 56, and 6 (the tens carried) make 62, and 2 make 64, therefore 2 is the other figure of the remainder. And in the same manner the other remainders are found.

29. When the divisor is a n[illegible] on the right, cut them off, and also the li[illegible] om the right of theividen[illegible] n divi[illegible] dividend by that of th[illegible] th[illegible] g down the figures [illegible] th[illegible] wh[illegible] after [illegible] mp[illegible] wi[illegible]

8. Divide 245035 by 2500.

25,00) 2450,35 ($98\frac{35}{2500}$ *quotient.*
225
――
200
200
――
Rem. 35

9. Divide 715640 by 6000.

6,000) 715,640
$119\frac{1640}{6000}$ *quotient,* the remainder being 1640.

10. Divide 6421 by 10.

1,0) 642,1
$642\frac{1}{10}$ *quotient,* the remainder being 1.

20. When the divisor is the product of two or more single figures, divide by one of those figures, and the quotient by another, and so on.

Ex. 11. Divide 332280 by 72, or 9 times 8. (See *Example* 7, in Multiplication.)

9) 332280
8) 36920
4615 quotient.

The method of finding the true quotient when there are remainders, belongs to Vulgar Fractions, to which we refer for an example.

Since the product of the divisor and quotient (without the fractional part, should there be any) gives the dividend lessened by the remainder, it is evident that division may be proved by casting out the 9's exactly in the same manner as multiplication.

OF VULGAR FRACTIONS.

21. THE operations by common arithmetic extend to integers only, unity or one being the least number in the computations. When parts or quantities less than 1 are the subject of consideration, it is called *Fractional Arithmetic.* A *fraction* there-

force is properly an expression for part of an *unit* or the integer 1. This integer 1 may represent a *whole* of any kind, and the parts into which it is broken, or supposed to be divided, are *fractions* of that *whole*.

Thus if 1 pound is the integer, and we divide it into 20 equal parts, 1 of these parts, or a shilling, will be represented by the fraction $\frac{1}{20}$ *(one twentieth)*; and 7 shillings by the fraction $\frac{7}{20}$ *(seven twentieths)*. If a foot in length is the integer, the expression for 1 inch will be $\frac{1}{12}$ *(one twelfth)*; but if we make a yard the integer, 1 inch will be denoted by $\frac{1}{36}$ *(one thirty-sixth)*, because 36 inches make a yard.

32. A fraction also arises from division in whole numbers when there is a remainder; or when the divisor is greater than the dividend: in the former case it is part of the quotient (see examples 2, 4, &c. in simple division), and in the latter, the quotient itself.

Thus if 5 be divided by 2 the quotient is $2\frac{1}{2}$. And 3 divided by 4 gives $\frac{3}{4}$ for the quotient. Here the fractions are $\frac{1}{2}$ and $\frac{3}{4}$: the former ($\frac{1}{2}$) being half, or 1 divided by 2; and the latter ($\frac{3}{4}$) three-fourths, or 3 divided by 4, or the 4th of 3.

33. The lower figure of a fraction (denoting the number of parts into which the integer or 1 is supposed to be divided) is called the *denominator*; and the upper figure (which shews the number of those parts expressed by the fraction) the *numerator*; thus 4 is the denominator, and 3 the numerator of the fraction $\frac{3}{4}$. Also both are generally named the terms of the fraction.

34. Fractions are either *proper*, *improper*, *simple*, or *compound*.

A *proper fraction* is when the numerator is less than the denominator, as $\frac{1}{2}$, or $\frac{3}{4}$, or $\frac{5}{11}$, &c. and therefore it is always less than 1.

An *improper fraction* has the numerator equal to, or greater than the denominator, and consequently its value must be equal to, or greater than 1. Thus $\frac{4}{4}$ is an improper fraction, because

II.

81. Find what is over the exact number of nines in the sum of the digits of each factor, then multiply the excesses together, and find the excess above nines in the digits of this product, which excess ought to be the same as the excess above nines in the digits of the whole product or answer.

Examp.	Multiply	836	—8, excess above nines
	By	727	—7, excess above nines
		5852	
		1672	
		5852	
	Product	607772	—2, excess above nines,

The product of the two excesses 8 and 7 is 56, which gives 2 for the excess above nines, the same as the excess in the whole product or amount.

This method of proof is founded on the following property of the number 9;—*every number, the sum of whose digits is an exact number of nines, is itself an exact number of nines.* This is easily proved as follows; any number containing an exact number of tens must consist of the same number of nines and of units; thus 1 nine and 1 unit make 1 ten; 2 nines and 2 units make 2 tens; 7 nines and 7 units make 7 tens; 60 nines and 60 units make 60 tens, &c.; consequently, if the nines are taken out of the tens in any number, the remainder will be as many units as there are tens in that number; for example, the nines taken from the tens in 670 will leave 67 units; and the nines taken from the 6 tens in 67 will leave 6 units, which, with the 7 units, make 13 the sum of the digits in 670; therefore if *all* the nines are cast out of 670, the remainder will be 4 (the difference of 13 and 9); and because 4 wants 5 of 9, it is evident that 675, and also the sum of its digits, are each an exact number of nines. And the same method of proof will extend to other numbers.

From hence it follows, that when the nines are cast out of any number, and also out of the sum of its digits, the remainders will be the same.

And in multiplication it is also evident, that when the sum of the digits in one factor is an exact number of nines, the sum of the digits in the product will be an exact number of nines.

In the foregoing example where the excesses above nines in the factors are 8 and 7, the product 607772 is the sum of 836 multiplied by 720, and

+ (*plus*) the character for *addition*:

Thus $2 + 3 = 5$, 2 added to 3 are equal to 5.
$4 + 6 = 7 + 3$, 4 added to 6 are equal to 7 and 3.

— (*minus*) signifies *subtraction*:

As $5 - 3 = 2$ 3 subtracted from 5 is equal to (or leaves) 2.
$4 - 3 = 2 - 1$, the difference of 4 and 3 is equal to that of 2 and 1.

× the character for *multiplication*:

$2 \times 3 = 6$, 2 multiplied by 3 is equal to (or produces) 6.
$2 \times 3 \times 4 = 24$, the continual product of 2, 3, and 4, is equal to 24.
$\frac{7 \times 3}{5 \times 4} = \frac{21}{20}$, the fraction $\frac{7 \times 3}{5 \times 4}$ is equal to the fraction $\frac{21}{20}$.

÷ the character sometimes used to signify *division*.

As $24 \div 4 = 6$, 24 divided by 4 is equal to (or produces) 6.
$5 \div 2 = 2\frac{1}{2}$, 5 divided by 2 is equal to $2\frac{1}{2}$.
$3 \div 4 = \frac{3}{4}$, 3 divided by 4 is equal to $\frac{3}{4}$.

37. But the proper method of abbreviating division is to set down the quotient in the form of a fraction by placing the divisor under the dividend; thus, 3 divided by 4 gives $\frac{3}{4}$ for the quotient; 5 divided by 2 gives the quotient $\frac{5}{2}$; and 1 divided by 4 produces $\frac{1}{4}$, or a quarter. In general, every fraction should be considered as the quotient arising from the division of the numerator by the denominator.

REDUCTION OF VULGAR FRACTIONS.

38. Reduction of Vulgar Fractions principally consists in changing them to a more commodious form for the operations of addition, subtraction, &c.

Case 1. *To abbreviate or reduce fractions to their lowest terms.*

39. If the terms of a fraction are multiplied or divided by any

number, its value will evidently remain the same as before; thus, the numerator and denominator of $\frac{3}{6}$ multiplied by 4 produces the fraction $\frac{12}{24}$, or divided by 3 gives $\frac{1}{2}$ (or half), the same as $\frac{12}{24}$ or $\frac{3}{6}$. Therefore to reduce a fraction to its lowest terms, divide the terms of the fraction by any number that will leave no remainder, and the quotients again by the same, or any other number, and so on, till 1 is the greatest divisor; then the fraction will be in its lowest terms.

Ex. 1. Reduce $\frac{1408}{1664}$ to its lowest terms.

This fraction may be reduced by a continual division by 2: thus

$$2)\ \frac{1408}{1664} = \frac{704}{832} = \frac{352}{416} = \frac{176}{208} = \frac{88}{104} = \frac{44}{52} = \frac{22}{26} = \frac{11}{13} \textit{ the lowest terms.}$$

Therefore $\frac{1408}{1664}$ is equal to $\frac{11}{13}$.

When 2 fails as a divisor, try 3, 5, or 7, because if a number is divisible by any digit, (1 excepted) it must be divisible by either 2, 3, 5, or 7.

Ex. 2. Reduce $\frac{1470}{2205}$ to its lowest terms.

$$5)\ \frac{1470}{2205} = 3)\ \frac{294}{441} = 7)\ \frac{98}{147} = 7)\ \frac{14}{21} = \frac{2}{3}.$$ *Ans.* where 5, 3, 7, 7, are the divisors.

Ex. 3. Reduce $\frac{36300}{231000}$ to its lowest terms.

$$\frac{36300}{231000} = \frac{363}{2310} = \frac{121}{770} = \frac{11}{70}.$$ *Ans.* where the divisors are 100, 3, and 11.

40. If the numerator and denominator are large numbers, find their greatest divisor, or common measure, by the following rule: *Divide the greater by the less, and the last divisor by the last remainder, and so on, till nothing remains; then the last divisor is the greatest common measure required.*

If 1 remains for the last divisor, the numerator and denominator (having 1 for their greatest common measure) are said

to be prime to each other; and the fraction is already in its lowest terms.

Ex. 4. Reduce $\frac{7631}{26415}$ to its lowest terms.

$$\begin{array}{l}
7631\,)\,26415\,(\,3 \\
\underline{22893} \\
3522\,)\,7631\,(\,2 \\
\underline{7044} \\
587\,)\,3522\,(\,6 \\
\underline{3522}
\end{array}$$

Therefore the last divisor 587 is the greatest number that will divide 7631 and 26415 without leaving any remainder.

$$587\Big)\frac{7631}{26415}\Big(\frac{13}{45} \text{ \textit{the fraction in its lowest terms.}}$$

In like manner the greatest divisor or common measure of three or more numbers may be found. For having found the greatest common measure of two of them, as above, find the greatest divisor of that common measure and another of the numbers, and so on. Thus 15 is the greatest common measure of 1995, 840, and 600.

The foregoing rule for finding the greatest common divisor of two numbers is founded on the following axiom; *If a number measures another number, and also a part of that number, it will measure the remaining part.* Thus 5 measures 40 (or 5 is contained in 40 an exact number of times), and it also measures 25 (a part of 40), therefore it measures 15 the other part. That the operation brings out the greatest divisor may be shewn from the 4th example, thus:—The denominator 26415 is equal to the numerator $7631 \times 3 + 3522$ (by the method of proving common division): now if there is a greater divisor than 587 which measures 7631, and $7631 \times 3 + 3522$, it must (by the preceding axiom) measure 3522. And for the like reason, if it measures 3522, it must measure 3522×2. And if it measures 7631 and 3522×2, it must (by the same axiom) measure their difference, or $7631 - 3522 \times 2$, or 587, viz. the greater measures the less, which is absurd.

Case 2. *To reduce an improper fraction to its equivalent whole or mixt number.*

41. This is evidently nothing more than common division. Therefore divide the numerator by the denominator, and the quotient will be the answer.

Ex. 1. Reduce $\frac{937}{43}$ to a whole, or mixt number.

$$43)937(21\tfrac{34}{43}\ \textit{Answer.}$$
$$\begin{array}{r} 86 \\ \hline 77 \\ 43 \\ \hline 34 \end{array}$$

2. Reduce [illegible] to its whole, or mixt number.

[illegible]) [illegible] (20 *Answer.*

3. Reduce [illegible] to its whole, or mixt number.

1000) [illegible] (7 [illegible] *Answer.*

Case 3. *To reduce a mixt number to its equivalent improper fraction.*

42. This operation is the reverse of the former; therefore multiply the whole number by the denominator of the fraction, and add the numerator to the product, then place the sum over the denominator for the fraction required.

Example. Reduce $21\frac{34}{43}$ to an improper fraction.

$$\begin{array}{r} 21 \\ 43 \\ \hline 63 \\ 84 \\ \hline 903 \\ 34 \\ \hline 937 \end{array} \qquad \frac{937}{43}\ \textit{Answer.}$$

Hence to reduce a whole number to an improper fraction having a given denominator: multiply the said number by the proposed denominator, and make the product the numerator of the required fraction.

Example. Let 13 be reduced to a fraction whose denominator is 7.

$13 \times 7 = 91$ the numerator. *Answer* $\frac{91}{7}$.
For $\frac{91}{7} = 13$ by the preceding article.

CASE 4. *To reduce a compound fraction to an equivalent simple one.*

43. MULTIPLY all the numerators together for the numerator, and all the denominators together for the denominator of the fraction required.

If part of the compound fraction be a mixt, or a whole number, reduce the former to an improper fraction, and make the latter a fraction by placing 1 under it as a denominator.

Ex. 1. Reduce $\frac{1}{2}$ of $\frac{3}{4}$ to a simple fraction.

$$\frac{1}{2} \times \frac{3}{4}, \text{ or } \frac{1 \times 3}{2 \times 4} = \frac{3}{8} \textit{ the fraction required.}$$

2. Reduce $\frac{2}{3}$ of $\frac{5}{7}$ of $3\frac{1}{3}$ of 4 to a simple fraction.

$$\text{First } 3\frac{1}{3} = \frac{10}{3}; \text{ and } 4 = \frac{4}{1};$$

$$\text{Then } \frac{2}{3} \times \frac{5}{7} \times \frac{10}{3} \times \frac{4}{1} = \frac{400}{63} \textit{ answer.}$$

44. When a like number of like factors are found in the numerator and denominator, cancel them in both.

Ex. 3. Reduce $\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{5}{7}$ of $\frac{3}{4}$ to a simple fraction.

$\frac{2 \times 1 \times 3 \times 5 \times 3}{3 \times 2 \times 5 \times 7 \times 4}$ here cancelling 2, 3, and 5, in both numerator and denominator, the fraction becomes $\frac{1 \times 3}{7 \times 4} = \frac{3}{28}$ the *answer*. This is reducing the fraction to lower terms by means of the divisors 2, 3, and 5. (39)

The rule for reducing compound fractions may be derived as follows:— Suppose a shilling to be the integer; then because 48 farthings make 1 shilling, the simple fraction denoting 3 farthings is $\frac{3}{48}$, and the compound fraction will be $\frac{3}{4}$ of $\frac{1}{12}$, $\left(\text{or } \frac{3}{4} \text{ of a penny}\right)$, and the respective products of the numerators, and the denominators give $\frac{3 \times 1}{4 \times 12}$ or $\frac{3}{48}$ the simple fraction.

Ex. 1. Reduce $\frac{957}{43}$ to a whole, or mixt number.

43) 957 ($22\frac{11}{43}$ *Answer.*
86
97
86
11

2. Reduce $\frac{54800}{2740}$ to its whole, or mixt number.

2740) 54800 (20 *Answer.*
5480
0

3. Reduce $\frac{7200}{1000}$ to its whole, or mixt number.

1000) 7200 ($7\frac{1}{5}$ *Answer.*
7000
200

CASE 3. *To reduce a mixt number to its equivalent improper fraction.*

42. THIS operation is the reverse of the former; therefore multiply the whole number by the denominator of the fraction, and add the numerator to the product, then place the sum over the denominator for the fraction required.

Example. Reduce $22\frac{11}{43}$ to an improper fraction.

22
43
66
88
946
11
957 $\frac{957}{43}$ *Answer.*

Hence to reduce a whole number to an improper fraction having a given denominator:– multiply the said number by the proposed denominator, and make the product the numerator of the required fraction.

Example. Let 13 be reduced to a fraction whose denominator is 7.

$13 \times 7 = 91$ the numerator. *Answer* $\frac{91}{7}$.
For $\frac{91}{7} = 13$ by the preceding article.

CASE 4. *To reduce a compound fraction to an equivalent simple one.*

43. MULTIPLY all the numerators together for the numerator, and all the denominators together for the denominator of the fraction required.

If part of the compound fraction be a mixt, or a whole number, reduce the former to an improper fraction, and make the latter a fraction by placing 1 under it as a denominator.

Ex. 1. Reduce $\frac{1}{2}$ *of* $\frac{3}{4}$ to a simple fraction.

$$\frac{1}{2} \times \frac{3}{4}, \text{ or } \frac{1 \times 3}{2 \times 4} = \frac{3}{8} \textit{ the fraction required.}$$

2. Reduce $\frac{2}{3}$ *of* $\frac{5}{7}$ *of* $3\frac{1}{3}$ *of* 4 to a simple fraction.

$$\text{First } 3\frac{1}{3} = \frac{10}{3}\text{; and } 4 = \frac{4}{1}\text{;}$$

$$\text{Then } \frac{2}{3} \times \frac{5}{7} \times \frac{10}{3} \times \frac{4}{1} = \frac{400}{63} \textit{ answer.}$$

44. When a like number of like factors are found in the numerator and denominator, cancel them in both.

Ex. 3. Reduce $\frac{2}{3}$ *of* $\frac{1}{2}$ *of* $\frac{3}{5}$ *of* $\frac{5}{7}$ *of* $\frac{3}{4}$ to a simple fraction.

$\frac{2 \times 1 \times 3 \times 5 \times 3}{3 \times 2 \times 5 \times 7 \times 4}$ here cancelling 2, 3, and 5, in both numerator and denominator, the fraction becomes $\frac{1 \times 3}{7 \times 4} = \frac{3}{28}$ the *answer*. This is reducing the fraction to lower terms by means of the divisors 2, 3, and 5. (39)

The rule for reducing compound fractions may be derived as follows:— Suppose a shilling to be the integer; then because 48 farthings make 1 shilling, the simple fraction denoting 3 farthings is $\frac{3}{48}$, and the compound fraction will be $\frac{3}{4}$ *of* $\frac{1}{12}$, $\left(\text{or } \frac{3}{4} \text{ of a penny}\right)$, and the respective products of the numerators, and the denominators give $\frac{3 \times 1}{4 \times 12}$ or $\frac{3}{48}$ the simple fraction.

Or more generally thus: let $\frac{3}{4}$ *of* $\frac{5}{7}$ be the compound fraction. Then because $\frac{4 \times 5}{4 \times 7}$ is $= \frac{5}{7}$, the fraction $\frac{5}{4 \times 7}$ will be $\frac{1}{4}$ *of* $\frac{5}{7}$, consequently $\frac{3 \times 5}{4 \times 7}$ will be 3 times $\frac{5}{4 \times 7}$ or $\frac{3}{4}$ *of* $\frac{5}{7}$. And in the same manner we may proceed with any number of fractions, first reducing two of them to a simple fraction, and then taking that and a third, and so on.

Hence it appears that the word *of* in a compound fraction signifies *multiplication*.

CASE 5. ***To reduce fractions of different denominators to equivalent fractions having a common denominator.***

45. THE general rule for this purpose may be derived thus. Let the fractions $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{1}{11}$ be proposed.

Multiply the terms of the fraction $\frac{2}{3}$ by the denominator 7, and we have $\frac{2 \times 7}{3 \times 7} = \frac{2}{3}$. (39)

And the terms of the fraction $\frac{5}{7}$ multiplied by the denominator 3 gives $\frac{3 \times 5}{3 \times 7} = \frac{5}{7}$.

Therefore the fractions $\frac{2 \times 7}{3 \times 7}$ and $\frac{3 \times 5}{3 \times 7}$ $\left(\text{or } \frac{14}{21} \text{ and } \frac{15}{21}\right)$ having the common denominator 21, are respectively equal to the fractions $\frac{2}{3}$ and $\frac{5}{7}$.

Next, taking $\frac{15}{21}$ and $\frac{1}{11}$, and multiplying the terms of the former fraction by 11, and those of the latter by 21, we get $\frac{15 \times 11}{21 \times 11} = \frac{15}{21}$, and $\frac{1 \times 21}{11 \times 21} = \frac{1}{11}$.

Therefore the fractions $\frac{15 \times 11}{21 \times 11}$ and $\frac{1 \times}{11 \times 21}$ having the common denominator 21×11, are respectively equal to $\frac{15}{21}$ and $\frac{1}{11}$, or $\frac{5}{7}$ and $\frac{1}{11}$.

And if the terms of the fraction $\frac{2 \times 7}{3 \times 7}$ $\left(\text{or } \frac{2}{3}\right)$ are multiplied by 11, we have $\frac{2 \times 7 \times 11}{3 \times 7 \times 11} = \frac{2 \times 7}{3 \times 7}$ $\left(\text{or } \frac{2}{3}\right)$.

Consequently the three fractions $\frac{2 \times 7 \times 11}{3 \times 7 \times 11}$, $\frac{11 \times 5 \times 3}{3 \times 7 \times 11}$, $\frac{1 \times 3 \times 7}{3 \times 7 \times 11}$, having the common denominator $3 \times 7 \times 11$, are equal to $\frac{2}{3}$, $\frac{5}{7}$, $\frac{1}{11}$ respectively. And the same method may be extended to any number of fractions.

Hence it appears that the new numerators are found by multiplying each numerator into all the denominators except its own, and that the common denominator is the continued product of all the denominators.

Ex. 2. Reduce $\frac{6}{7}$, $\frac{5}{9}$, and $\frac{2}{3}$ to equivalent fractions having a common denominator.

$$\left.\begin{array}{l} 6 \times 9 \times 3 = 162 \\ 5 \times 7 \times 3 = 105 \\ 2 \times 9 \times 7 = 126 \end{array}\right\} \text{the } \textit{numerators}.$$

$$7 \times 9 \times 3 = 189 \text{ the } \textit{common denominator}.$$

And the fractions are $\frac{162}{189}$, $\frac{105}{189}$, $\frac{126}{189}$, or $\frac{54}{63}$, $\frac{35}{63}$, $\frac{42}{63}$, when abbreviated.

When any factors in the new numerators and common denominator have a common measure or divisor, resolve them into other factors, then (39) reject the like number of like factors in the numerators and denominator, and the fractions will be reduced to the lowest terms which admit of a common denominator.

Ex. 3. Let $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{9}$, be reduced to a common denominator.

The fractions with a common denominator are $\frac{6 \times 9}{4 \times 6 \times 9}$, $\frac{4 \times 9}{4 \times 6 \times 9}$, and $\frac{4 \times 6}{4 \times 6 \times 9}$: now 2, and 3, are the respective divisors of 4 and 6, and 6 and 9; therefore if 6 in the first and third fractions, and 6, 4 and 9 in

the second, are resolved into the factors 2 and 3, the fractions will be $\frac{2\times3\times9}{4\times2\times3\times9}$, $\frac{2\times2\times3\times3}{4\times2\times3\times9}$, and $\frac{4\times2\times3}{4\times2\times3\times9}$, and rejecting 2×3 in the numerators and denominators, we have $\frac{9}{4\times9}$, $\frac{2\times3}{4\times9}$, and $\frac{4}{4\times9}$, or $\frac{9}{36}$, $\frac{6}{36}$, and $\frac{4}{36}$; where the common denominator 36 is the least common multiple or number divisible by 4, 6, and 9. And in the same manner the least common multiple of other proposed numbers may be found, first making them the denominators of fractions having 1 for each numerator.

46. But the least common multiple is readily found by the following rule. (See *art.* 212. vol. 2.)

Write down the proposed numbers in a line, and divide by the prime number 2 as long as it will divide two or more of them without a remainder, and set down the quotients together with the undivided numbers in a line below.——Divide this second line by 2, and also the third line, &c. in the same manner, if they will divide. This done, proceed with 3 the next prime number, and so on to 5, or 7, &c. till there are no two numbers that can be thus divided: Then the continued product of the divisors, the last quotients, and the undivided numbers, is the multiple sought.

Examp. 1. To find the least common multiple of 7, 24, 40, 45, and 72.

2)	7	24	40	45	72
2)	7	12	20	45	36
2)	7	6	10	45	18
3)	7	3	5	45	9
3)	7	1	5	15	3
5)	7	1	5	5	1
	7	1	1	1	1

Then $2\times2\times2\times3\times3\times5\times7 = 2520$ is the multiple required; or the least number divisible by 7, 24, 40, 45, and 72.

Examp. 2. Required the least common multiple of 27, 66, 135, 275, and 675.

3)	27	66	135	275	675
3)	9	22	45	275	225
3)	3	22	15	275	75
5)	1	22	5	275	25
5)	1	22	1	55	5
11)	1	22	1	11	1
	1	2	1	1	1

Then $3 \times 3 \times 3 \times 5 \times 5 \times 11 \times 2 = 14850$ the multiple sought.

47. When the least denominator of two fractions exactly divides the greatest, multiply the terms of that fraction which hath the least denominator by the quotient.

Thus $\frac{3}{4}$ and $\frac{7}{8}$ are brought to a common denominator by multiplying the numerator and denominator of $\frac{3}{4}$ by 2 (the quotient of 8 divided by 4).

And $\frac{2}{3}$, $\frac{5}{6}$, $\frac{11}{12}$ are brought to a common denominator by multiplying the terms of $\frac{2}{3}$ by 4; and those of $\frac{5}{6}$ by 2; the three required fractions being $\frac{8}{12}$, $\frac{10}{12}$, $\frac{11}{12}$.

Or thus:

48. Having reduced the given fractions to their lowest terms, find the least common multiple of the denominators, which divide by the denominators, and multiply the numerators by the corresponding quotients; then the products placed over the said multiple give the fractions in their lowest terms.

Thus, let it be required to reduce the fractions $\frac{3}{14}$, $\frac{5}{22}$, and $\frac{10}{121}$, to equivalent fractions having the least common denominator.

The least common multiple of 14, 22, and 121 is 1694:

$$\left.\begin{array}{l} \frac{1694}{14} = 121 \\ \frac{1694}{22} = 77 \\ \frac{1694}{121} = 14 \end{array}\right\} \text{the three quotients or multipliers.}$$

Then $3 \times 121 = 363$
$5 \times 77 = 385$
$10 \times 14 = 140$ } the three numerators.

And $\frac{363}{1694}$, $\frac{385}{1694}$, $\frac{140}{1694}$, are the fractions required.

ADDITION OF VULGAR FRACTIONS.

49. REDUCE compound fractions to simple ones; and all the fractions to a common denominator. Then add the numerators together and place the sum over the common denominator for the answer.

When the fractions are large, or numerous, it will be best to reduce them to the least common denominator.

Examp. 1. What is the sum of $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$?

$1 + 2 + 3 = 6$. *Ans.* $\frac{6}{4}$ or $1\frac{2}{4}$.

2. Required the sum of $\frac{2}{11}$, $\frac{4}{11}$, and $\frac{5}{11}$?

$2 + 4 + 5 = 11$. *Ans.* $\frac{11}{11}$ or 1.

3. What is the sum of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$?

The fractions when brought to a common denominator will be $\frac{20}{60}$, $\frac{15}{60}$, and $\frac{12}{60}$:

$20 + 15 + 12 = 47$. *Ans.* $\frac{47}{60}$.

4. Required the sum of $\frac{6}{7}$, and $\frac{2}{3}$ *of* $\frac{1}{4}$?

$\frac{2}{3}$ *of* $\frac{1}{4} = \frac{6}{18} = \frac{1}{3}$;

$\frac{1}{3}$ and $\frac{6}{7}$ brought to a common denominator are $\frac{7}{18}$ and $\frac{12}{18}$:

then $7 + 12 = 19$.

Ans. $\frac{19}{18} = 1\frac{1}{18}$.

50. When mixed numbers, or mixed numbers and fractions are to be added together, bring the fractions to a common denominator, then set down the integers as in common addition, and the fractions on the right hand:

Add the fractions together, and carry the integers (if any) from

the sum, to the numbers on the left, which add up as in common addition.

Ex. 5. What is the sum of $421\frac{7}{9}$, $67\frac{1}{2}$, and $\frac{5}{6}$?

$$\begin{array}{r} 421\frac{7}{9} \\ 67\frac{1}{2} \\ \frac{5}{6} \\ \hline 490\frac{1}{9} \end{array} \quad \textit{Answer.}$$

6. Required the sum of $1000\frac{3}{5}$, $74\frac{2}{9}$, and $6\frac{8}{45}$?

The fractions $\frac{3}{5}$, $\frac{2}{9}$, $\frac{8}{45}$ when brought to a common denominator are $\frac{27}{45}$, $\frac{10}{45}$, $\frac{8}{45}$:

$$\begin{array}{rr} & 1000\frac{27}{45} \\ & 74\frac{10}{45} \\ & 6\frac{8}{45} \\ \hline \text{Sum} & 1081 \end{array}$$

SUBTRACTION OF VULGAR FRACTIONS.

31. Let the fractions be prepared the same as for Addition: then the difference of the numerators set over the common denominator will give the difference of the proposed fractions.

Ex. 1. What is the difference of $\frac{1}{4}$ and $\frac{3}{4}$?

The difference of the numerators 1 and 3 is 2; therefore the required difference is $\frac{2}{4}$ or $\frac{1}{2}$.

2. Required the difference of $\frac{11}{19}$ and $\frac{2}{19}$?

$$\frac{11}{19} - \frac{2}{19} \text{ or } \frac{11-2}{19} = \frac{9}{19} \quad \textit{Ans.}$$

3. Required the difference of $\frac{4}{5}$ and $\frac{3}{4}$?

$\frac{4}{5}$ and $\frac{3}{4}$ brought to a common denominator, are $\frac{16}{20}$ and $\frac{15}{20}$; therefore $\frac{16}{20} - \frac{15}{20} = \frac{1}{20}$. *Ans.*

4. What is the difference of $\frac{7}{9}$ and [illegible]?

$\frac{7}{9}$ and [illegible] reduced to a common denominator are [illegible] and [illegible] *therefore the fractions are equal.*

5. From $\frac{4}{9}$ of $\frac{3}{4}$ take $\frac{2}{3}$ of $\frac{1}{4}$.

$\frac{4}{9}$ of $\frac{3}{4} = \frac{12}{36} = \frac{1}{3}$. $\frac{2}{3}$ of $\frac{1}{4} = \frac{2}{12} = \frac{1}{6}$.

$\frac{1}{3}$ and $\frac{1}{6}$ reduced to a common denominator are $\frac{2}{6}$ and $\frac{1}{6}$;
hence $\frac{2}{6} - \frac{1}{6} = \frac{1}{6}$ *Ans.*

82. When the difference of two mixt numbers, or a mixt number and a fraction is required, bring the fractions to a common denominator as before; then place the less number under the greater and take their difference for the answer. But if the lower fraction is greater than the upper one, subtract the numerator of the former from the sum of the terms of the latter, then set down the difference for the numerator of the remaining fraction, and carry 1 to be subtracted.

Ex. 6. From $74\frac{3}{5}$
Take $16\frac{1}{5}$
Rem. $58\frac{2}{5}$

7. From $59401\frac{6}{11}$
Take $7624\frac{8}{11}$
Rem. $51776\frac{9}{11}$

8. Required the difference of $17\frac{2}{7}$ and $1\frac{1}{2}$.

The fractions $\frac{2}{7}$ and $\frac{1}{2}$ reduced to a common denominator are $\frac{4}{14}$ and $\frac{7}{14}$

$17\frac{4}{14}$
$1\frac{7}{14}$
$15\frac{11}{14}$ *Ans.*

9. From 104290
Take $5610\frac{3}{11}$
Rem. $98679\frac{8}{11}$. In this example I take $\frac{3}{11}$ from $\frac{11}{11}$ or 1. And in the preceding example, 7 is taken from 18 (the sum of the terms of the fraction $\frac{4}{14}$), which is the same thing as subtracting $\frac{7}{14}$ from $\frac{4}{14}$ added to $\frac{14}{14}$; for in either case 1 is borrowed, and evidently for the same reason that we borrow 10 in the subtraction of whole numbers when the figure to be subtracted is greater than that above it.

83. The reason why fractions must be brought to a common denominator for the purposes of addition and subtraction, will be evident, if we consider that in order to compare their several values, it is necessary to exhibit them in like parts of the integer.

Thus to compare $\frac{2}{3}$ with $\frac{3}{4}$, if we suppose the integer 1 to be divided into

12 equal parts, $\frac{2}{3}$ will be $\frac{8}{12}$, and $\frac{3}{4}$ will be $\frac{9}{12}$; now the values being expressed in 12*ths* (instead of 3*ds* and 4*ths*) it appears that $\frac{2}{3}$ is less than $\frac{3}{4}$ by $\frac{1}{12}$: also, that both together make $\frac{17}{12}$.

MULTIPLICATION OF VULGAR FRACTIONS.

54. REDUCE mixt numbers to improper fractions; and whole numbers to the form of fractions, by putting 1 for the denominators. Then multiply the numerators together for the numerator, and the denominators together for the denominator of the product. This rule is the same as that for reducing a compound fraction to a simple one; for when the multiplier is a fraction, the product will be a part or parts *of* the multiplicand: thus $\frac{1}{2}$ *of* $\frac{1}{2}$ is $\frac{1}{4}$ or $\frac{1 \times 1}{2 \times 2}$; and $\frac{2}{3}$ *of* $\frac{3}{4}$ is $\frac{1}{2}$ or $\frac{2 \times 3}{3 \times 4}$; and therefore the fractions to be multiplied may be set down in the form of a compound fraction, aud the product found in the same manner as that is reduced to a simple one.

Examp. 1. What is the product of $\frac{3}{7}$ and $\frac{5}{8}$?

$$\frac{3 \times 5}{7 \times 8} = \frac{15}{56} \quad \textit{Ans.}$$

2. Required the product of $\frac{4}{9}$ and $\frac{18}{19}$?

$$\frac{4 \times 18}{9 \times 19} = \frac{4 \times 2 \times 9}{9 \times 19} = \frac{4 \times 2}{19} = \frac{8}{19} \quad \textit{Ans.}$$

3. What is the continued product of 4, $7\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ of $\frac{6}{7}$?

First $4 = \frac{4}{1}$; and $7\frac{1}{2} = \frac{15}{2}$.

Then,

$$\frac{4 \times 15 \times 2 \times 5 \times 6}{1 \times 2 \times 3 \times 6 \times 7} = \frac{4 \times 15 \times 5}{3 \times 7} = \frac{4 \times 3 \times 5 \times 5}{3 \times 7} = \frac{4 \times 5 \times 5}{7} = \frac{100}{7} = 14\frac{2}{7} \quad \textit{Ans.}$$

4. What is $\frac{2}{3}$ of 29?

$\frac{2}{3} \times \frac{29}{1} = \frac{58}{3} = 19\frac{1}{3}$ the *answer*. Therefore to find the product of a fraction and a whole number, multiply by the numerator, and divide by the denominator.

55. When one factor is a whole, and another a mixt number, or if one is a small fraction, and another a large mixt number, multiply the parts of the latter separately, and add the products together.

Ex. 5. Required the product of $6742\frac{1}{121}$ by 8?

$$\begin{array}{r} 6742\frac{1}{121} \\ 8 \\ \hline 53936\frac{8}{121} \end{array} \quad \textit{Ans.}$$

6. What is the product of $597\frac{4}{5}$ and 24?

$$\begin{array}{rl} 597 \times 24 = & 14328 \\ \frac{4}{5} \times 24 = & 19\frac{1}{5} \\ \hline \text{Sum} & 14347\frac{1}{5} \end{array} \quad \textit{Ans.}$$

7. What is $\frac{3}{4}$ of $9614273\frac{1}{18}$?

$$\begin{array}{r} 9614273 \\ 3 \\ \hline 4\,)\,28842819 \\ \hline 7210704\frac{3}{4} \end{array} \qquad \frac{3}{4} \textit{ of } \frac{1}{18} = \frac{1}{24}$$

$$7210704\tfrac{3}{4} + \tfrac{1}{24} = 7210704\tfrac{19}{24} \quad \textit{Ans.}$$

56. And when both factors are mixt numbers, the product may be found by multiplying the parts separately, as in the next example.

Ex. 8. Required the product of $574\frac{1}{2}$ by $485\frac{3}{4}$?

$$\begin{array}{rl} 574 \times 485 = & 278390 \\ \frac{1}{2} \times 485 = & 242\frac{1}{2} \\ 474 \times \frac{3}{4} = & 430\frac{1}{2} \\ \frac{1}{2} \times \frac{3}{4} = & \frac{3}{8} \\ \hline \text{Sum} & 279063\frac{3}{8} \end{array} \quad \textit{Ans.}$$

DIVISION OF VULGAR FRACTIONS.

57. PREPARE the fractions the same as for multiplication; then divide the terms of the dividend by the respective terms of the divisor, if they will exactly divide; but if not, then invert the divisor and proceed as in multiplication.

When the terms exactly divide, the truth of the rule is manifest from the principles of common division. And the reason for inverting the divisor in the other case will be evident if we consider that division is the reverse of multiplication: thus the product of $\frac{1}{2}$ and 4 is $\frac{1}{2} \times \frac{4}{1} = 2$ or the half of 4; but 4 divided by $\frac{1}{2}$ will give 8, because $\frac{1}{2}$ is contained 8 times in 4, the quotient being $\frac{4}{1} \times \frac{2}{1}$, where $\frac{2}{1}$ is the divisor $\frac{1}{2}$ inverted.

As a second example, let $\frac{3}{7}$ be divided by $\frac{2}{5}$; or suppose it is required to find how often $\frac{2}{5}$ is contained in $\frac{3}{7}$. Now if we divide 3 by $\frac{1}{5}$, the quotient will be $\frac{3}{1} \times \frac{5}{1}$ or 15, (becaus $\frac{1}{5}$ is contained 15 times in 3); but when the divisor is twice $\frac{1}{5}$, or $\frac{2}{5}$, the quotient will be only $\frac{1}{2}$ of 15, or $\frac{3 \times 5}{2}$ the quotient of 3 divided by $\frac{2}{5}$, consequently the 7*th.* of 3 (or $\frac{3}{7}$) will give but a 7*th.* of that quotient, or $\frac{3 \times 5}{2 \times 7}$; therefore the quotient $\frac{3}{7}$ divided by $\frac{2}{5}$ is truly expressed by $\frac{3 \times 5}{2 \times 7}$ equal to $1\frac{1}{14}$.

Ex. 3. Divide $\frac{16}{21}$ by $\frac{4}{7}$

$\frac{4}{7}$) $\frac{16}{21}$ ($\frac{4}{3}$ *quotient or answer.*

4. Required the 5*th* part of $\frac{10}{13}$?

$\frac{5}{1}$) $\frac{10}{13}$ ($\frac{2}{13}$ *Ans.*

5. Divide $\frac{9}{35}$ by $\frac{3}{5}$?

$$\frac{5}{3} \times \frac{9}{35} = \frac{5 \times 9}{7 \times 3 \times 5} = \frac{9}{7 \times 3} = \frac{9}{21} \textit{ Ans.}$$

6. Divide $\frac{5}{6}$ *of* $\frac{3}{4}$ by $\frac{5}{4}$ *of* $\frac{4}{7}$?

The divisor $\frac{5}{4}$ *of* $\frac{4}{7}$ when inverted is $\frac{4}{5} \times \frac{7}{4}$;

$$\frac{5 \times 7 \times 3 \times 4}{6 \times 4 \times 4 \times 5} = \frac{7 \times 3}{6 \times 4} = \frac{7 \times 3}{3 \times 2 \times 4} = \frac{7}{2 \times 4} = \frac{7}{8} \textit{ Ans.}$$

7. Let $3\frac{2}{5}$ be divided by $4\frac{1}{6}$?

$3\frac{2}{5} = \frac{17}{5}$, and $4\frac{1}{6} = \frac{25}{6}$;

$\frac{6}{25} \times \frac{17}{5} = \frac{102}{125}$ *Ans.*

8. Divide 1 by $\frac{4}{7}$?

$\frac{1}{1} \times \frac{7}{4} = \frac{7}{4}$ *quotient:* this is called the *reciprocal* of the divisor $\frac{4}{7}$.

9. Divide $\frac{3}{8}$ by 3?

$\frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$ *quotient*. Therefore to divide a fraction by a whole nber, multiply the denominator by that number, except it will divide numerator, as in *Ex.* 4.

58. If the divisor is a *whole*, and the dividend a large *mixt* nber, divide the parts separately, and then add the quotients ether.

Ex. 10. Required the *5th.* part of 4561412$\frac{4}{5}$?

$$
\begin{array}{rrl}
5\,) & \underline{4561412} & \\
 & 912282\frac{2}{5} & \text{the integral part divided by 5.} \\
 & \underline{\frac{4}{25}} & \text{the fraction } \frac{4}{5} \text{ divided by 5.} \\
\text{Sum} & \underline{912282\frac{14}{25}} & \text{the } \textit{answer}.
\end{array}
$$

59. When the divisor is a small fraction and the dividend a large mixt number, multiply the latter (without reducing it an improper fraction) by the denominator of the divisor, and ide the product by the numerator.

Ex. 11. Divide 6421078 $\frac{2}{11}$ by $\frac{5}{6}$?

$$
\begin{array}{rrl}
 & 6421078\frac{2}{11} & \\
 & \underline{6} & \\
5\,) & \underline{38526469\frac{1}{11}} & \text{product by the denominator 6.} \\
 & 7705293\frac{4}{5} & \text{the whole number divided by 5.} \\
 & \underline{\frac{1}{55}} & \text{the fraction } \frac{1}{11} \text{ divided by 5.} \\
\text{Sum} & \underline{7705293\frac{4}{5}\frac{1}{55}} & \textit{Ans.}
\end{array}
$$

60. In like manner the quotient is found in the contracted thod of division of whole numbers when the divisor is the duct of two or more factors. (30. *Ex.* 11.)

Ex. 12. Let 8783 be divided by 56, or 7 times 8.

$$
\begin{array}{rrl}
7\,) & 8783 & \\
8\,) & \underline{1254\frac{5}{7}} & \text{quotient by 7.} \\
 & 156\frac{6}{8} & \text{the whole number divided by 8.} \\
 & \underline{\frac{5}{56}} & \text{the fraction } \frac{5}{7} \text{ divided by 8.} \\
\text{Sum} & \underline{156\frac{6}{8}\frac{5}{56}} & \textit{Ans.}
\end{array}
$$

OF DECIMALS.

61. DECIMALS are Fractions in the form of whole numbers, but whose values decrease from the place of units progressively to the right hand in the same decuple or tenfold proportion as the common scale of whole numbers increase to the left. They are usually separated from the integers by a comma or dot, the decimals being on the right hand.

Thus the mixt number $21\frac{2}{10}$ when the fraction is set down decimally will be 21·2; the 2 on the right of the 1, or dot, denotes 2 *tenths*, whereas the other 2 on the left are 2 *tens*. Another 2 on the left will be 2 *hundreds*, but on the right 2 *hundredths*, ($\frac{2}{100}$), and the whole or 221·22 is $221\frac{22}{100}$, because $\frac{2}{10}$ and $\frac{2}{100}$ together make $\frac{22}{100}$. A third figure on the left will be *thousands*, but on the right, the like number of *thousandth parts*. Thus 5008·005 is the same as $5008\frac{5}{1000}$; and 5000·005 the same as $5000\frac{5}{1000}$. Consequently a decimal fraction has always either 10, 100, 1000, &c. for its denominator; *viz.* the number of equal parts into which the *integer* or *whole* is supposed to be divided. For example, let a foot in length be the integer, and conceive it to be divided into 100 equal parts; then ·25 (or 25 with a dot on the left) will be the decimal part of a foot denoting 3 inches or $\frac{1}{4}$ ($\frac{25}{100}$ being $= \frac{1}{4}$). And $1\frac{1}{2}$ inches or $\frac{1}{8}$ of a foot will be ·125 of a foot, because 125 is $\frac{1}{8}$ of 1000; the foot in this case is supposed to be divided into 1000 equal parts.

Therefore to read, or set down a proposed decimal, it is only necessary to remember that the denominator is 1 with as many ciphers annexed as there are decimal places, or that the same number of figures to the right of the decimal point have always the same common denominator. Thus the denominator of the

fractions ·5000, ·0746, ·0005, is 10000. And hence it appears that the value of a decimal fraction is not altered by ciphers on the right hand; for ·5000 (or $\frac{5000}{10000}$) when reduced to its lowest terms is the same as ·5, each being equal to $\frac{1}{2}$.

ADDITION AND SUBTRACTION OF DECIMALS.

62. PLACE the numbers so that the decimal points may stand directly under each other; then add, and subtract, as in whole numbers, and set the decimal point in the sum or difference directly under the points above.

Ex. 1. Required the sum of ·7, ·014, and ·1246?

	·7
	·014
	·1246
Sum	·8386

By placing the decimal points under each other, tenths are brought under tenths, hundredths under hundredths, &c. whence the method of addition becomes the same as that for whole numbers.

The decimals in the foregoing example set down as vulgar fractions are $\frac{7}{10}$, $\frac{14}{1000}$, and $\frac{1246}{10000}$, and when brought to a common denominator will be $\frac{7000}{10000}$, $\frac{140}{10000}$, and $\frac{1246}{10000}$;

hence	7000
	140
	1246
	8386

the sum of the numerators, and $\frac{8386}{10000}$ *the sum of the fractions* as before: but this is evidently nothing more than reducing the decimals to a common denominator by annexing ciphers on the right hand:

Thus	·7000
	·0140
	·1246
Sum	·8386

Ex. 2. What is the sum of ·0139, 54·77 and 9·299?

	·0139
	54·77
	9·299
Sum	64·0829

Ex. 3. Required the sum of 9 *tenths*, 19 *hundredths*, 18 *thousandths*, 211 *hundred thousandths*, and 19 *millionth parts*?

$$\begin{array}{rl} & \cdot 9 \\ & \cdot 19 \\ & \cdot 018 \\ & \cdot 00211 \\ & \underline{\cdot 000019} \\ \textit{Sum} & 1\cdot 110129 \end{array}$$

4. Required the difference of ·406 and ·11?

$$\begin{array}{rl} & \cdot 406 \\ & \underline{\cdot 11} \\ & \cdot 296 \ \textit{Ans.} \end{array}$$

5. What is the difference of 49·01 and ·9078?

$$\begin{array}{rl} & 49\cdot 01 \\ & \underline{0\cdot 9078} \\ & 48\cdot 1022 \ \textit{Ans.} \end{array}$$

6. What is the difference of 1 and ·042?

$$\begin{array}{rl} & 1 \\ & \underline{0\cdot 042} \\ & 0\cdot 958 \ \textit{Ans.} \end{array}$$

7. Required the difference of 594·0012 and 24·98?

$$\begin{array}{rl} & 594\cdot 0012 \\ & \underline{24\cdot 98} \\ & 569\cdot 0212 \ \textit{Ans.} \end{array}$$

MULTIPLICATION OF DECIMALS.

63. MULTIPLY as in whole numbers, and point off as many places for decimals in the product as there are decimals in both multiplier and multiplicand; but if there should not be so many, put ciphers on the left to supply the defect.

Ex. 1. Required the product of ·2 and ·03?

$$\begin{array}{rl} & \cdot 03 \\ & \underline{\cdot 2} \\ & \cdot 006 \ \textit{Ans.} \end{array}$$

The decimals ·2 and ·03 when set down as vulgar fractions will be $\frac{2}{10}$ and $\frac{3}{100}$, and their product $\frac{2}{10} \times \frac{3}{100} = \frac{6}{1000}$ or 6 thousandth parts, as before. Hence the truth of the rule is evident.

Other Examples.

Multiply	·621
By	·26
	3726
	1242
Product	·16146

Multiply	·043
By	·003
Product	·000129

Multiply	621
By	·26
	3726
	1242
Product	161·46

Multiply	·0023
By	1700
	161
	23
Product	3·9100 or 3·91.

Multiply	·642
By	10000
Product	6420·000.

Therefore multiplying by 10, 100, 1000, &c. is only removing the decimal point so many places to the right as there are ciphers in the multiplier. Thus 82·1 multiplied by 10 is 821; 4·4 multiplied by 1000 is 4400, &c.

64. There is a method of contracting the operation so as to retain only a proposed number of decimals in the product. Let ·5849 be multiplied by 7·26, and the product have only 3 decimal places.

·5849	
7·26	
35	094
116	98
4094	3
4·246	374

The required product is 4·246. But to omit setting down the figures on the right of the perpendicular bar, yet retain the product to the left, it is evident that the multiplication by the integer 7 must begin at 4 in the multiplicand or the 3d. place in the decimal from the left (3 being the number of decimals to be retained); the multiplication by 2 must begin at the 8; and that by 6 at the 5, remembering to carry from the figures omitted on the right hand, as in common multiplication. But when the figures of the multiplier are set down

in a contrary order, and the units place (7) is under (4) from the left, the figures in the multiplier will stand direct in the multiplicand where the respective multiplications mu

'3849
67·7
4094
117
35
4·246 Here 6 is carried to 7 times 4, because 7 figure omitted) is 63;—1 is carried to 2 times 8, because t figure on the right of 8) exceeds half 10.—And 5 is carried to since 6 times 8 make almost 5 tens.

As a further illustration of this method of contraction, take ing examples.

Multiply 8467·73912 by 0·725184, reserving only 4 decima product.

8467·73912
4815270
59274174 Here (0) the units place stands under the 4th from the left.
1693548
423387
8468
6774
338
6140·6689 *Product.*

Multiply 3842·63 by 79·6543, retaining the integers only.

3842·63
3456·97 Here units stand under units, no decimals being requi in the product.
268984
34584
2306
192
13
1
306082 *Product.*

DIVISION OF DECIMALS.

63. DIVIDE as in whole numbers and point off as many decimals in the quotient as the number of decimals in the dividend exceed those in the divisor. But if the number of figures in

the quotient are not so many as the rule requires, prefix ciphers on the left to supply the defect.

If the number of decimals in the divisor exceed those in the dividend, annex ciphers to the latter before you begin the division.

When the divisor is 1 with ciphers on the right hand, remove the decimal point in the dividend as far to the left as there are ciphers.—But when the divisor is any other number with ciphers annexed, first divide by 10, 100, or 1000, &c. according to the number of ciphers; then divide the quotient by the remaining figure or figures. (60)

N. B. Should there be a remainder after division, ciphers may be annexed to it, and the division continued as far as is necessary.

Examples.

Divide ·01728 by 14·4?

14·4) ·01728 (·0012 *quotient.*
144
288
288

Proof
14·4
·0012
Product ·01728 Hence it appears that the number of decimals in the divisor and quotient must be equal to those in the dividend; and therefore the truth of the rule is manifest.

Divide 17·28 by 14·4?

14·4) 17·28 (1·2 *quotient.*
144
288
288

Divide ·2123 by ·84?

·84) ·2123 (·25 &c. *quotient.*
168
443
420
23

Divide 172·8 by ·144?

```
·144 ) 172·800 ( 1200 quotient.
       144
       ---
       288
       288
       ---
```

Divide 192 by 5·423?

```
5·423 ) 192·0000 ( 35·4 &c. quotient.
        162 69
        ------
         29 310
         27 115
         ------
          2 1950
          2 1692
          ------
             258
```

Divide 542·3 by 10?

```
10 ) 542·3
     54·23 quotient.
```

Divide 29·74 by 1000?

```
1000 ) 29·74
       ·02974 quotient.
```

Divide 6·48 by 200?

```
100 ) 6·48
  2 ) ·0648   quotient by 100.
      ·0324   Answer.
```

Divide 64·9 by 7000?

```
1000 ) 64·9
   7 ) ·0649   quotient by 1000.
       ·0092   &c. Ans.
```

Divide 594·27 by 470?

```
10 ) 594·27
47 ) 59·427   quotient by 10.
     47       (1·2644 &c. Ans.
     ---
     124
      94
     ---
     302
     282
     ---
      207
      188
      ---
       190
       188
       ---
         2
```

Divide 290·6 by 24000?

```
1000 ) 290·6
   4 ) ·2906
   6 ) ·07265
       ·0121083 &c. Ans.
```

66. When a certain number of decimals only are wanted in the quotient, the division may be contracted in the following manner:

Take the divisor one figure more than the number of figures required to be in the quotient.

Make each remainder a new dividend, and for every such dividend leave out a figure on the right hand of the divisor, remembering to carry for the increase of the figures omitted as in the contraction of multiplication. (64)

Let 94·78 be divided by 2·84671281 so as to have 4 decimals in the quotient.

The number of figures in the quotient will be 6, viz. 2 integers and 4 decimals, therefore we must take 7 figures for the divisor.

```
2·846712 | 81 ) 94·78000 ( 33·2945 + quotient.
                8540138
       284671 )  937862
                 854014
        28467 )   83848
                  56934
         2846 )   26914
                  25620
          284 )    1294
                   1138
           28 )     156
                    142
                     14
```

The two right hand figures (81) of the given divisor are cut off, and 2 are carried for the product of 8 by 3. And instead of bringing down each divisor (as above) the figures may successively be pointed off. It is also evident when the number of figures in the divisor is less than the number required in the quotient, that ciphers must be added to the former.

To reduce a Vulgar Fraction to an equivalent Decimal.

67. ADD ciphers to the numerator and divide by the denominator, then point off as many decimal places in the quotient for the answer as there were ciphers annexed. This is continuing the division of whole numbers when there is a *remainder*, by which means we get a *decimal* in the quotient instead of a vulgar fraction.

For example, if 97 be divided by 32, the quotient is $\frac{97}{32}$ or $3\frac{1}{32}$, but if ciphers are added we shall have 3·03125 for the quotient.

Thus,

```
32 ) 97·00000 ( 3·03125 quotient.
     96
      100
       96
        40
        32
         80
         64
         160
         160
```

The ciphers annexed only point out the number of decimal places, and therefore 97·00000 is the same as 97, consequently 97·00000 and 97 when divided by 32 must give equal quotients, and therefore the decimal ·03125 is equivalent to $\frac{1}{32}$. Which also will be evident by taking the decimal as a vulgar fraction, for $\frac{3125}{100000}$ reduced to its lowest terms is $\frac{1}{32}$.

Other Examples.

Reduce $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ to equivalent decimals?

4) 1·00 — ·25 2) 1·0 — ·5 4) 3·00 — ·75 *Ans.* ·25, ·5, and ·75.

Reduce $\frac{3}{625}$ to a decimal?

625) 3·0000 (·0048 *Ans.*
2 500
5000
5000

Reduce $\frac{2}{12500}$ to a decimal?

12500) 2·00000 (·00016 *Ans.*
1 2500
75000
75000

Reduce $\frac{946}{10000}$ to a decimal?

10000) 946
·0946 *Ans.*

Reduce $\frac{10}{12}$ to a decimal?

12) 10·000 (·833 &c. *Ans.*
9 6
40
36
40
36
4

Divide 98 by 24000; or which is the same thing, reduce $\frac{98}{24000}$ to a decimal?

1000) 98
4) ·098
6) ·0245
·0040833 &c. *Ans.*

Reduce $\frac{1}{7}$ to a decimal?

7) 1·000 &c.
·1428571428 &c. *Ans.*

Reduce $\frac{1}{9}$ to a decimal?

9) 1·0000 &c.
·1111111 &c. *Ans.*

The decimals in the last examples are called circulating, recurring, or repeating decimals, because the same figure or figures are regularly repeated.

68. When an improper fraction is to be reduced, the answer will be a mixt number. Thus $\frac{37}{4}$ = 9·25; and $\frac{38}{3}$ = 12·666 &c.

69. If the right hand figure of the denominator of the fraction to be reduced is either 1, 3, 7, or 9, it is manifest there will always be a remainder after division, and consequently the decimal can never be brought exactly equal to the vulgar fraction: in such cases continue the division to as many places of figures as may be thought necessary.

OF DUODECIMALS.

70. DUODECIMALS are vulgar fractions whose denominators are 12, or 144, or 1728, &c. The division and sub-division of the integer being in 12*ths* instead of 10*ths* as in the decimal scale. The method of operating with these fractions (sometimes called *cross multiplication*) seems to have been introduced for the purpose of computing the contents of artificer's work when the dimensions are in *feet* and *inches*, or in *feet*, *inches*, and *lines* or 12*ths* of an *inch*.

71. *Rule.* Set down the dimensions to be multiplied together so that feet may stand under feet, inches under inches, &c.

Begin with the feet in the multiplier and the lowest dimension in the multiplicand. Carry 1 to the left for every 12 in each product, setting down the remainders directly under the corresponding terms of the multiplicand. Next, multiply by the inches, and carry 1 for every 12 as before, remembering to set down the result one place to the right of the product by the feet. And in the same manner proceed with a third dimension, &c. Then add the several products together, carrying 1 for every 12 as before, and the sum will be the answer.

		F.	*in.*	*lines.*
Examp. 1.	Multiply	7	9	0
	By	5	6	0
		38	9	0
		3	10	6
	Product	42	7	6

Or $42 + \frac{7}{12} + \frac{6}{144} = 42\frac{90}{144}$ *Feet.*

The reason for setting down the products in the manner directed by the rule will be evident if we retain the denominators, but multiply by the parts separately as before (see examp. 8. in the multiplication of vulgar fractions):

Thus,

	Feet.	
Multiply	$7\frac{9}{12}$	
By	$5\frac{6}{12}$	
	$38\frac{9}{12}$	..................Product by 5.
	$3\frac{10}{12}\ \frac{6}{144}$	........Product by $\frac{6}{12}$.
Product	$42 + \frac{7}{12} + \frac{6}{144}$	

First, $\frac{9}{12} \times 5 = \frac{45}{12} = 3\frac{9}{12}$ or 3 to be carried and $\frac{9}{12}$ to set down, and $7 \times 5 = 35$ which with 3 make 38.

Secondly, $\frac{9}{12} \times \frac{6}{12} = \frac{54}{144}$ or $\frac{4}{12}$ to be carried and $\frac{6}{144}$ to set down (because 12 of any denomination make 1 of the next superior or that on the left):

And $7 \times \frac{6}{12} = \frac{42}{12}$ which with $\frac{4}{12}$ (that were carried) make $\frac{46}{12}$, or $3\frac{10}{12}$ to set down.

The answer however, in this example, is sooner obtained by the usual method of vulgar fractions, thus:

F.	*in.*		*F.*		*F.*
7	9	=	$7\frac{3}{4}$	=	$\frac{31}{4}$
5	6	=	$5\frac{1}{2}$	=	$\frac{11}{2}$

And $\frac{11}{2} \times \frac{31}{4} = \frac{341}{8} = 42\frac{5}{8}$ *feet*, as before.

Ex. 2. Required the number of square feet in a rectangular board whose length is 17 *F.* $5\frac{1}{2}$ *in.*, and breadth 1 *F.* $8\frac{3}{4}$ *in.*?

N. B. The two dimensions multiplied together give the answer.

	F.	*in.*	*lines.*				
	17	5	6				
	1	8	9				
	17	5	6				
	11	7	8	0			
	1	1	1	1	6	*Sq. F.*	*Sq. in.*
Product	30	2	3	1	6	or 30	and $27\frac{1}{8}$.

Ex. 3. Required the number of cubic feet in a rectangular ditch whose breadth is $8\frac{1}{2}$, depth $4\frac{2}{3}$, and length 121 feet?

Here the three dimensions must be continually multiplied together.

F.	*In.*
4	8
8	6
37	4
2	4
39	8
121	0
4799	8, or $4799\frac{8}{12}$, *cubic feet.*

Or thus,

121	
$4\frac{2}{3}$	
484	.... product by 4.
$40\frac{1}{3}$	.... product by $\frac{1}{3}$.
$40\frac{1}{3}$	.... ditto.
$564\frac{2}{3}$	.... $= 121 \times 4\frac{2}{3}$.
$8\frac{1}{2}$	
$4517\frac{1}{3}$	.... product by 8.
$282\frac{1}{3}$	.... product by $\frac{1}{2}$.
$4799\frac{2}{3}$	the *answer*, as before.

TABLES

12. OF MONEY, *WEIGHTS*, *AND* MEASURES.

OF MONEY.

		Marked.
Farthings	....	*qrs.*
4 Farthings	.... = 1 Penny	*d.*
12 Pence	.... = 1 Shilling	*s.*
20 Shillings	.... = 1 Pound	£.
240 Pence, or 960 Farthings	.. = 1 £.	

PENCE TABLE.

d.		s.	d.
20	is	1	8
30		2	6
40		3	4
50		4	2
60		5	0
70		5	10
80		6	8
90		7	6
100		8	4
110		9	2
120		10	0

TROY WEIGHT.

		Marked.
Grains		*gr.*
24 Grains	= 1 Pennyweight	*dwt.*
20 Pennyweights	= 1 Ounce	*oz.*
12 Ounces	= 1 Pound	*lb.*

By this weight are weighed Gold, Silver, Jewels, and some Liquids. Jewellers sometimes express the weight of a diamond in carats of 4 grains (troy-weight) each. A carat however, signifies the $\frac{1}{24}$ of any mass of gold, or of gold with alloy, and is generally used to denote its degree of fineness.

APOTHECARIES WEIGHT.

		Marked.
Grains		*gr.*
20 Grains	*make* 1 Scruple	*sc.* or ℈
3 Scruples	1 Dram	*dr.* or ʒ
8 Drams	1 Ounce	*oz.* or ℥
12 Ounces	1 Pound	*lb.* or ℔

The Pound is the same as the Pound Troy, only differently divided.

Apothecaries use this weight in compounding their medicines, but buy and sell their Drugs by Avoirdupois Weight.

AVOIRDUPOIS WEIGHT.

				Marked.	
	Drams			*dr.*	
16	Drams	.... *make*	1 Ounce	*oz.*	*gr.*
16	Ounces		1 Pound	*lb.*	= 7000 Troy
28	Pounds		1 Quarter	*qr.*	
4	Quarters, or 112lb.		1 Hundred Weight	*cwt.*	
20	Hundred		1 Ton	*ton.*	

And 8lb. is a Stone in the London Markets.
14lb. a Stone, Horseman's Weight.
28lb. a Tod.

By this Weight are weighed all Groceries, Chandler's Wares, some Liquids, and all Metals except Gold and Silver.

LONG MEASURE.

3	Barley Corns	*make*	1 Inch.
12	Inches		1 Foot.
3	Feet		1 Yard.
6	Feet		1 Fathom.
5½	Yards, or 16½ Feet		1 Rod, Pole or Perch.
40	Rods		1 Furlong.
8	Furlongs, or 1760 Yards		1 Mile.
3	Miles		1 League.
69½	Miles (nearly)		1 Degree.
360	Degrees the Earth's circumference.		

Also,

4	Inches	*make*	1 Hand, or hands breadth.
5	Feet		1 Geometrical Pace.
4	Poles, or 66 Feet,		1 Chain.
100	Links, each $7\frac{92}{100}$ *in.*		1 Chain.

CLOTH MEASURE.

2¼	Inches	*make*	1 Nail.
4	Nails		1 Quarter of a Yard.
4	Quarters		1 Yard.
3	Quarters		1 Ell Flemish.
5	Quarters		1 Ell English.
6	Quarters		1 Ell French.

SQUARE MEASURE.

144	Square Inches	*make*	1 Foot Square.
9	Square Feet		1 Yard.
30¼	Square Yards		1 Pole.
40	Square Poles		1 Rood.
4	Roods, or 160 Square Poles		1 Acre.
4840	Square Yards		1 Acre.
10	Square Chains		1 Acre.
100000	Square Links		1 Acre.

By this Measure, Land, and all Works which have length and breadth only, are measured.

CUBIC OR SOLID MEASURE.

1728	Cubic Inches	*make*	1 Foot.
27	Cubic Feet		1 Yard.

By this Measure, Stone, Timber, and all Works of three dimensions (length, breadth, and depth) are measured.

DRY OR CORN MEASURE.

2	Pints	*make*	1 Quart.
2	Quarts		1 Pottle.
2	Pottles, or 4 Quarts		1 Gallon.

76. When a greater denomination is to be reduced to a less (as pounds to shillings, or feet to inches) the process is by Multiplication. But less denominations are brought to greater by Division.

Ex. 1. Reduce £84 to shillings, pence, and farthings?

By the first of the foregoing tables it is evident that
Pounds multiplied by 20 give shillings.
Shillings multiplied by 12 give pence.
Pence multiplied by 4 give farthings.

Consequently,
Farthings divided by 4 give pence.
Pence divided by 12 give shillings.
Shillings divided by 20 give pounds.

$$
\begin{array}{rl}
84 & \\
20 & \\
\hline
1680 & \text{the shillings.} \\
12 & \\
\hline
20160 & \text{the pence.} \\
4 & \\
\hline
80640 & \textit{the farthings.}
\end{array}
$$

Ex. 2. Reduce 80640 farthings to pounds?

$$
\begin{array}{r|rl}
4\,) & 80640 & \\
12\,) & 20160 & \text{the pence.} \\
20\,) & 1680 & \text{the shillings.} \\
 & 84 & \textit{the pounds.}
\end{array}
$$

Or because 960 farthings are equal in value to £1, if 80640 be divided by 960 the quotient will be the number of pounds required.

$$
\begin{array}{rl}
 & \qquad\; £ \\
960\,)\;80640\;(& 84 \textit{ as before.} \\
7680 & \\
\hline
3840 & \\
3840 & \\
\hline
\end{array}
$$

Ex. 3. Reduce 26779 farthings to pounds?

$$
\begin{array}{r|rll}
4\,) & 26779 & & \\
12\,) & 6694 & \text{———} \tfrac{3}{4} & \\
20\,) & 557 & \text{———} 10 & \qquad £\;\; s.\;\; d. \\
 & 27 & \text{———} 17 & \textit{Ans.}\; 27\;\, 17\;\, 10\tfrac{3}{4}
\end{array}
$$

24 Hours	1	Natural Day.
365 Days, 6 Hours	1	Julian Year.
365 *Days*, 5 *h.* 48 *min.* 48 *sec.*	1	Solar Year.

73. FOREIGN MEASURES OF LENGTH.

The Rhynland Foot	= 1·033	English
Rhynland Rood, 12 Rhynland Feet	= 12·396	Feet.

	Yards.
The French Toise, 6 Paris Feet	2·1315
Common French League, 2000 Toises	4263
Common French League, 25 to a degree	4869
Brabant League, 2800 Toises (nearly)	5968
Italian Mile, 60 to a Degree	2029
German Mile, 15 to a Degree	8116

The scales to the French and the German Military Maps and Plans are commonly in Leagues, Miles, Toises, or Rhynland Roods. But the "*mean*" and "*common*" German Miles seem to be of no determinate lengths; according to the Table in Teilke's *Field Engineer*, they vary from 19020 to 28530 Paris feet. And we sometimes find a scale denominated, "a mile, or 2 hours walk on the road."

74. From the measurements lately carried on through France and part of Spain, the French Mathematicians conclude (according to a particular hypothesis) that ¼ of the whole terrestrial meridian is 5130740 *Toises* in length; and the *ten millionth part*, or ·5130740 of a *Toise* is the "*Metre,*" or standard for the measures of length now adopted in France. This Metre is equal to 3·280832 English Feet.

OF REDUCTION.

75. THE operation of changing numbers from one name or denomination to another without altering their value, is called *Reduction*.

76. When a greater denomination is to be reduced to a less (as pounds to shillings, or feet to inches) the process is by Multiplication. But less denominations are brought to greater by Division.

Ex. 1. Reduce £84 to shillings, pence, and farthings?

By the first of the foregoing tables it is evident that
Pounds multiplied by 20 give shillings.
Shillings multiplied by 12 give pence.
Pence multiplied by 4 give farthings.

Consequently,
Farthings divided by 4 give pence.
Pence divided by 12 give shillings.
Shillings divided by 20 give pounds.

$$\begin{array}{rl} 84 & \\ 20 & \\ \hline 1680 & \text{the shillings.} \\ 12 & \\ \hline 20160 & \text{the pence.} \\ 4 & \\ \hline 80640 & \textit{the farthings.} \end{array}$$

Ex. 2. Reduce 80640 farthings to pounds?

$$\begin{array}{r|rl} 4\,) & 80640 & \\ 12\,) & 20160 & \text{the pence.} \\ 20\,) & 1680 & \text{the shillings.} \\ & 84 & \textit{the pounds.} \end{array}$$

Or because 960 farthings are equal in value to £1, if 80640 be divided by 960 the quotient will be the number of pounds required.

$$\begin{array}{rl} & \quad\;\; £ \\ 960\,)\,80640\,(& 84 \textit{ as before.} \\ 7680 & \\ \hline 3840 & \\ 3840 & \\ \hline \end{array}$$

Ex. 3. Reduce 26779 farthings to pounds?

$$\begin{array}{r|rlll} 4\,) & 26779 & & & \\ 12\,) & 6694 & \text{——} \tfrac{3}{4} & & \\ 20\,) & 557 & \text{——} 10 & & £\ \ s.\ \ d. \\ & 27 & \text{——} 17 & \textit{Ans.} & 27\ 17\ 10\tfrac{3}{4} \end{array}$$

15. If 10000 men have each 40 rounds of cartridge with ball, what is the whole weight of lead, the balls being an ounce each?

$$10000 \times 40 = \overset{oz.}{400000}.$$

$$\frac{400000}{16} = \overset{lb.}{25000}.$$

$$\frac{25000}{112} = \overset{C.}{223}\ \overset{lb.}{24} = \overset{ton.}{11}\ \overset{C.}{3}\ \overset{lb.}{24}\ \textit{Ans.}$$

16. What is ·95 of an hundred weight?

$$\begin{array}{lr}
 & \cdot95 \\
 & 112 \\
\hline
 & 190 \\
 & 95 \\
 & 95 \\
\hline
lb. & 106\cdot40 \\
 & 16 \\
\hline
oz. & 6\cdot40 \\
 & 16 \\
\hline
dr. & 6\cdot40
\end{array}$$

Ans. 106 lb. 6 oz. 6·4 dr.

17. Reduce 2·24 feet to the decimal of a yard?

$$3\,)\,2\cdot24$$
$$\cdot7466 \text{ \&c. } \textit{Ans.}$$

18. Reduce $\frac{3}{7}$ of a mile to yards, &c.?

$$\begin{array}{r}
3 \\
1760 \\
\hline
7\,)\,5280\,(\,754 \\
2 \\
36 \\
\hline
7\,)\,72\,(\,10\tfrac{2}{7}
\end{array}$$

Ans. 754 yds. $10\frac{2}{7}$ in.

19. What is ·625 of a yard?

$$\begin{array}{r}
\cdot625 \\
3 \\
\hline
1\cdot875 \\
12 \\
\hline
10\cdot500
\end{array}$$

Ans. 1 ft. 10·5 in.

20. Reduce 59·74 square inches to the decimal of a square foot?

$$\frac{59\cdot74}{144} = \cdot4148 \text{ \&c. } \textit{Ans.}$$

9. Reduce $5\frac{3}{4}$d. to the fraction of a shilling?

$5\frac{3}{4}$d. = 23 farthings, which divided by 48 (the farthings in a shilling) gives $\frac{23}{48}$ the *Answer*.

10. Reduce $\frac{3}{11}$ of a guinea to the denomination or fraction of a crown?

$\frac{3}{11} \times 21 = \frac{63}{11}$ sh., which divided by 5 gives $\frac{63}{55}$ the *Answer*.

11. Reduce £0·93 to farthings?

·93 × 960 = 892·8 farthings, the *Answer*.

12. What is ·885 of a £?—Or to find the value of the decimal ·885 of a pound.

	·885
	20
Shillings	17·700
	12
Pence	8·400
	4
Farthings	1·600

Ans. 17 s. 8 d. 1·6 qrs.

13. Bring 9·84 pence to the decimal of a £.

$$
\begin{array}{r}
240\ )\ 9{\cdot}840\ (\ {\cdot}041\ £\ \textit{Ans.} \\
\underline{960} \\
240 \\
\underline{240}
\end{array}
$$

77. In like manner other denominations are reduced by means of the numbers in the foregoing tables, remembering to multiply, or divide, as the case may require.

Ex. 14. How many guineas weigh a *lb.* Troy, each being 5 dw. $9\frac{1}{2}$ gr.?

12 × 20 × 24 = 5760 gr. = 1 *lb.*

5 dw. $9\frac{1}{2}$ gr. = $129\frac{1}{2}$ gr. = $\frac{259}{2}$ gr.

5760 divided by $\frac{259}{2}$, is $\frac{2}{259} \times 5760 = \frac{11520}{259} = 44\frac{124}{259}$ *Ans.*

15. If 10000 men have each 40 rounds of cartridge with ball, what is the whole weight of lead, the balls being an ounce each?

$$10000 \times 40 = \overset{oz.}{400000},$$

$$\frac{400000}{16} = \overset{lb.}{25000}.$$

$$\frac{25000}{112} = \overset{C.}{223}\ \overset{lb.}{24} = \overset{ton.}{11}\ \overset{C.}{3}\ \overset{lb.}{24}\ \textit{Ans.}$$

16. What is ·95 of an hundred weight?

```
         ·95
         112
        ----
         190
         95
         95
       ------
  lb.  106·40
           16
         ----
  oz.    6·40
           16
         ----                  lb. oz. dr.
  dr.    6·40           Ans. 106  6   6·4
```

17. Reduce 2·24 feet to the decimal of a yard?

```
3 ) 2·24
    ·7466 &c. Ans.
```

18. Reduce $\frac{3}{7}$ of a mile to yards, &c.?

```
        3
      1760
7 )  5280 ( 754
        2
       36                      yds. in.
7 )    72 ( 10 2/7        Ans. 754 10 2/7
```

19. What is ·625 of a yard?

```
   ·625
      3
  -----
  1·875
     12
  ------                ft. in.
  10·500           Ans. 1  10·5
```

20. Reduce 59·74 square inches to the decimal of a square foot?

$$\frac{59{\cdot}74}{144} = {\cdot}4148\ \&c.\ \textit{Ans.}$$

21. Reduce ⅚ of a cubic yard to cubic feet?

$$\tfrac{5}{6} \times 27 = \tfrac{135}{6} = 22\tfrac{1}{2} \quad \textit{Ans.}$$

22. Reduce 64·984 cubic inches to the decimal of a cubic foot?

$$\frac{64 \cdot 984}{1728} = \cdot 0376 \text{ \&c.} \quad \textit{Ans.}$$

23. Reduce 500 Rhynland roods to English miles?

$$3\,)\,\underline{12 \cdot 396}$$
$$4 \cdot 132 \text{ yards} = 1 \text{ rood.}$$

$$4 \cdot 132 \times 500 = 2066 \text{ yards} = \overset{m.}{1}\ \overset{yds.}{306} \quad \textit{Ans.}$$

24. Reduce an English mile to toises

$$\frac{1760}{2 \cdot 1315} = 825 \cdot 709 \text{ \&c.} \quad \textit{Ans.}$$

25. Reduce 5 French leagues (25 to a degree) to English miles?

$$\frac{4869 \times 5}{1760} = \overset{m.}{13}\ \overset{yds.}{1465} \quad \textit{Ans.}$$

COMPOUND ADDITION.

78. Compound Addition is the collecting several numbers of different denominations into one sum.

79. *Rule.* Reduce fractional quantities of different denominations to like denominations. And fractions having different denominators to a common denominator. Then set down the numbers, so that those of the same denomination may stand directly under each other, as pounds under pounds, shillings under shillings, feet under feet, &c.

Add up the figures in the lowest denomination, and find by the rule of Reduction how many units of the next higher denomination are contained in the sum. Set down the remainder and carry the units to the next denomination, which add up

in the same manner as before; and so on till the whole is finished.

Examples.

1. Required the sum of £15 18s. $2\frac{3}{4}$d., £5 10s. $11\frac{1}{2}$d., £74 17s. $8\frac{3}{4}$d., and £29 19s. $5\frac{1}{4}$d.?

	£	s.	d.
	15	18	$2\frac{3}{4}$
	5	10	$11\frac{1}{2}$
	74	17	$8\frac{3}{4}$
	29	19	$5\frac{1}{4}$
Sum	126	6	$4\frac{1}{4}$

The number of farthings are 9, which make 2 pence to carry to the pence, and 1 farthing to set down. The pence in the next column are 26, and 2 carried make 28, or 4 pence over 2 shillings. The sum of the shillings 64, with 2 carried make 66, or 6 shillings to set down and 3 to carry to the pounds.

2. What is the sum of £4 16s. $9\frac{1}{4}$d., £5 0s. $0\frac{1}{2}$d., and £10 3s. $2\frac{1}{4}$d.?

	£	s.	d.
	4	16	$9\frac{1}{4}$
	5	0	$0\frac{1}{2}$
	10	3	$2\frac{1}{4}$
Sum	20	0	0

3. Required the sum of £$11\frac{1}{9}$, and £0 17s. $8\frac{3}{4}$d.?

£$\frac{1}{9}$ = 2s. 2d. $2\frac{2}{3}$ qrs.

	£	s.	d.	qrs.
	11	2	2	$2\frac{2}{3}$
	0	17	8	3
Sum	11	19	11	$1\frac{2}{3}$

4. What is the sum of £$\frac{1}{7}$ and $19\frac{1}{3}$s.?

£$\frac{1}{7}$ = $2\frac{6}{7}$s.

$$\begin{aligned} 2\tfrac{6}{7}\text{s.} &= 2\tfrac{18}{21}\text{s.} \\ 19\tfrac{1}{3} &= 19\tfrac{7}{21} \\ \hline & 22\tfrac{4}{21} \text{ } \textit{Ans.} \end{aligned}$$

1 2

5. Add £ 2·29 and s. 17·241 together?

·29
20
5·80

	£		£	s.
	2·29	=	2	5·8
				17·241
Sum			3	3·041

6. What is the sum of 77 guineas, 13 half guineas, three 15*l.* notes, 11 half-crowns, and 29 dollars at 4*s.* 1½*d.* each?

77 guineas......................	=	80	17	0
13 half-guineas.................	=	6	16	6
Three 15*l.* notes..............	=	45	0	0
11 half-crowns..................	=	1	7	6
29 dollars, at 4*s.* 1½.......	=	6	0	2½
		140	1	2½ *Ans.*

7. Add *lb. oz. dwts. gr.* 2 10 18 19, *oz. dwts. gr.* 11 15 16, and *lb. oz. dwts.* 3 4 14 together?

	lb.	*oz.*	*dwts.*	*gr.*
	2	10	18	19
		11	15	16
	3	4	14	0
Sum	7	3	8	11

8. Add ℔ ℥ ʒ ℈ *gr.* 6 10 7 2 19, ℔ ℥ ʒ ℈ *gr.* 1 6 6 2 18, and ʒ ℈ *gr.* 7 1 10 together.

℔	℥	ʒ	℈	*gr.*	
6	10	7	2	19	
1	6	6	2	18	
		7	1	10	
8	6	6	1	7	*Ans.*

9. Let *cwt. qr. lb. oz.* 2 1 27 15, *cwt. qr. lb. oz.* 5 3 26 14, and *qr. lb. oz. dr.* 3 10 12 15 be added together.

cwt.	*qr.*	*lb.*	*oz.*	*dr.*	
2	1	27	15	0	
5	3	26	14	0	
	3	10	12	15	
9	1	9	9	15	*Ans.*

10. Let 4·56 cwt. and 104·44 lb. be added together?

cwt. lb.
·56 × 112 = 62·72

cwt.	lb.
4	62·72
0	104·44
5	55·16 *Ans.*

11. Add ·262 yd., 2·4 feet, and $\frac{7}{9}$ of an inch together?

		inch.
yd. ·262 × 36	=	9·432
feet. 2·4 × 12	=	28·8
in. $\frac{7}{9}$	=	0·7777, &c.
	Inches	39·0097 *Ans.*

12. The contents of three fields A, B, C, were as below; required the whole number of acres?

		ac.	roods	pol.
viz.	A	14	3	37
	B	16	3	30
	C	10	2	24
		42	2	11 *Ans.*

13. A field having been measured with the chain in 4 divisions, the contents were found as below. Required the whole number of acres?

		chains	links
viz.		43	2842
		23	6173
		14	7292
		91	5463
Acres	17	3	9070 *Ans.*

14. The several contents of a piece of work are 24*f.* 124*in.* 14*f.* 100*in.* 99*f.* 29*in.* and 16*f.* 99*in.*; what is the whole content?

F.	in.
24	124
14	100
99	29
16	99
155	64 *Ans.*

15. The cubic contents of three pieces of timber are 29*f.* 1629*in.* 24*f.* 1561*in.* and 19*f.* 1104*in.* how many feet in the whole?

F.	*in.*	
29	1629	
24	1561	
19	1104	
74	838	*Ans.*

16. Add $3\frac{3}{7}$ cubic yards, and $21\frac{3}{8}$ cubic feet together?

$$\overset{yd.}{\tfrac{3}{7}} \times 27 = \tfrac{81}{7} = \overset{f.}{11\tfrac{4}{7}} = \overset{f.}{11\tfrac{32}{56}}, \text{ and } \overset{f.}{21\tfrac{3}{8}} = \overset{f.}{21\tfrac{21}{56}}$$

	yd.	*f.*
	3	$11\frac{32}{56}$
		$21\frac{21}{56}$
Sum	4	$5\frac{53}{56}$

COMPOUND SUBTRACTION.

80. *Rule.* Prepare the numbers and set them down as in Addition, only let the less stand under the greater.

Begin at the right hand, and take each number in the lower line from that above it and set the remainder directly under: but if any number in the lower line be greater than that above it, instead of adding 10 to the upper one, as in simple subtraction, increase it by as many as make one of the next higher denomination, then subtract the lower number from the sum, and set down the remainder. Carry 1 for that borrowed to the next number in the lower line, and proceed as before till the whole is finished.

Examples.

		£	s.	d.
1.	From	64	16	3
	Take	12	4	2
	Rem.	52	12	1
	Proof	64	16	3

		£	s.	d.
2.	From	19	0	$9\frac{3}{4}$
	Take	6	0	$9\frac{1}{2}$
	Rem.	13	0	$0\frac{1}{4}$
	Proof	19	0	$9\frac{3}{4}$

		£	s.	d.
3.	From	5	2	$1\frac{1}{4}$
	Take	1	3	$2\frac{1}{2}$
	Rem.	3	18	$10\frac{3}{4}$

Here 3 farthings being greater than 1 farthing, I borrow 1 penny or 4 farthings, which added to $\frac{1}{4}$ in the upper line make 5 farthings, then 3 from 5 leave 2 farthings or $\frac{1}{2}$ a penny to set down. Next, 1 that was borrowed and 2 make 3, which taken from 13 (because I borrow 1*s.* or 12 pence, and add it to the 1 in the upper line) and 10 remains. Carrying 1 that was borrowed to the 3 shillings and the sum is 4, which subtracted from 22 (because I borrow 20) leaves 18. Lastly, 1 carried to 1, and the sum taken from 5 gives 3 the last remainder.

		£	*s.*	*d.*
4.	From	27	0	0
	Take	25	19	$11\frac{3}{4}$
	Rem.	1	0	$0\frac{1}{4}$

		£	*s.*	*d.*
5.	From	1	0	$0\frac{1}{2}$
	Take	0	0	$0\frac{3}{4}$
	Rem.	0	19	$11\frac{3}{4}$

6. What is the difference of £$\frac{1}{7}$ and $\frac{2}{7}$*s.*?

$$\frac{1}{7} \times 20 = \frac{20}{7} = 2\tfrac{6}{7}s. = £\tfrac{1}{7}.$$

$$\begin{array}{rl} & 2\frac{6}{7} \\ & \frac{2}{7} \\ \hline \textit{Shill.} & 2\frac{4}{7} \ \textit{Ans.} \end{array}$$

7. What is the difference of $\frac{2}{9}$ of a guinea and $\frac{7}{30}$ of a pound?

$\frac{2}{9}$ of a guinea $= \frac{2}{9} \times \overset{s.}{21} = \frac{42}{9} = 4\frac{2}{3}$ *shillings.*

$\overset{£}{\frac{7}{30}} = \frac{7}{30} \times \overset{s.}{20} = \frac{140}{30} = 4\frac{2}{3}$ *shillings.* Therefore the difference is nothing.

8. Required the difference between $\overset{£}{0{\cdot}232}$ and $\overset{s.}{5{\cdot}218}$?

$$\begin{array}{rr} & £ \\ & 0{\cdot}252 \\ & 20 \\ \hline \textit{Shill.} & 5{\cdot}040 \end{array} \qquad \begin{array}{r} s. \\ 5{\cdot}218 \\ 5{\cdot}04 \\ \hline 0{\cdot}178 \end{array} \quad \textit{Ans.}$$

		yd.	*f.*	*in.*
9.	From	17	1	$4\frac{1}{2}$
	Take	15	2	11
	Diff.	1	1	$5\frac{1}{2}$

		yd.	*f.*	*in.*
10.	From	6	0	4·75
	Take	2	1	5·84
	Diff.	3	1	10·91

11. From $11\frac{1}{7}$ square feet take $100\frac{1}{7}$ square inches.

$\frac{1}{7} \times 144 = \frac{144}{7} = 20\frac{4}{7}$ square inches.

	F.	*in.*	
	11	$20\frac{4}{7}$	
	0	$100\frac{1}{7}$	
Diff.	10	$64\frac{3}{7}$	*Ans.*

Or thus,

$$\overset{in.}{100\tfrac{1}{7}} = \overset{f.}{\frac{701}{7 \times 144}} = \overset{f.}{\tfrac{701}{1008}}$$

$$\overset{feet}{11\tfrac{1}{7}} = \overset{f.}{11\tfrac{144}{1008}}$$

$$0\tfrac{701}{1008} \text{ sub.}$$

$$10\tfrac{451}{1008} \text{ the } \textit{answer} \text{ in square feet.}$$

12. What is the difference between ·58 of a solid yard and 16·66 solid feet?

$$\cdot 58 \times 27 = \overset{feet}{15\cdot 66}$$

$$16\cdot 66$$

$$\textit{Diff.}\ \ 1$$

COMPOUND MULTIPLICATION AND DIVISION.

81. COMPOUND Multiplication and Division are compendious methods of Compound Addition and Subtraction.

I. *When the multiplier is a whole number.*

82. *Rule.* Multiply the number in the lowest denomination, and find, by the rule of Reduction, how many integers of the next superior denomination are contained in the product, and set down the remainder if any. Carry the integers thus found to the product of the next higher denomination, with which proceed as before till the whole is multiplied.

II. *When the divisor is a whole number.*

83. *Rule.* Divide the highest denomination by the divisor and set down the quotient; and if there be any remainder, find how many integers of the next denomination it is equal to, and add them to the number (if any) which stands in that denomination. Divide the number thus found by the divisor, and set down the quotient under its proper denomination. Reduce the remainder to the next lower denomination; and so on, till the whole is finished.

Examples in Multiplication by whole numbers

1. What cost 7 quarters of oats at £1 9s. 10d. per quarter?

£	s.	d.	
1	9	10	
		7	
10	8	10	*Ans.*

2. At £2 19s. 10¾d. per barrel, what is the cost of 10 barrels of gunpowder?

£	s.	d.	
2	19	10¾	
		10	
29	18	11½	*Ans.*

3. What is the whole length of 9 planks, each being 13 f. 5·7 in.?

f.	in.	
13	5·7	
	9	
121	3·3	*Ans.*

84. When the multiplier is the product of two or more single figures, the answer may be found by multiplying successively by those figures instead of the whole at once. (19)

4. What cost 25 chaldron of coals at £2 7s. 9d. per chaldron?

$5 \times 5 = 25$

£	s.	d.	
2	7	9	
		5	
11	18	9	
		5	
59	13	9	*Ans.*

5. What must be paid for 105 hundred weight of bullets, at 5s. 7¾d. per hundred weight?

$3 \times 5 \times 7 = 105$

£	s.	d.	
	5	7¾	
		7	
1	19	6¼	
		5	
9	17	7¼	
		3	
£29	12	9¾	*Ans.*

85. If the multiplier cannot be produced by the multiplication of two or more single figures, take the nearest number to it which can be so produced, and multiply by its factors as before. Then augment, or diminish the result by as many times the multiplicand as the said number is less or greater than the multiplier.

6. At £4 1s. 10¼d. per thousand, what is the cost of 58 thousand bricks?

	£	s.	d.	
	4	1	10¼	
			8	
	32	14	10	
			7	
	229	3	10	............price of 56.
	8	3	8½	..........price of 2 add.
Ans.	237	7	6½	

Or thus,

	£	s.	d.	
	4	1	10¼	
			10	
	40	18	6½	
			6	
	245	11	3	............price of 60.
	8	3	8½	....of 2 subtract.
Ans.	237	7	6½	as before.

yds. f. in.
7. Multiply 2 2 10·7 by 29.

yd.	f.	in.	
2	2	10·7	
		7	
20	2	2·9	
		4	
82	2	11·6	....product by 28.
2	2	10·7	....add.
Product 8[illegible]	[illegible]	10·3	

86. *Examples in Division by whole numbers.*

£ s. d.
1. When oats are at 1 17 9 per quarter, what is that per bushel?

	£	s.	d.	qr.	
8)	1	17	9	0	
	0	4	8	$2\frac{1}{2}$	*Ans.*

2. If the interest of £100 for a year be £$3\frac{3}{8}$, what is the interest of £70 for that time?

£$3\frac{3}{8}$ = £3 7s. 6d.

	£	s.	d.	
10)	3	7	6	
	0	6	9	the interest for 10.
			7	
	2	7	3	*Ans.*

87. When the divisor is the product of two or more simple numbers, divide by them separately. (30)

£ s. d.
3. If a chaldron of coals cost 2 10 3, what is that per bushel?

	£	s.	d.	
6)	2	10	3	
6)	0	8	$4\frac{1}{2}$	
	0	1	$4\frac{3}{4}$	*Ans.*

4. At 3 guineas the hundred weight, what is that per lb.?

2 × 7 × 8 = 112

	£	s.	d.	
2)	3	3	0	
7)	1	11	6	
8)	0	4	6	
	0	0	$6\frac{3}{4}$	*Ans.*

5. What is the 24th part of 19 yds. 2 f. 9·5 in.?

```
       y.  f.  in.
  4 ) 19   2   9·5
  6 )  4   2  11·375
Ans.   0   2   5·895833 &c.
```

88. **If the divisor cannot be resolved into small factors, divide by the whole at once after the manner of long division.**

6. If the whole pay of 179 men for 61 days be £625 11s. 4½d., what is the daily pay of each?

```
        £    s.  d.     £   s.  d.
179 ) 625   11  4½  ( 3 . 9 . 10¾  the whole pay of 1 man.
      537
       88
       20
     1771
     1611
      160
       12
     1924
      179
      134
        4
      537
      537
                                £  s.  d.     s. d.
                           61 ) 3  9   10¾ ( 1  1¾  Answer.
                                20
                                69
                                61
                                 8
                                12
                               106
                                61
                                45
                                 4
                               183
                               183
```

89. III. **When the multiplier, or the divisor, is a vulgar fraction, it is evident that the product in the former case, and the quotient in the latter, will each be obtained by both multiplication and division, except the numerator of the fraction be 1.— For the product of the numerator and multiplicand divided by the denominator will give the answer in multiplication. And the product of the denominator and dividend divided by the numerator is the quotient in division.**

Examples.

Ton. C. lb.
1. If 86 17 100 of provisions will serve a garrison 12 months, what quantity will be necessary for 8 months?

8 months = $\frac{2}{3}$ of 12 months.

	T.	*C.*	*lb.*	
	86	17	100	
			2	
3)	173	15	88	
	57	18	66$\frac{2}{3}$	*Ans.*

£ s. d.
2. If I agree to give a labourer 1 6 6 for working 10 days, what must I pay him for 7 days?

Here 7 days is $\frac{7}{10}$ of the whole time.

	£	*s.*	*d.*	*qr.*	
	1	6	6	0	
			7		
10)	9	5	6	0	
	0	18	6	2$\frac{2}{5}$	*Ans.*

yds. f. in.
3. What is $\frac{3}{5}$ of 79 8 54·7 square measure?

	y.	*f.*	*in.*
	79	8	54·7
			3
5)	239	7	20·1
	47	8	90·42

£ s. d.
4. If 7 hundred weight cost 6 13 4, what is that per ton?

cwt.
Here 7 is $\frac{7}{20}$ of a ton, therefore $\frac{7}{20}$ is the divisor.

	£	*s.*	*d.*	
	6	13	4	
			2	
	13	6	8	
			10	
7)	133	6	8	...... product by 20.
	19	0	11$\frac{3}{7}$	*Ans.*

90. When the multiplier is a mixt number, the multiplication may be made by the parts separately and the products added together for the answer. If the divisor is a mixt number reduce it to an improper fraction. And when decimals are in the multiplier or divisor, reduce the multiplicand or dividend to the lowest denomination, and find the answer by the rules of Reduction.

OF ALIQUOT PARTS.

91. An *aliquot part* of a number is any other number which will divide it without leaving a remainder. Thus if the aliquot parts are confined to integers, 1, 2, and 3, will be all the aliquot parts of 6; 1 being the $\frac{1}{6}$, 2 the $\frac{1}{3}$, and 3 the $\frac{1}{2}$ of 6. Fractions and mixt numbers however, are aliquot parts, as $\frac{1}{5}$ or the *5th* of 1 is an aliquot part of 1; $3\frac{1}{3}$ or $\frac{1}{3}$ of 10, an aliquot part of 10; $4\frac{1}{2}$ or $\frac{1}{3}$ of $13\frac{1}{2}$, an aliquot part of $13\frac{1}{2}$, &c. Also 3*s*. 4*d*. and 2*s*. 6*d*, are aliquot parts of a pound, the former being $\frac{1}{6}$, and the latter $\frac{1}{8}$. 4 inches is an aliquot part of a foot and also of a yard, being $\frac{1}{3}$ of the former, and $\frac{1}{9}$ of the latter, &c.

The principal use of aliquot parts is to abridge the operations in compound multiplication, or when several numbers of different denominations are to be multiplied together. The method by aliquot parts is also called *Practice*.

Examples.

1. What is the product of 144 and $6\frac{3}{4}$?

```
      144
       6¾
      ———
      864 ...... product by 6.
       72 ...... ½ of 144.
       36 ...... ½ of 72.
      ———
Ans.  972
```

Here, instead of multiplying by $\frac{3}{4}$, I take $\frac{1}{2}$, the multiplicand, and again the $\frac{1}{2}$ of that $\frac{1}{2}$, or $\frac{1}{4}$; therefore both these parts together make $\frac{3}{4}$ of the multiplicand to be added to the product by 6.

2. Required the product of 782 and 20¾?

```
   782
    20¾
 15640  ......  product by 20.
   391  ......  ½ or 4/8 of the multiplicand.
    97¾ ......  ¼ of 391, or ¼ of ½ or ⅛ of the multiplicand.
Ans. 16128¾
```

3. What will be the expense of a brick wall 785 yards long at 3s. 9d. per yard?

3s. 9d. may be divided into two aliquot parts of a pound, viz. 2s. 6d. or ⅛, and 1s. 3d. or ½ of ⅛. And therefore it is evident that ⅛ of 785, and ½ of that ⅛ when added together will be the answer in pounds, &c.

```
s. d.
2 6 = ⅛ ........... 8 ) 785
1 3 = ½ of ⅛ ... 2 )  98  2  6
                      49  1  3
                   £ 147  3  9
```

Or the aliquot parts may be taken as follows:

```
s.       £
2   =  1/10 ............ 10 ) 785
1s. =  ½ 2s. ..........  2 )  78 10
6d. =  ½ 1s. ..........  2 )  39  5
3d. =  ½ 6d. ..........  2 )  19 12 6
                               9 16 3
                          £ 147  3 9  the answer as before.
```

4. If gunpowder is £5 13s. 6d. the hundred weight, what will 8 cwt. 1 qr. 20 lb. cost?

```
 £  s. d.
 5 13  6
       8
45  8  0  cost of 8 cwt.
```

```
                         £  s.  d.
1 qr. = ¼ cwt. ...... 4 ) 5  13  6
                          1   8  4½ ...... cost of 1 qr.
16 lb. = ⅐ cwt. ......    0  16  2½ ........... of 16 lb.
4 lb. = ¼ of 16 lb. ...   0   4  0 9/14 ....... of 4 lb.
                         45   8  0  ........... of 8 cwt.
                      £  47  16  7¾  Ans.
```

In the last example I find the price of 8 cwt. by compound multiplica-

tion; and that of 1*qr.* 20*lb.* by the aliquot parts of a hundred weight: Thus 1*qr.* is $\frac{1}{4}$ of a hundred, and its price $\frac{1}{4}$ of £5 13*s.* 6*d.*—16*lb.* is $\frac{1}{7}$ of a hundred, therefore its price is $\frac{1}{7}$ of £5 13*s.* 6*d.* and the price of 4*lb.* (making up the 20*lb.*) is $\frac{1}{4}$ of that $\frac{1}{7}$.

5. If I pay 4*s.* 10$\frac{1}{4}$*d.* per yard, what will be the expense of 79 *yds.* 2 *f.* 6 *in.*?

	s. d.	
1$\frac{1}{2}$ foot = $\frac{1}{2}$ a yard.......	2) 4 10$\frac{1}{4}$	
	2 5$\frac{1}{8}$	expense of 2 *f.* 6 *in.* at 4*s.* 10$\frac{1}{4}$*d.* per *yd.*
1 foot = $\frac{1}{3}$ of a yard	1 7$\frac{5}{12}$	

	79		
	4	10$\frac{1}{4}$	
shill.	316	9	.... 79*yds.* at 4*s.*
6*d.* = $\frac{1}{2}$ 1*s.*..........	39	6	expense of 79 *yds.* at 10$\frac{1}{4}$*d.*
3*d.* = $\frac{1}{2}$ of 6*d.*......	19	9	
1*d.* = $\frac{1}{3}$ of 3*d.*......	6	7	
1 *fur.* = $\frac{1}{4}$ of 1*d.*....	1	7$\frac{3}{4}$	
	2	5$\frac{1}{8}$	expense of 2 *f.* 6 *in.* at 4*s.* 10$\frac{1}{4}$*d.*
	1	7$\frac{5}{12}$	
shill.	387	6$\frac{1}{12}$	
£	19 7	6$\frac{1}{12}$	*Ans.*

But the result may be obtained more concisely thus: Since 79 *yds.* 2 *f.* 6 *in.* is only 6 inches or $\frac{1}{6}$ of a yard short of 80 yards, if $\frac{1}{6}$ of 4*s.* 10$\frac{1}{4}$*d.* be deducted from the expense of 80 yards, the remainder will evidently be the answer required.

	£	s.	d.	
		4	10$\frac{1}{4}$	
			8	
	1	18	10	
			10	
	19	8	4	80 yards at 4*s.* 10$\frac{1}{4}$*d.*
$\frac{1}{6}$ of 4*s.* 10$\frac{1}{4}$*d.* = ...	0	0	9$\frac{17}{24}$	subtract.
£	19	7	6$\frac{1}{12}$	*Ans.* as before

6. What is the product of 16 *f.* 7$\frac{1}{2}$ *in.* by 22 *f.* 10 *in.*?

	f.	in.		
	16	7$\frac{1}{2}$		
	22	10		
	352	0	= 16 × 22.	
6*in.* = $\frac{1}{2}$ a foot	11	0	= $\frac{1}{2}$ of 22	= 22 *f.* × 7$\frac{1}{2}$ *in.*
1$\frac{1}{2}$*in.* = $\frac{1}{4}$ of 6*in.*	2	9	= $\frac{1}{4}$ of 11	
6 *in.* = $\frac{1}{2}$ a foot	8	3$\frac{3}{4}$	= $\frac{1}{2}$ of 16 *f.* 7$\frac{1}{2}$ *in.*	= 16 7$\frac{1}{2}$ × 10 *in.*
4 *in.* = $\frac{1}{3}$ a foot	5	6$\frac{1}{2}$	= $\frac{1}{3}$ of 16 *f.* 7$\frac{1}{2}$ *in.*	
Ans.	379	7$\frac{1}{4}$		

This product is square measure, and therefore 379 are square feet, and $7\frac{1}{4}$ are 12ths of a square foot, equal to $7\frac{1}{4} \times 12$ or 87 square inches.

7. Required the product of 46 *f.* $5\frac{1}{2}$ *in.* and 8 *f.* $9\frac{3}{4}$ *in.*?

	f.	*in.*	
	46	$5\frac{1}{2}$	
	8	$9\frac{3}{4}$	
	371	8	Product by 8,
6*in.* = ½ a foot	23	$2\frac{3}{4}$	= ½ the multiplicand.
3*in.* = ½ of 6*in*	11	$7\frac{3}{8}$	= ½ of 23 $2\frac{3}{4}$
¾*in.* = ¼ of 3*in.*	2	$10\frac{27}{32}$	= ¼ of 11 $7\frac{3}{8}$.
	409	$4\frac{31}{32}$	or 409*sq.f.* and $59\frac{5}{8}$ *sq. in.*

8. Let 31 *yds.* 2 *f.* 9 *in.* be multiplied by 10 *yds.* 1 *f.* 10 *in.*?

	yds.	*f.*	*in.*	
	31	2	9	
	10	1	10	
	319	0	6	product by 10.
1*f.* = ⅓ of a yard	10	1	11	= ⅓ the multiplicand.
6*in.* = ½ a foot	5	0	$11\frac{1}{2}$	= ½ of 10 1 11.
4*in.* = ⅓ a foot	3	1	$7\frac{2}{3}$	= ⅓ of 10 1 11.
	338	2	$0\frac{1}{6}$	*Ans.*

Here the principal integer being a yard, 338 will be square yards, each of the units in the next denomination ⅓ of a square yard, and each unit under inches $\frac{1}{12}$ of ⅓ or $\frac{1}{36}$ of a square yard. And the whole 338*y.* 6*f.* 6*in.* square measure.

This method by Aliquot parts will frequently be more expeditious than Duodecimals for obtaining the contents of the sections of field works, &c.

OF THE RULES OF PROPORTION.

I. *Of Direct Proportion.*

92. If 4 numbers are such that the first divided by the second is equal to the third divided by the fourth, or the second divided

by the first equal to the fourth divided by the third, they are said to be directly proportional.

Let the numbers be 2, 4, 5, 10: Then $\frac{2}{4} = \frac{5}{10}$; and $\frac{4}{2} = \frac{10}{5}$.

The fraction $\frac{2}{4}$ denotes the *ratio*, or rather the *exponent* of the *ratio* of 2 to 4, or of 5 to 10, because $\frac{5}{10} = \frac{2}{4}$: And $\frac{4}{2}$ the ratio of 4 to 2, or of 10 to 5.

The numbers or terms of the proportion are usually set down thus 2 : 4 :: 5 : 10, and read thus, as 2 is to 4, so is 5 to 10; which signifies that 2 bears the same proportion to 4, as 5 does to 10: This is evident, since 2 is the half of 4, and 5 is the half of 10; or 2 is contained in 4 the same number of times as 5 is contained in 10.

Hence if two fractions are equal, their terms are proportional;

For $\frac{2}{4} = \frac{5}{10}$; and 2 : 4 :: 5 to 10.

93. Since equal numbers multiplied by equal numbers must give equal products, if the equal fractions $\frac{4}{2}$, $\frac{10}{5}$ are multiplied by 5 (or any other number) the products will be equal; namely, $\frac{4 \times 5}{2} = \frac{10 \times 5}{5}$, or (when the fraction $\frac{10 \times 5}{5}$ is abridged) $\frac{4 \times 5}{2} = \frac{10}{1}$, or $\frac{4 \times 5}{2} = 10$; therefore *the product of the second and third terms of the proportion*, 2 : 4 :: 5 : 10, *divided by the first term, gives the fourth term* 10. Consequently the product of the two middle terms 4×5, is equal to 2×10, the product of the other two.

94. Hence the rule of proportion is called the RULE OF THREE; because from *three given numbers*, a *fourth* may be found, which shall have the same proportion to one of the three, as there is between the other two.

For example: If a body of troops in 2 hours march 4 miles; how far would they march in 5 hours at the same rate?

Here it is evident that the two distances will be in the same direct proportion as the times 2 and 5, or that 5 will have the same proportion to the required distance or 4*th.* term, as 2 has to the distance 4.

Therefore having set down the three given terms or numbers in the order they are proposed, *multiply the 2d. and 3d. together, and divide the product by the first, for the answer.*

$$\overset{h.}{2} : \overset{h.}{4} :: \overset{m.}{5} : \overset{m.}{\frac{4 \times 5}{2}} = \overset{m.}{10}, \text{ the } \textit{answer.}$$

Or because $5 \times 4 = 4 \times 5$, the proportion may stand thus,

$$\overset{h.}{2} : \overset{m.}{5} :: \overset{h.}{4} : \overset{m.}{\frac{5 \times 4}{2}} = 10, \text{ as before.}$$

The terms 2 and 4 are called the terms of *supposition,* and 5 that of *demand:* therefore in setting down the three given numbers of a proportion, or stating the question, always make that number the first term, which is of the same kind as the term of demand.

95. Since the terms of two equal fractions, $\frac{1}{2}$, $\frac{5}{10}$, are proportionals (92), if $\frac{5}{10}$ is reduced to its lowest terms we have $\frac{1}{2} = \frac{5}{10}$, therefore as $1 : 2 :: 5 : 10$; or $1 : 5 :: 2 : 10$. Hence when 4 numbers are directly proportional, if the first and second terms, or the first and third terms are divided (or multiplied) by any number, the 4*th.* term will still be the same:

Thus, $\frac{1}{2} : \frac{2}{2} :: 5 : 10$. And $\frac{1}{2} : 2 :: \frac{5}{2} : 10$.

Therefore like *multiples* or *sub-multiples* of any numbers, are in the same proportion as the numbers themselves.

96. Hence the operations in working proportions may sometimes be abridged, as in the following question:

If 500 men require 15000 rations of bread for a month, how many rations will a garrison of 1170 men require?

$$\text{As } \overset{m.}{500} : \overset{r.}{15000} :: \overset{m.}{1170} : \overset{r.}{35100} \text{ the } \textit{answer;}$$

Or, dividing the two first terms by 500,

As 1 : 30 :: 1170 : 30 $\times$ 1170 = 35100 : where the product of the 2d. and 3d. terms is the 4th. term or answer.

97. Hence, if to several numbers we respectively add other numbers in the same proportion, the *sums* will also be in that same proportion. For the latter numbers may be considered as *like* multiples or sub-multiples of the former.

Thus, if to 3, 4, 6, we add 1, $1\frac{1}{3}$, 2 (having the same proportion) respectively, the sums will be 4, $5\frac{1}{3}$, 8, which are in the same proportion as 3, 4, 6. And the like is also evident with respect to the *differences*.

98. Hence also, it appears that fractions having a common denominator are in the same proportion as their numerators.

Thus the fractions $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, are in the same proportion as 6, 8, 9. But the fractions when reduced to their lowest terms are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, consequently those fractions are in the same proportion as 6, 8, 9.

99. If 4 numbers are directly proportional; then, as the sum of the 1st. and 2d. is to the 2d. (or 1st.) so is the sum of the 3d. and 4th. to the 4th. (or 3d.)

Suppose 2 : 4 :: 5 : 10.
Then 2 + 4 : 4 :: 5 + 10 : 10.
And 2 + 4 : 2 :: 5 + 10 : 5.

For $\frac{2}{4} = \frac{5}{10}$; and since equal numbers added to equal numbers must give equal sums, if the fraction $\frac{4}{4}$ (or 1) be added to $\frac{2}{4}$, and $\frac{10}{10}$ (or 1) added to $\frac{5}{10}$, the sums must be equal,

viz. $\frac{2}{4} + \frac{4}{4} = \frac{5}{10} + \frac{10}{10}$, or $\frac{2+4}{4} = \frac{5+10}{10}$; these fractions being equal, their terms will be proportional,

Or 2 + 4 : 4 :: 5 + 10 : 10

In like manner, by adding $\frac{2}{2}$ and $\frac{5}{5}$ to $\frac{4}{2}$ and $\frac{10}{5}$ respectively, we have 2 + 4 : 2 :: 5 + 10 : 5. (And if we subtract the equal fractions $\frac{2}{2}$, $\frac{5}{5}$, instead of adding them, it may be proved that the *differences* are proportional).

Since $2 + 4 : 5 + 10 :: 2 : 5$, therefore $\frac{2+4}{5+10} = \frac{2}{5}$. Now, if for $\frac{2}{5}$ we take any fraction equal to it, as $\frac{8}{20}$, we have $\frac{2+4}{5+10} = \frac{8}{20}$;

Therefore $2 + 4 : 5 + 10 :: 8 : 20$; or $2 + 4 : 8 :: 5 + 10 : 20$;

Hence, as $2 + 4 + 8 : 8 :: 5 + 10 + 20 : 20$;

or $2 + 4 + 8 : 5 + 10 + 20 :: 8 : 20$.

$:: 4 : 10.$

$:: 2 : 5.$

100. Hence is derived the method of dividing a number into a proposed number of parts having given proportions. Let 35 (or $5 + 10 + 20$) be divided into 3 parts which shall be as 2, 4, and 8: Then,

$2 + 4 + 8 : 35 :: 2 : 5$

$2 + 4 + 8 : 35 :: 4 : 10$ *Ans.* 5, 10, and 20.

$2 + 4 + 8 : 35 :: 8 : 20$

Again: Suppose it is required to divide 100 into 3 parts having the proportions of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

The three fractions when brought to a common denominator are $\frac{6}{12}$, $\frac{4}{12}$, and $\frac{3}{12}$; therefore (98) the three parts will have the same proportions as the three numerators 6, 4, and 3:

$$6 + 4 + 3 = 13,$$

Then, $\left.\begin{array}{l} 13 : 100 :: 6 : 46\frac{2}{13} \\ 13 : 100 :: 4 : 30\frac{10}{13} \\ 13 : 100 :: 3 : 23\frac{1}{13} \end{array}\right\}$ the 3 parts required.

As a third example, let $10\frac{1}{2}$ be divided into 2 parts having the same ratio as the decimals ·05 and ·075:

$$\begin{array}{r} \cdot05 \\ \cdot075 \\ \hline \cdot125 \end{array}$$

Then $\left.\begin{array}{l} \cdot125 : 10\cdot5 :: \cdot05 : 4\cdot2 \\ \cdot125 : 10\cdot5 :: \cdot075 : 6\cdot3 \end{array}\right\}$ the two required parts.

II. Of Inverse or Reciprocal Proportion.

101. When 4 numbers ($2 : 4 :: 5 : 10$) are in direct proportion, (as above) the product of the two middle terms (4×5) is equal to that of the other two (2×10). But if the proportion is inverse or reciprocal, the product of the two first terms will be equal to the product of the two last; or the ratio of the first term to the third is equal to that of the fourth to the second.

For example: If 4 men can do a piece of work in 6 days, in what time would 8 men do the same?

Taking the proportion direct, the answer comes out 12 days; but the true time is evidently no more than 3 days, because 8 men will require but half the time which 4 require.

$$\overset{m.}{4} : \overset{d.}{6} :: \overset{m.}{8} : \overset{d.}{3}.$$ As 4 : 6 :: 8 : 3. Here $\frac{4 \times 6}{8} = 3$; *viz.* the product of the two first terms divided by the third gives the fourth term or answer. Hence this

Rule. Multiply the terms of supposition together and divide the product by the term of demand for the fourth term or answer.

As another example; Suppose 40 men stand 5 in a rank, then if a yard is allowed to each rank they will extend 8 yards. But if the same 40 stand 4 in a rank, the extent will be 10 yards (allowing a yard to each rank as before). In this case it is evident that the lengths are *inversely* or *reciprocally* as the number of men in front.

Therefore $\overset{m.}{5} : \overset{yds.}{8} :: \overset{m.}{4} : \overset{yds.}{10}$. Here $\frac{5 \times 8}{4} = 10$. And the ratio of 5 to 4 is equal to that of 10 to 8; or $\frac{5}{4} = \frac{10}{8}$. The terms of supposition being 5 and 8, and that of demand 4.

102. To discover when a proportion should be wrought inversely, consider if *more* requires *less*, or *less* requires *more*, or if one number *increases* in the same proportion as another *diminishes*, for in either case the inverse rule must be used.

103. *N.B.* When the two terms of a proportion which are of the same kind, are given in different denominations, reduce them to the same denomination. Thus if one is pounds, &c. and the other pence, &c. reduce them both to pounds, or to pence. If one is feet and inches, and the other inches, reduce them both to feet or to inches, &c. And the fourth term or answer will always be in that denomination to which the given term of the same kind is reduced.

Questions in *Compound Proportion* or the *Double Rule of Three*, may always be answered by two or more single statings.

Examples.

1. To find a 3d. proportional to 5 and 33.

As 5 : 33 :: 33 : 217·8 *Ans.*

```
        33
        99
       99
 5 ) 1089
     217·8
```

The work is proved by reversing the question.—Thus, to find a 3d. proportional to 217·8 and 33.

$$\text{As } 217{\cdot}8 : 33 :: 33 : \frac{33 \times 33}{217{\cdot}8} = 5 \text{ the } Ans.$$

2. To find a 4th. proportional to 11·5, ·0769 and 1000.

As 11·5 : ·0769 :: 1000 : 6·6869 &c. *Ans.*

```
          1000
 11·5 ) 76·9000 ( 6·6869 &c.
        69 0
         7 90
         6 90
         1000
          920
          800
          690
         1100
```

3. Let a *4th.* proportional to the three fractions [illegible], [illegible], and [illegible], be required.

As [illegible] : [illegible] :: [illegible] : [illegible] $\times$ [illegible] $\times$ [illegible] $=$ [illegible] $\times$ [illegible] $\times$ [illegible] $= \frac{12}{15}$ the *Ans.*

N. B. It will be advisable in most cases to set down the 4th. term in the form of a vulgar fraction, and then reduce it to its lowest terms, as in the last example.

4. What are coals per chaldron when three bushels cost 4 shillings?

36 *bush.* = 1 chaldron. Therefore we have to find a *4th.* proportional to 3, 4, and 36:

$$\overset{bush.}{\text{As } 3} : \overset{shill.}{4} :: \overset{bush.}{36} : \overset{shill.}{\frac{4 \times 36}{3}} = 48 \textit{ shill.} \text{ the } \textit{Ans.}$$

5. The quick time, or step in marching being 2 paces *per* second or 120 *per* minute at $2\frac{1}{4}$ feet ea ; at what rate *per* hour does a troop march: and what time is taken up in marching 6 miles?

$120 \times 2\frac{1}{4} \times 60 = 18000$ feet *per* hour $= 3\frac{9}{22}$ miles.

$$\overset{mil.}{\text{As } 3\tfrac{9}{22}} : \overset{min.}{60} :: \overset{mil.}{6} : \overset{min.}{\frac{60 \times 9 \times 22}{75}} = \frac{4 \times 6 \times 22}{5} = \overset{min.}{105\tfrac{3}{5}} = \overset{h.\ m.}{1\ 45\tfrac{3}{5}}.$$

Ans. $\begin{cases} 3\frac{9}{22} \text{ m. per hour.} \\ 1 \text{ h. } 45\frac{3}{5} \text{ m.} \end{cases}$

6. What will the tax on £514 15 s. be at 1 s. 8 d. in the pound?

$$\overset{£}{\text{As } 1} : \overset{s.}{1} \ \overset{d.}{8} :: \overset{£}{514} \ \overset{s.}{15} :$$

$$\overset{d.}{\text{Or as } 240} : \overset{d.}{20} :: \overset{£}{514} \ \overset{s.}{15} :$$

Or (96) dividing the two first terms by 20, we have

$$\text{As } 12 : 1 :: \overset{£}{514} \ \overset{s.}{15} : \frac{\overset{£}{514} \ \overset{s.}{15}}{12} = \overset{£}{42} \ \overset{s.}{17} \ \overset{d.}{11} \ \textit{Ans.}$$

	ac.	roods	pol.		£	s.	d.
7. What is the rent *per ann.* of	140	3	20	at	1	10	8

per acre?

$$\overset{ac.}{140}\ \overset{r.}{3}\ \overset{p.}{20} = 22540 \text{ poles.}$$

$$\overset{£}{1}\ \overset{s.}{10}\ \overset{d.}{8} = 368 \text{ pence.}$$

As $\overset{p.}{160} : \overset{d.}{368} :: \overset{p.}{22540} : \dfrac{368 \times 22540}{160} = \overset{d.}{51842} = \overset{£}{216}\ \overset{s.}{0}\ \overset{d.}{2}$ the answer.

8. A sets out from Oxford to London at the same time that B leaves London for Oxford, the former travels 5, and the latter 6 miles an hour; now supposing Oxford to be 58 miles from London, how far from the latter place will they meet if they travel the same road?

If the whole distance be divided into two parts having the proportion of 5 to 6, it is evident those parts will be the respective distances travelled.

(100) As $5 + 6 : 58 :: 6 : \overset{m.}{31\tfrac{7}{11}}$ travelled by B.
$5 + 6 : 58 :: 5 : 26\tfrac{4}{11}$ travelled by A.

9. A detachment sets out at 6 in the morning, marching at the rate of $1\tfrac{3}{4}$ miles an hour; 3 hours after, another detachment from the same place follows them, but their march is $2\tfrac{1}{2}$ miles an hour. In what time will the latter overtake the former; and what distance will they have marched?

$1\tfrac{3}{4} \times 3 = \overset{m.}{5\tfrac{1}{4}}$ the first detachment is a-head when the other begins its march.

The difference of $\overset{m.}{2\tfrac{1}{2}}$ and $\overset{m.}{1\tfrac{3}{4}}$ is $\overset{m.}{\tfrac{3}{4}}$, what the latter gains on the former per hour.

But it has to gain $\overset{m.}{5\tfrac{1}{4}}$ in the whole.

Therefore, as $\overset{m.}{\tfrac{3}{4}} : \overset{h.}{1} :: \overset{m.}{5\tfrac{1}{4}}$
or, as $\tfrac{3}{4} : 1 :: \tfrac{21}{4}$

$$\text{(98) or, as } 3 : 1 :: 21 : \tfrac{21}{3} \overset{h.}{=} 7 \text{ the } \textit{time required.}$$

And $2\frac{1}{2} \times 7 = 17\frac{1}{2}$ miles the *distance required.*

10. The hour and minute hands of a watch are together at 12 o'clock; at what time are they next together?

The minute hand moves 1 circumference on the dial plate in 1 hour;
but the hour hand moves only $\frac{1}{12}$:
the difference is $\frac{11}{12}$ which the minute hand gains per hour.

But at setting off at 12 o'clock we may consider the hour hand as being 1 circumference before the minute hand;
Therefore the minute hand has to gain 1 circumference:

As $\frac{11}{12}$: $1h.$ :: 1 : $\frac{12}{11}h. = 1\frac{1}{11}h.$ the *answer*.

11. There is an Island 29 miles in circumference, and three travellers all start together to travel the same way about it; A goes 3 miles per hour, B 5, and C 7; when will they all be together again?

B gains 2 miles an hour upon A;
Therefore as $\overset{m.}{2} : \overset{h.}{1} :: \overset{m.}{29} : \overset{h.}{14\frac{1}{2}}$ the time from starting when B overtakes A.

C gains 4 miles an hour upon A;
Hence $\overset{m.}{4} : \overset{h.}{1} :: \overset{m.}{29} : \overset{h.}{7\frac{1}{4}}$ the time when C overtakes A.

And since C will overtake A at the end of every $7\frac{1}{4}$ hours, they will be together at the end of twice $7\frac{1}{4}$ hours, or $14\frac{1}{2}$ hours:

Therefore all three will be together again at the end of $14\frac{1}{2}$ hours from the time of starting.

12. Suppose a clock has 4 hands, A, B, C, D; and that A goes round once in *5d. 20h.* B in *7d. 14h.* C in *10d. 20h.* and D in *18d. 23h.* Now if the hands are all together at any particular time, how long will it be before they come in conjunction again?

d.	h.		h.	
5	20	=	140	the times of revolution.
7	14	=	182	
10	20	=	260	
18	23	=	455	

Now it is evident that at the end of any number of hours which is a common multiple of 140, 182, 260, and 455 (the times of 1 revolution) the hands will be together again: but the least common multiple is 13 × 5 × 7 × 4 or 1820 (46); therefore in 1820 hours,

A will have moved	13	times round.
B	10	
C	7	
D	4	

Consequently at the end of every 1820 hours the hands are together *at the same place.*

Therefore since the hands come together at every like whole multiple of 13, 10, 7, 4 revolutions (as twice, thrice, four times, &c.), it follows, that if we can find like sub-multiples or aliquot parts of 13, 10, 7, and 4, *having like fractions,* the hands must have been in conjunction without performing entire revolutions: Thus, if we divide 13, 10, 7, and 4 by 3, we get $4\frac{1}{3}$, $3\frac{1}{3}$, $2\frac{1}{3}$, $1\frac{1}{3}$ revolutions for the elapsed time, or $\frac{1820}{3} = 606\frac{2}{3}$ *hours* the time required.

Or thus.

A moves $\frac{1}{140}$, B $\frac{1}{182}$, C $\frac{1}{260}$, and D $\frac{1}{455}$ of the circumference in 1 hour, respectively.

Now if we proceed according to *Examp.* 7.
we have $\frac{1}{140} - \frac{1}{455} = \frac{9}{1820}$ of the circumference which A gains on D in 1 hour:
Therefore $\frac{9}{1820}$ *circumf.* : 1*h.* :: 1 *circumf.* : $202\frac{2}{9}$*h.* the time in which A is overtaking D.

And $\frac{1}{182} - \frac{1}{455} = \frac{3}{910}$ circumf. which B gains on D in 1 hour:
As $\frac{3}{910}$: 1*h.* :: 1 : $303\frac{1}{3}$*h.* the time in which B is overtaking D.

Also $\frac{1}{260} - \frac{1}{455} = \frac{3}{1820}$ circumf. which C gains on D in 1 hour;
And $\frac{3}{1820}$: 1*h.* :: 1 : $606\frac{2}{3}$*h.* the time in which C is overtaking D.

Now it is evident that the least common multiple of $202\frac{2}{9}$, $303\frac{1}{3}$, and $606\frac{2}{3}$ will be the time when A, B, and C will first overtake D together; but $606\frac{2}{3}$ is the least common multiple; for twice $303\frac{1}{3}$ is $606\frac{2}{3}$, and three times $202\frac{2}{9}$ is $606\frac{2}{3}$; therefore $606\frac{2}{3}$ *hours* is the time, as before.

13. What length must be cut off a rectangular board that is $7\frac{1}{2}$ inches broad, to make a foot or 144 square inches?

In other words——What number is that which multiplied by $7\frac{1}{2}$ shall make 12 times 12, or 144?

Here the proportion will be inverse;

$$\text{As } \overset{broad}{12} : \overset{long}{12} :: \overset{broad}{7\tfrac{1}{2}} : \frac{12 \times 12}{7\frac{1}{2}} = 19\tfrac{1}{5} \text{ inches, } answer.$$

14. A garrison of 488 men have provisions for 39 weeks, how long will those provisions last if the garrison be increased to 732 men?

It is evident that the provisions will last a less time, therefore the proportion must be wrought inversely:

$$\text{As } \overset{m.}{488} : \overset{w.}{39} :: \overset{m.}{732} : \frac{488 \times 39}{732} = 26 \text{ weeks, } answer.$$

15. If 1000 men besieged in a town with provisions for 28 days, allowing 18 ounces a day per man, be reinforced with 600 men, and supposing that they cannot be relieved till the end of 42 days; how many ounces a day must each man have that the provisions may last that time.

$1000 \times 18 \times 28$ ounces, the whole quantity of provisions. This quantity is to last 1600 men 42 days.

Divide by 1600, and we have $\frac{1000 \times 18 \times 28}{1600}$ *ounces* the quantity which must last 1 man 42 days: this divided by 42 will give the allowance *per day* for 1 man: *viz.*

$$\frac{1000 \times 18 \times 28}{1600 \times 42} = \frac{10 \times 18 \times 28}{16 \times 42} = \frac{10 \times 9 \times 2}{8 \times 3} = \frac{5 \times 3}{2} = 7\tfrac{1}{2} \text{ oz.}$$

the *answer*.

16. If the carriage of 71 *cwt.* 3 *qr.* of baggage amounts to £5 16*s.* for 40 miles; what will be the expence of 6 *ton* 17 *cwt.* for 94 miles at the same rate?

ton cwt.
6 17 = 548 *qrs.*

£ s.
5 16 = 116*sh.*

cwt. qrs.
71 3 = 287 *qrs.*

As $\overset{qrs.}{287} : \overset{sh.}{116} :: \overset{qrs.}{548} : \overset{sh.}{\dfrac{548 \times 116}{287}}$ the expence of $\overset{ton.}{6}\ \overset{cwt.}{17}$ for 40 miles.

As $\overset{m.}{40} : \overset{sh.}{\dfrac{548 \times 116}{287}} :: \overset{m.}{94} : \overset{sh.}{\dfrac{548 \times 116 \times 94}{287 \times 40}} = \dfrac{137 \times 116 \times 47}{287 \times 5} =$

$520\frac{724}{1435}$ *sh.* $= \overset{£}{26}\ \overset{s.}{0}\ \overset{d.}{6}\frac{78}{1435}$ the *answer*.

17. If a company of 160 men in six days of 11 hours each, can dig a trench 230 yards long, $5\frac{1}{2}$ wide, and $1\frac{1}{2}$ deep; in how many days of 8 hours long would another company consisting of 96 men dig a trench 220 yards long, $3\frac{1}{2}$ wide, and 1 deep; supposing the hardness of the ground in the former case is to that in the latter as 5 to 7, and that 4 men of the latter company can do as much work as 5 of the former in the same time?

$230 \times 5\frac{1}{2} \times 1\frac{1}{2} = 1897\frac{1}{2}$ (by mensuration) the cubic yards in the first trench.

$220 \times 3\frac{1}{2} \times 1 = 770$ the cubic yards in the other.

Now if we suppose the labour necessary to raise a like quantity of earth to be directly proportional to the hardness of the ground, it is evident that the strength required to dig the former trench, will be to that required for the latter, as $\overset{yds.}{1897\frac{1}{2}} \times 5$ to $\overset{yds.}{770} \times 7$.

And, as $\overset{m.}{4} : \overset{m.}{5} :: \overset{m.}{96} : \overset{m.}{120}$, therefore the labour of 120 men of the first company is equal to that of the 96 men.

Hence the question is reduced to the following.

If 160 men in 66 Hours (6×11) can dig $\overset{yds.}{1897\frac{1}{2}} \times 5$ in what time would 120 men dig $\overset{yds.}{770} \times 7$?

As $\overset{m.}{160} : \overset{yds.}{1897\frac{1}{2}} \times 5 :: \overset{m.}{120} : \overset{yds.}{\dfrac{1897\frac{1}{2} \times 5 \times 120}{160}}$, the yards which 120 men could dig in 66 hours.

$$\text{As } \frac{\overset{yds.}{1897\frac{1}{2}} \times 5 \times 120}{160} : \overset{h.}{66} :: \overset{yds.}{770} \times 7 : \frac{66 \times 770 \times 7 \times 160}{1897\frac{1}{2} \times 5 \times 120} =$$

$\frac{23 \times 154 \times 7 \times 8}{3795}$ *hours*, which divided by 8 gives $6\frac{2448}{3795}$ days the *answer*.

Questions of this kind however, may be answered in the following manner: Set down the several proportions in succession, remembering to make the term of supposition which is of the same kind as the required answer, the second term of each proportion; then if the proportions are compounded (140) it will be reduced to a single stating.

Thus, the required answer being *days*, 6 days will be the second or middle term.

		d.		
As 160*m.*	:	6	::	96*m.* *(inverse)*
11*h.*	:		::	8*h.* *(inverse)*
230*l.*	:		::	220*l.*
$5\frac{1}{2}$*br.*	:		::	$3\frac{1}{2}$*br.*
$1\frac{1}{4}$*d.*	:		::	1*d.*
5*har.*	:		::	7*har.*
5*m.*	:		::	4*m.*

And the divisors are 96, 8, 230, $5\frac{1}{2}$, $1\frac{1}{4}$, 5, and 5, therefore

$$\text{As } 96 \times 8 \times 230 \times 5\tfrac{1}{2} \times 1\tfrac{1}{4} \times 5 \times 5 : \overset{d.}{6} :: 160 \times 11 \times 220 \times 3\tfrac{1}{2} \times 1 \times 7 \times 4 : \frac{6 \times 160 \times 11 \times 220 \times 3\frac{1}{2} \times 1 \times 7 \times 4}{96 \times 8 \times 230 \times 5\frac{1}{2} \times 1\frac{1}{4} \times 5 \times 5}$$

days, which reduced to its lowest terms is $\frac{24716}{3795}$ *days*, the *answer* as before.

The three last questions and others of the same kind, belong to what is usually denominated the *Double Rule of Three.*

18. A detachment consisting of 4 companies being sent into a garrison in which the duty requires 60 men a day; what number must each company furnish in proportion to its strength; the first consisting of 42 men, the second of 49, the third of 56, and the fourth of 63?

It is evident that 60 must be divided into 4 numbers having the proportions of 42, 49, 56, and 63.

42
49
56
63
Sum 210

	men		men	
As 210 :	60 ::	42 :	12	from the 1st. company.
210 :	60 ::	49 :	14	2d.
(100). 210 :	60 ::	56 :	16	3d.
210 :	60 ::	63 :	18	4th.

19. Two troops of horse rent a field for which they pay £8[illegible] one troop sent 64 horses for 25 days, and the other sent [illegible] horses for 30 days. How much of the rent must each tro[op] pay?

Suppose 1 is the quantity of grass which a horse eats in 1 day:

Then 64 horses will eat 64 × 25 (1600) such quantities in 25 days.

And 56 horses will eat 56 × 30 (1680) such quantities in 30 days.

Now it is evident that the shares of the rent will be in the same di[rect] proportion as the quantities consumed, or as 1600 and 1680. Hence [the] following rule for questions of this kind:

Multiply each stock by the time of its continuance, then divide [the] whole quantity to be parted into shares in the same proportion as th[e] products.

(100) 1600
1680
3280

	£		£	
As 3280 :	82 ::	1600 :	40	what one troop must pay.
3280 :	82 ::	1680 :	42	what the other must pay.

The last question, and others of the same kind, belong to [the] rule called *Double Fellowship.*

20. To divide 108 into three such parts, that ½ the fir[st,] ⅓ of the second, and ¼ of the third may be equal each other.

Assume 3 numbers which shall be in the same proportion as the re[quired] parts:

Suppose $\left\{\begin{matrix}2\\3\\4\end{matrix}\right\}$ where the ½, ⅓, and ¼ are equal.

Then (100)

$$\text{As } 2+3+4\ (9) : 108 :: 2 : 24$$
$$9 : 108 :: 3 : 36$$
$$9 : 108 :: 4 : 48$$
the three parts required.

21. A general after detaching $\frac{4}{11}$ of his army to occupy a certain height, and $\frac{7}{11}$ of the remainder to watch the enemy's motions, had only 700 men left. Query the whole number of troops?

If we suppose the army to be 1, then $\frac{7}{11}$ will be left when $\frac{4}{11}$ is detached.

And $\frac{7}{11}$ *of* $\frac{7}{11}$ or $\frac{49}{121}$ will be the strength of the 2d. detachment.

And $\frac{4}{11} + \frac{49}{121} = \frac{93}{121}$ will be both detachments; this taken from 1, and $\frac{28}{121}$ of the army remains, which by the questions is equal to 700:

Therefore as $\frac{28}{121} : 700 :: 1 : \frac{121 \times 700}{28} = 3025$ the number required.

Questions which can be answered in a manner similar to the two last, are generally classed under the Rule of *Single Position*.

22. Sold a horse for 40 guineas, by which I lost 4 *per cent.* whereas in dealing I ought to have gained 10 *per cent.* How much was it sold for under its value?

$$\begin{array}{r l} \pounds & \\ 100 & \\ 4 & \text{subtract} \\ \hline 96 & \end{array}$$

$$\text{As } \overset{\pounds}{96} : \overset{\pounds}{100} :: \overset{G.}{40} : \overset{G.}{\frac{100 \times 40}{96}} \text{ the prime cost.}$$

$$\text{And, as } \overset{\pounds}{100} : \overset{\pounds}{110} :: \overset{G.}{\frac{100 \times 40}{96}} : \overset{G.}{\frac{110 \times 100 \times 40}{100 \times 96}} = 45\tfrac{5}{6} \text{ guineas its}$$

price at 10 *per cent* profit.

$$\begin{array}{l l} 45\frac{5}{6} & \\ 40 & \text{sub.} \\ \hline 5\frac{5}{6} & \text{guineas, the } \textit{answer}. \end{array}$$

23. Suppose on a march, a party of foot is 1000 paces before another of horse, and the rate of marching is 6 paces by the foot to 5 by the horse; now if two horse's steps be equal to $2\frac{1}{2}$ of a man's, how many paces will the horse take to come up with the foot?

Because 1 horse's pace is equal to $1\frac{1}{4}$ man's paces, 5 paces of a horse will be equal to $6\frac{1}{4}$ man's paces:

Therefore the horse at every 5 paces gains $\frac{1}{4}$ of a man's pace; and at this rate the party of horse have to gain 1000 man's paces;

$$\text{Hence, as } \overset{m.p.}{\tfrac{1}{4}} : \overset{h.p.}{5} :: \overset{m.p.}{1000} : \overset{h.\,paces.}{20000}, \text{ the answer.}$$

24. A can do a piece of work in 7 days, and B can do the like in 5 days; in what time would it be done if they work together?

$$\text{As } \overset{d.}{7} : \overset{w.}{1} :: \overset{d.}{5} : \overset{w.}{\tfrac{5}{7}} \text{ what A can do in 5 days.}$$

Therefore both together can do $1\frac{5}{7}$ in 5 days.

$$\text{As } \overset{w.}{1\tfrac{5}{7}} : \overset{d.}{5} :: \overset{w.}{1} : \overset{d.}{\tfrac{35}{12}} = 2\tfrac{11}{12} \text{ days, the answer.}$$

25. A and B can perform a piece of work in 2 days; A and C in 3 days; and B and C in 5 days: in what time would each do it by himself?

$$\text{As } \overset{d.}{3} : \overset{w.}{1} :: \overset{d.}{2} : \overset{w.}{\tfrac{2}{3}} \text{ what A and C can do in 2 days.}$$
$$\text{As } 5 : 1 :: 2 : \tfrac{2}{5} \text{ what B and C can do in 2 days.}$$

	d.	w.
By A and B in	2	1
By A and C in	2	$\frac{2}{3}$
By B and C in	2	$\frac{2}{5}$
Sum		$2\frac{1}{15}$

but in doing this, each of the three must evidently work 4 days, therefore the three together would do half of $2\frac{1}{15}$ or $1\frac{1}{30}$ in 2 days.

$$\text{Hence } \left.\begin{array}{lll} 1\tfrac{1}{30} - 1 = \tfrac{1}{30} & \text{what C} \\ 1\tfrac{1}{30} - \tfrac{2}{3} = \tfrac{11}{30} & \text{what B} \\ 1\tfrac{1}{30} - \tfrac{2}{5} = \tfrac{19}{30} & \text{what A} \end{array}\right\} \text{can do in 2 days.}$$

Therefore, as $\frac{1}{10}$: 2 :: 1 : 60 days the *time* by C.
$\frac{1}{11}$: 2 :: 1 : $5\frac{1}{11}$ .... by B.
$\frac{1}{16}$: 2 :: 1 : $3\frac{3}{4}$ .... by A.

96. The plan of a fortified town and its environs in the Netherlands is 15 inches long and 12 broad. The *scale* annexed to it is 800 toises, and is 4·7 inches in length. Now if the plan be enlarged to a scale of 6 inches the English mile; what will be the length and breadth?

A toise = 2·1315 yards.
2·1315 × 800 = 1705·2 yards the scale.

As $\overset{yds.}{1705{\cdot}2} : \overset{in.}{4{\cdot}7} :: \overset{yds.}{1760} : \overset{in.}{\dfrac{4{\cdot}7 \times 1760}{1705{\cdot}2}}$ the length of a mile on the scale of toises.

And since the dimensions will be in the same proportion as the respective scales, we have,

As $\overset{in.}{\dfrac{4{\cdot}7 \times 1760}{1705{\cdot}2}} : \overset{in.}{6} :: \overset{in.}{15} : \overset{in.}{\dfrac{1705{\cdot}2 \times 6 \times 15}{4{\cdot}7 \times 1760}} = \overset{in.}{18{\cdot}55}$ the *required length.*

And $\dfrac{4{\cdot}7 \times 1760}{1705{\cdot}2} : 6 :: 12 : \dfrac{1705{\cdot}2 \times 6 \times 12}{4{\cdot}7 \times 1760} = 14{\cdot}84$ *inch* the *breadth.*

97. In what time would 16 battalions of infantry each consisting of 510 men, with two field pieces, 4 horses to each, pass through a defilé $1\frac{1}{2}$ miles long, supposing the march is in open column with 6 men in front, and the rate 75 paces (of $2\frac{1}{2}$ feet each) *per minute*, being that of ordinary time?

Suppose a battalion in line of 3 ranks; then $\frac{510}{3}$ = 170 men in each rank; and 22 inches or $1\frac{5}{6}$ feet being the allowance for each man in front, we have 170 × $1\frac{5}{6}$ = $311\frac{2}{3}$ feet the extent of the front or line, which also is the estimated extent of the same battalion when in open column.

$311\frac{2}{3}$
160 *feet*, extent of 2 field pieces with 4 horses to each.
Sum $471\frac{2}{3}$ *feet*, extent of 1 battalion with 2 field pieces.

And $471\frac{5}{8} \times 16 = 7546\frac{2}{8}$ *feet*, extent of the 16 battalions; equal to 3019 *paces* of $2\frac{1}{2}$ *feet* each.

$$\begin{array}{rl} 3019 & \\ \underline{3696} & \textit{paces} = 1\frac{3}{4} \text{ miles.} \\ 6715 & \textit{paces}, \text{ extent of column and defile.} \end{array}$$

As 75*pa.* : 1*min.* :: 6715*pa.* : $89\frac{8}{15}$*min.* *Ans.*

28. Suppose 18 battalions each consisting of 560 men, with 18 mounted officers, and 2 field pieces (each with 4 horses) have to pass two defilés; one is a bad road, 1 mile in length; the other a good road, $1\frac{3}{4}$ miles long; each defilé admitting of 3 men to march in front; how many battalions must pass each defilé that the whole march through them may be made in the least time,

allowing { 6 feet in front to each rank of foot;
12 feet to a rank of horse;
80 feet for the extent of a field piece with 4 horses;
$2\frac{1}{2}$ feet the pace of a man:

And that infantry march { 80 paces *per minute* in a good road.
50 .... in a bad road.

In order that the whole march may be made in the least time, it will be necessary to divide the 18 battalions into two columns whose lengths shall be such that their rears may quit the defilés at the same time; or that the march of one column through one defilé must be made in the same time as that of the other column through the other defilé. This will evidently be when the length of one column added to a mile, is to the length of the other column added to $1\frac{3}{4}$ miles, as 50 to 80, the rates of marching in the defilés.

$$\begin{array}{rrl} & 3\,)\,560 & \\ & \overline{187} & \text{ranks.} \\ & 6 & \\ \textit{feet} & \overline{1122} & \text{extent of 187 ranks.} \\ & 160 & \text{for 2 field pieces.} \\ & \underline{108} & \text{for 9 ranks of officers riding two and two.} \\ & 1390 & \textit{feet}, \text{extent of 1 battalion} = 556 \textit{ paces} \text{ of } 2\frac{1}{2} \textit{ feet} \text{ each.} \end{array}$$

$556 \times 18 = 10008$ *paces*, extent of 18 battalions.
2112 *paces* = 1 mile.
3168 *paces* = $1\frac{1}{2}$ miles.
15288 *paces*, extent of both columns and defilés.

$80 + 50 = 130$.

(100) As 130 : 15288 :: 80 : 9408 *paces*, length of the $1\frac{1}{2}$ mile defilé with its column.

And

As 130 : 15288 :: 50 : 5880 *paces*, the length of the 1 mile defilé with its column.

9408
3168 *paces* = $1\frac{1}{2}$ miles.
6240 *paces*, length of the column which must pass the longest defilé; this divided by 556 the length of 1 battalion, gives 11 (the nearest whole number) for the number of battalions which must march through the $1\frac{1}{2}$ mile defilé.

5880
2112 *paces* = 1 mile.
3768 *paces*, length of the column to pass the 1 mile defilé.

And $\frac{3768}{556} = 7$ (the nearest integer) for the number of battalions which must march through the shortest defilé.

29. Suppose the same 18 battalions have to pass two defilés of equal extent, one admitting of 3, the other of 4 men in front; how must the 18 battalions be divided that the whole march through them may be performed in the least time, if the roads are equally good?

Since the rate of marching in each defile is the same, the extent of the columns must be equal. And therefore 18, the number of battalions, must be divided into two parts having the same proportion as the length of a battalion marching 3 men in front, to the length when 4 men march in front.

443 *paces*, extent of a battalion 4 men in front.
556 .... extent, 3 men in front *(see the last question)*.
999

As 999 : 18 :: 443 : 8 nearly.
999 : 18 :: 556 : 10 nearly.

Therefore 10 battalions must march through the widest defilé; and 6 through the other.

30. To divide 20 into 2 such parts that the product of the first part by 5, shall be to the product of the other part by 6, in the proportion of 10 to 3?

It is evident that the two required parts will be in the same proportion as $\frac{10}{5}$ and $\frac{3}{6}$, because if the former of those fractions is multiplied by 5, and the latter by 6, the products will be in the given proportion; therefore 20 must be divided into two parts having the proportion of $\frac{10}{5}$ and $\frac{3}{6}$.

Hence (100) as $\frac{10}{5} + \frac{3}{6} : 20 :: \frac{10}{5} : 16$ the first part; consequently 4 is the other part.

In like manner any other number may be divided into a proposed number of parts such, that their products by given numbers may obtain given proportions.

31. Suppose 8 battalions have to pass 2 defilés, one $\frac{3}{4}$, the other $1\frac{3}{4}$ miles in length; the former admitting 6, and the latter 4 men to march in front; now if the length of a battalion (including 2 field pieces) be 330 *paces* of $2\frac{1}{2}$ *feet* each, when 6 men march in front, and 440 when 4 men march in front; how many battalions must pass each defilé that the whole march through them may be made in the least time, supposing the rate of marching in the shortest defilé is 50, and in the other 80 *paces per* minute?

It follows from *examp.* 28, that the length of one column added to $1\frac{3}{4}$ miles must be to the length of the other column added to $\frac{3}{4}$ mile, in the proportion of 80 to 50, the rates of marching.

$1\frac{3}{4}$m. = 3696 *paces.*
$\frac{3}{4}$m. = 1584 *paces.*

As 80 : 50 :: 3696 : 2310 *paces.*
1584 *paces* = $\frac{3}{4}$m.
726 *paces*, diff. and $\frac{726}{330} = 2\frac{66}{330}$ battal.

Therefore if the extent of $2\frac{66}{330}$ battal. (726 *paces*) be added to the shortest defilé, the sum will be to the longest defilé, in the proportion of 50 to 80 the rates of marching. Consequently $5\frac{264}{330}$ battal. (the difference of 8 and $2\frac{66}{330}$) must be divided into 2 such parts that the product of one

part by 330 shall be to the product of the other part by 440, in the proportion of 50 to 80.

Hence (by the last example):

As $\frac{80}{330} + \frac{50}{440}$: 8 :: $\frac{50}{440}$: 3 (the nearest integer) for one of the parts required; (97) which part is the number of battalions that must march through the longest defilé; consequently 5 have to march through the other.

$\frac{1584 + 330 \times 5}{50} = 64\frac{17}{25}$ *min.* the time of marching through the shortest defilé.

$\frac{3696 + 440 \times 3}{80} = 62\frac{7}{10}$ *min.* time of marching through the longest.

N. B. In this and the 28*th.* and 29*th.* examples it is supposed that the fronts of the columns enter the defilés nearly at the same time.

32. Suppose 40*lb.* of gunpowder at 1*s.* *per lb.* be mixt with 60*lb.* at 1*s.* 3*d.* *per lb.* what is 20*lb.* of the mixture worth?

40*lb.*	at 1*s* .... is	40*s.*	
60	at 1*s.* 3*d.* ...... is	75*s.*	
100		115*s.*	Therefore the value of 100*lb.* is 115*s.*

Hence, as 100*lb.* : 115*s.* :: 20*lb.* : 23*s.* the *answer.*

33. If the strength or quality of three sorts of gunpowder (or other ingredients) be denoted by 10, 15, and 16; how much of each must be taken that the proportionate quality of the mixture may be 12?

Or, putting the question in more familiar terms: Suppose 10, 15, and 16 pence are the prices *per* pound; what quantity of each will make a mitxure worth 12 pence *per* pound?

Because every *lb.* at 10*d.* gives 2*d.* *less,* and every *lb.* at 15*d.* cost 3*d.* *more* than 12*d.* the mean price, therefore 3*lb.* at 10*d.* to 2*lb.* at 15*d.* will make the defect below 12*d.* equal to the excess above it.

Thus 3*lb.* at 10*d.* will give 6*d.* less than 3*lb.* at 12*d.*
And 2*lb.* at 15*d.* will give 6*d.* more than 2*lb.* at 12*d.*

Hence the quantities will be reciprocally as the differences between the mean and extreme prices.

Therefore 3lb. at 10*d.* and 2lb. at 15*d.* will together be worth 12*d.* per lb.

Again, the difference of 10*d.* and 12*d.* is 2*d.*
and that of 16*d.* and 12*d.* is 4*d.*
Therefore 4lb. at 10*d.* and 2lb. at 16*d.* will together be worth 12*d.* per lb.

Consequently 7lb. (4 + 3) at 10*d.*, 2 .... at 15, 2 .... at 16 — will be worth 12*d.* per lb. Or any quantities in the same proportion as 7, 2, and 2.

And in the same manner the proportional quantities of any number of ingredients may be found.

When the whole mixture is to be of a certain weight, find the quantity of each ingredient by the rule of proportion. Thus, suppose in the foregoing example a mixture of 30lb. is required.

Then,

	lb.		lb.	
As 7 + 2 + 2 or 11 :	30 ::	7 :	$19\frac{1}{11}$	the *quantities required.*
		2 :	$5\frac{5}{11}$	
		2 :	$5\frac{5}{11}$	

Questions of this kind when proposed to be solved arithmetically, come under the rule called *Alligation.* It is easy to perceive that they admit of a great variety of answers, which cannot however, be readily discovered without Algebra.

Of INTEREST.

105. INTEREST is the sum allowed for the loan or forbearance of money. It is reckoned at so much *per cent. per* [illegible] the *rate.* Thus if £4 is paid for the use of [illegible] year, £4 is the interest; and the rate is 4 *per cent.* [illegible] if £9 is paid for the use of £300 for a year, [illegible] is 3 *per cent. per annum;* and

Interest is distinguished into two kinds, *Simple*, and *Compound*.

106. *Simple Interest* is the allowance for the first sum or principal *only* for the whole time. So the simple interest of £100 for 3 years at 4 *per cent.* will be £12. Therefore the interest of any sum for a given time will be directly proportional to the principal.

Hence,
As £100
Is to its interest for any given time;
So is any other principal,
To its interest for that time.

Examples of Simple Interest.

1. What is the interest of £270 for 1 year at 4 *per cent?*

$$\text{As } \overset{\pounds}{100} : \overset{\pounds}{4} :: \overset{\pounds}{270} : \frac{4 \times 270}{100}\overset{\pounds}{} = \overset{\pounds}{10}\ \overset{s.}{16}. \quad \textit{Ans.}$$

2. What is the interest of £524 10s. for 5 years at 3 *per cent?*

$5 \times 3 = £15$ the interest of £100 for 5 years.

$$\text{As } \overset{\pounds}{100} : \overset{\pounds}{15} :: \overset{\pounds}{524\tfrac{1}{2}} : \overset{\pounds}{78}\ \overset{s.}{13}\ \overset{d.}{6}. \quad \textit{Ans.}$$

3. How much is the interest of £122 15s. for 240 days at 5 *per cent?*

$$\text{As } \overset{d.}{365} : \overset{\pounds}{5} :: \overset{d.}{240} : \overset{\pounds}{\frac{5 \times 240}{365}} \text{ the interest of £100 for 240 days.}$$

$$\text{As } \pounds 100 : \frac{5 \times 240}{365} :: \pounds 122\tfrac{3}{4} : \frac{5 \times 240 \times 122\frac{3}{4}}{100 \times 365} = \overset{\pounds}{4 \cdot 036}. \quad \textit{Ans.}$$

4. What will 218*l.* amount to in $2\frac{3}{4}$ years at $3\frac{1}{2}$ *per cent?*

$2{\cdot}75 \times 3{\cdot}5 = £9{\cdot}625$ the interest of £100 for 2·75 years.
100
Sum 109·625 amount of £100 in 2·75 years.

$$\text{As } \overset{\pounds}{100} : \overset{\pounds}{109{\cdot}625} :: \overset{\pounds}{218} : \overset{\pounds}{238}\ \overset{s.}{19}\ \overset{d.}{7{\cdot}8}. \quad \textit{Ans.}$$

Required the discount of £80 due 2½ years hence at 5 *per cent*?

Here the amount (£80) is given, and the interest or discount is required.

5 × 2½ = £12½ the interest of £100 for 2½ years.
100
112½ the amount of £100 in 2½ years.

£ £ £ £ s.
As 112½ : 12½ :: 80 : 8 17¾ the answer.

6. What is the purchase of £2000 bank-stock at 106⅜ *per cent.* or what £106⅜ must be given for £100 stock?

£ £ £ £ s.
As 100 : 106⅜ :: 2000 : 2127 10. *Ans.*

7. When the 3 *per cent. consols* are done at 56⅝, what is the interest of money?

As 56⅝ : 3 :: 100 : $5\frac{45}{151}$ *per cent.* *Ans.*

COMPOUND INTEREST.

107. When the amount at Simple Interest is forborn, the interest arising from that sum is called Compound Interest. And therefore any succeeding amount may be found as in the 4*th* example of Simple Interest; only repeating the operation.

Examples.

1. What is the amount of £120 in 4 years at 3 *per cent. per annum* compound interest?

The amount of £100 in 1 year is £103. Hence,

$$\text{As } £100 : £103 :: £120 : £\frac{103 \times 120}{100}$$ the amount at the end of the 1*st.* year.

Or dividing the two first terms of the proportion by 100. (96.)

As 1 : 1·03 :: 120 : 1·03 × 120, the amount at the end of the 1*st.* year.
1 : 1·03 :: 1·03 × 120 : 1·03 × 1·03 × 120 at the end of the 2*d.*
1 : 1·03 :: 1·03 × 1·03 × 120 : 1·03 × 1·03 × 1·03 × 120 at the end of the 3*d.*

$1 : 1{\cdot}03 :: 1{\cdot}03 \times 1{\cdot}03 \times 1{\cdot}03 \times 120 : 1{\cdot}03 \times 1{\cdot}03 \times 1{\cdot}03 \times 1{\cdot}03 \times 120$, at the end of the *4th*.

$1{\cdot}03 \times 1{\cdot}03 \times 1{\cdot}03 \times 1{\cdot}03 = 1{\cdot}1255$ (retaining 4 decimals only)

$$\begin{array}{r} 1{\cdot}1255 \\ 120 \\ \hline \text{Ans. } £135{\cdot}0600 \end{array}$$

Ans. £135·0600, or £135 1·2*s.* the amount.

2. What is the compound interest of £242 10*s.* forborn $2\frac{1}{2}$ years at 4 *per cent. per ann.* the interest payable half yearly?

The interest of £100 for $\frac{1}{2}$ a year is £2.

Therefore the amount of £100 at the end of $\frac{1}{2}$ a year is £102

As £100 : £102 :: £242·5 :

Or dividing the two first terms by 100:

As $1 : 1{\cdot}02 :: 242{\cdot}5 : 1{\cdot}02 \times 242{\cdot}5$ the amount at the end of the first $\frac{1}{2}$ year.

And proceeding in the same manner for 5 half years, we have

$1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 \times £242{\cdot}5$ for the whole amount.

$1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 \times 1{\cdot}02 = 1{\cdot}10408$ (retaining only 5 decimals)

$$\begin{array}{rll} \text{And } 1{\cdot}10408 \times 242{\cdot}5 = & £267{\cdot}7394 & \text{the amount.} \\ & 242{\cdot}5 & \text{the principal, subtract,} \\ \hline & 25{\cdot}2394 & \text{the } \textit{interest.} \end{array}$$

But the operations in compound interest are much more expeditiously performed by means of Logarithms.

OF POSITION.

108. **Position** or the *Rule of False* is a method of solving questions by means of assumed or false numbers; and is of two kinds, *single*, and *double*.

Questions which require but one assumption, or where the results are proportional to the suppositions, belong to single position; such as the 20*th*. and 21*st*. examples in the Rules of Proportion.

109. WHEN two assumptions are made for answerin question, it is called Double Position; and sometimes the thod of Trial-and-Error.

Rule. Make two suppositions, and proceed with each cording to the conditions of the question. Then find differences between the results and the result in the question.

Multiply the first supposition by the second differenc error; and the second supposition by the first differenc error.

Then, if the errors are *alike* (*viz.* both too great, or little) divide the *difference* of the products by the *differ* of the errors, and the quotient will be the answer.

But if the errors are *unlike* (or one too great, and the o to little) divide the *sum* of the products by the *sum* of the rors, for the answer.

This rule is founded on the supposition that the differe between the true and supposed numbers are directly pro tional to the respective differences between the true and errone results (vol. 2. *art.* 128. *examp.* 8.) When that is not the c the rule cannot give the exact answer.

Examples.

1. What two numbers are those whose sum is 10, and the greate vided by the less gives the quotient 20?

Suppose the numbers are	$\left\{ \begin{array}{c} 20 \\ 1 \end{array} \right\}$	the quotient being 20;
their sum	21	
But the sum should be	10	
Difference or *first error*	11	*too great.*

1 : 1·03 :: 1·03 × 1·03 × 1·03 × 120 : 1·03 × 1·03 × 1·03 × 1·03 × 120, at the end of the 4th.

1·03 × 1·03 × 1·03 × 1·03 = 1·1255 (retaining 4 decimals only)
120
Ans. £135·0600, or £135 1·2s. the amount.

£ s.
What is the compound interest of 242 10 forborn 2½ years at 4 *per cent. per ann.* the interest payable half yearly?

The interest of £100 for ½ a year is £2.

Therefore the amount of £100 at the end of ½ a year is £102

∴ £100 : £102 :: £242·5 :

Or dividing the two first terms by 100;

As 1 : 1·02 :: 242·5 : 1·02 × 242·5 the amount at the end of the first half year.

And proceeding in the same manner for 5 half years, we have

£
1·02 × 1·02 × 1·02 × 1·02 × 1·02 × 242·5 for the whole amount.

1·02 × 1·02 × 1·02 × 1·02 × 1·02 = 1·10408 (retaining only 5 decimals)

£
And 1·10408 × 242·5 = 267·7394 the amount.
242·5 the principal, subtract,
25·2394 the *interest.*

But the operations in compound interest are much more expeditiously performed by means of Logarithms.

OF POSITION.

108. POSITION or the *Rule of False* is a method of solving questions by means of assumed or false numbers; and is of two kinds, *single*, and *double*.

Questions which require but one assumption, or where the results are proportional to the suppositions, belong to single position; such as the 20*th*, and 21*st*, examples in the Rules of

Again, suppose the numbers $\left\{ \begin{matrix} 40 \\ 2 \end{matrix} \right\}$ the quotient being 20,

their sum	42	
But the sum should be	10	
Difference or *second error*	32	*too great.*

Now the greater suppositions 20 and 40 will give the greater of the two required numbers; and the other suppositions 1 and 2 will bring out the less.

First supposition	20	Second supposition	40
Second error	32	First error	11
Product	640	Product	440

$$\begin{array}{r} 640 \\ 440 \\ \hline 200 \end{array}$$

Difference of products 200 which divided by 21 the *difference* of the errors 32 and 11, gives $\frac{200}{21} = 9\frac{11}{21}$ the greatest of the two required numbers.

Again (for the least of the two numbers).

First supposition	1	Second supposition	2
Second error	32	First error	11
Product	32	Product	22

$$\begin{array}{r} 32 \\ 22 \\ \hline 10 \end{array}$$

Difference of products 10 which divided by 21 the *difference* of the errors gives $\frac{10}{21}$ the least of the two required numbers. But when either number is found, the other will be given, because their sum is given.

To verify the rule when the errors are *unlike*. Let the first suppositions be 20 and 1 as before; then the *first error* will be 11 *too great.*

For the second suppositions let the numbers be .... $\left\{ \begin{matrix} 5 \\ 0\frac{1}{4} \end{matrix} \right\}$ the quotient being 20.

Sum	$5\frac{1}{4}$	
But the sum should be	10	
Difference or *second error*	$4\frac{3}{4}$	*too little.*

First supposition	20	Second supposition	5
Second error	$4\frac{3}{4}$	First error	11
Product	95	Product	55

$$\begin{array}{r} 95 \\ 55 \\ \hline 150 \end{array} \text{ sum of products.} \qquad \begin{array}{r} 11 \\ 4\frac{3}{4} \\ \hline 15\frac{3}{4} \end{array} \text{ sum of errors.}$$

Then $\frac{150}{15\frac{1}{2}} = 9\frac{21}{31}$ the greater number as before.

2. What number is that which added to its square shall make the sum 12?

Suppose the number to be	5	Next, suppose	4
Its square is 5 × 5 .... =	25	Its square is 4 × 4 =	16
Sum	30	Sum	20
But the sum should be	12		12
Error	18 *too great.*	Error	8 *too great.*

Products $\begin{cases} 5 \times 8 = 40 \\ 4 \times 18 = 72 \end{cases}$

Difference of errors 10) 32 *diff.* of products
3·2 quotient.

But the required number is 3, (for 3 added to 9, the square of 3, make 12) therefore the rule fails in this example. The true answer, however, may be approximated to any assigned degree of accuracy by repeating the operation, and constantly making the last quotients or approximations, the assumed numbers:

Then,

Let one supposition be	4	And the other supp.	3·2
Its square (as before)	16	Its square 3·2 × 3·2 =	10·24
Sum	20	Sum	13·44
	12		12
Error as before	8 *too great.*	Error ...	1·44 *too great.*
		Error ...	8
		Diff. error ...	6·56

Products $\begin{cases} 4 \times 1{\cdot}44 = 5{\cdot}76 \\ 3{\cdot}2 \times 8 = 25{\cdot}6 \end{cases}$

19·84 *diff.* products.

$\frac{19{\cdot}84}{6{\cdot}56} = 3{\cdot}02$ the *second approximation.*

Next, making 3·2 and 3·02 the assumptions we have

3·02 × 3·02 = 9·1204 the square of 3·02

3·02

Sum 12·1404

12

Error 0·1404 *too great.*

1·44 first error, as before.

1·2996 *diff.* errors.

Products $\begin{cases} 3{\cdot}2 \times 0{\cdot}1404 = 0{\cdot}44928 \\ 3{\cdot}02 \times 1{\cdot}44 = 4{\cdot}3488 \end{cases}$

3·89952 *diff.*

= 3·0005 the *third approximation.*

sitions be 3·02 and 3·0005; and the next approxi-
000001. And if the operation be repeated with
the result will be 3·0000000002, &c.

rule may frequently be applied with success in very

OF INVOLUTION.

number is multiplied into itself a certain
is called Involution, or raising of powers.

nultiplied is the root; and the products are

oot,

2 × 2 = 4 is the 2*d* power or square of 2.
2 × 2 = 8 is the 3*d* power or cube of 2.
2 × 2 = 16 is the 4*th* power or biquadrate.
2 × 2 = 32 is the 5*th* power or sursolid.
&c.

2	3	4	5	6	7	8	9
4	9	16	25	36	49	64	81
8	27	64	125	216	343	512	729

to which a number is to be raised is usually
figure called the *index* or *exponent.*

lenotes the 3*d* power or cube of 5.
the 4*th* power of 7.
the square of 10. Here the *indices*
e powers are 3, 4, and 2.

2 × 2 × 2 = 32 is the 5*th* power of the
nat the 5*th* power is the product of the square

For $2 \times 2 = 4$ is the square; and $2 \times 2 \times 2 = 8$ is the cube; therefore $4 \times 8 = 32$ the *5th* power.

Hence $2^2 \times 2^3 = 2^5$; consequently the addition of the indices 2 and 3 answer to the multiplication of the powers; viz. $2^2 \times 2^3 = 2^{2+3}$.

Also $3^3 \times 3^4 = 3^7$. For 3^3 is 27; and 3^4 is 81; and 27×81 is equal to $2187 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$.

Other Examples.

What is the square of 100?

$100 \times 100 = 10000$. *Ans.*

What is the square of $\frac{3}{11}$?

$\frac{3}{11} \times \frac{3}{11} = \frac{9}{121}$. *Ans.*

Required the cube of the decimal ·013?

$·013^3 = ·013 \times ·013 \times ·013 = ·000002197$. *Ans.*

What is the *4th* power of 2·01?

$2·01^2 = 2·01 \times 2·01 = 4·0401$ the square, which squared is $4·0401^2 = 4·0401 \times 4·0401 = 16·32240801$, the *Answer*.

EVOLUTION.

112. EVOLUTION is the extraction or finding the roots of any given powers, being the reverse of Involution.

Every number which is a known power will have a determinate root called a rational root: thus the number 8 is a cube number whose root is 2; and the number 9 is a square having 3 for the root: but 10 is not an exact power of any kind, because its root can never be accurately obtained. By the help of decimals however, the roots of any numbers may be approximated to any assigned degree of exactness; these approximate

roots are called *irrational* or *surd roots.* Thus any root of 10 will be a surd. And the square root of 8, and the cube root of 9 are both surds.

To Extract the SQUARE ROOT.

113. *Rule.* Begin at the units place and point the number into periods of two figures each.

Find the greatest square in the first period on the left hand and set its root on the right of the given number, in the same manner as a quotient figure in division.

Subtract the square from the period above it, and to the remainder bring down the next period, for a dividend.

Double the aforesaid root, and find how often it is contained in the dividend, exclusive of its right-hand figure, and set the result in the quotient, and also on the right of the divisor.

Multiply the augmented divisor by this last quotient figure, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

Then find a new divisor by doubling the figures of the quotient; and proceed as before till all the periods are brought down.

The best way of doubling the root or quotient is by adding the last figure always to the last divisor.

Examples.

1. Required the square root of 41409225?

```
           41409225 ( 6435 root or quotient,
           36
 124 )     540
   4       496
 1283 )    4492
    3      3849
 12865 )   64325
           64325
```

5. Required the square root of the decimal ·4?

```
        ·40 ( ·632 &c. root.
         36
  123 ) 400
    3   369
 1262 ) 3100
        2524
         576
```

6. What is the square root of ·00095?

```
  ·000950 ( ·03082 &c. root.
       9
  608 ) 5000
    8   4864
 6162 ) 13600
        12324
         1276
```

116. *To extract the square root of a Vulgar Fra*

Reduce it to its lowest terms: then the roots of the num and denominator will form the fractional root required.

Thus the square root of $\frac{9}{16}$ is $\frac{3}{4}$.

And the square root of $\frac{18}{32}$ is $\frac{3}{4}$; for $\frac{18}{32} = \frac{9}{16}$ whose root is $\frac{3}{4}$.

Also, the square root of $\frac{4}{1600}$ is $\frac{1}{20}$; for $\frac{4}{1600} = \frac{1}{400}$ whose root is

When the terms of the fraction are not perfect squares, it may duced to a decimal, and its root extracted.

Thus, suppose the square root of $\frac{5}{7}$ is required?

$\frac{5}{7} = \cdot714285714$ &c. whose root is ·84515 &c.

Or because $\frac{5}{7} = \frac{5 \times 7}{7 \times 7} = \frac{35}{49}$, therefore the square root of 35 by 7 (the square root of 49) will be the root required.

The square root of 35 is 5·91608 nearly.

Therefore $\frac{5 \cdot 91608}{7} = \cdot 84515$ &c. the *root*, as before.

A *Mixt Number* may be brought to an improper fractio its root extracted as above.

thousands, hundreds, and tens, added to the units: hence the reason for doubling the root. And because a cipher in the divisor, and another in the quotient, will make two in the product, if the ciphers are omitted in both, it is evident that only two figures must be brought down at a time in order to form the dividend, which is the reason for pointing the number from the right to the left into periods of two figures each: for it is manifest from the formation of the square, that the root will consist of as many figures as there are points or periods.

2. Required the square root of 100861849?

```
        100861849 ( 10043 root.
        1
2004 )  008618
   4      8016
20083 )    60249
           60249
```

3. What is the square root of 5·9049?

```
      5·9049 ( 2·43 root.
      4
 44 ) 190
  4   176
483 ) 1449
      1449
```

4. Required the square root of 8?

```
        8 ( 2·828 &c. root.
        4
  48 )  400
   8    384
 562 )  1600
   2    1124
5648 )  47600
        45184
         2416
```

Thus by annexing periods of two ciphers each to the remainders, the extraction may be continued to any number of decimals in the root. And the integral part of the root will consist of as many figures as there are points over the integers in the number whose root is required.

115. *The root of a proper fraction is greater than its square:* Therefore decimals are pointed at every second figure from the left-hand.

5. Required the square root of the decimal ·41?

```
        ·41 ( ·632 &c. root.
         36
 123 )  500
   3    369
1262 ) 3100
       2524
        576
```

6. What is the square root of ·00095?

```
     ·000950 ( ·03082 &c. root.
          9
 608 )  5000
   8    4864
6162 ) 13600
       12324
        1276
```

116. *To extract the square root of a Vulgar Fraction.* Reduce it to its lowest terms: then the roots of the numerator and denominator will form the fractional root required.

Thus the square root of $\frac{9}{16}$ is $\frac{3}{4}$.

And the square root of $\frac{18}{128}$ is $\frac{3}{8}$; for $\frac{18}{128} = \frac{9}{64}$ whose root is $\frac{3}{8}$.

Also, the square root of [illegible] is [illegible]; for [illegible] = [illegible] whose root is [illegible].

When the terms of the fraction are not perfect squares, it may be reduced to a decimal, and its root extracted.

Thus, suppose the square root of $\frac{5}{7}$ is required?

$\frac{5}{7} = ·714285714$ &c. whose root is ·84515 &c.

Or because $\frac{5}{7} = \frac{5 \times 7}{7 \times 7} = \frac{35}{49}$, therefore the square root of 35 divided by 7 (the square root of 49) will be the root required.

The square root of 35 is 5·91608 nearly.

Therefore $\frac{5·91608}{7} = ·84515$ &c. the *root*, as before.

A *Mixt Number* may be brought to an improper fraction, and its root extracted as above.

The 4 decimals in the root are found by annexing 4 periods of three ciphers each.

3. Let the cube root of the decimal ·07 be required.

Here the periods or points are placed over every 3d. figure from the left hand.

$$
\begin{array}{rl}
 & \cdot 07\dot{0}00\dot{0}\ \text{\&c.}\ (\cdot 412128\ \text{\&c. root.} \\
4^3 = & \underline{64} \\
4^2 \times 300 = 4800\ ) & \overline{6000}\ (\ 1 \\
 & \underline{70000} \\
41^3 = & \underline{68921} \\
41^2 \times 300 = 504300\ ) & 1079000\ (\ 2 \\
 & \overline{70000000} \\
412^3 = & \underline{69934528} \\
412^2 \times 300 = 50923200\ ) & 65472000\ (\ 1 \\
 & \overline{70000000000} \\
4121^3 = & \underline{69985463561} \\
4121^2 \times 300 = 5094792300\ ) & 14536439000\ (\ 2 \\
 & \overline{70000000000000} \\
41212^3 = & \underline{69995653640128} \\
41212^2 \times 300 = 509528683200\ ) & 4346359872000\ (\ 8 \\
\text{\&c.} & \text{\&c.}
\end{array}
$$

The reason for pointing the number into periods of 3 figures each is manifest from the principles of common multiplication; for any number with one or more ciphers on the right hand, must have exactly 3 times as many ciphers in its cube.

118. But all the usual or common rules for extracting the cube and higher roots are extremely prolix. The following general method of approximation however, derived from the *rational formulæ* of Dr. Halley, (vol. 2, *art.* 111) is more expeditious, and easily remembered.

To extract the Root of any Power.

Assume the root (the nearer the true root the better), then raise this root to the power whose root is required, and call it the assumed power.

Then take the sum of

The assumed power multiplied by its index added to 1;

And the given number multiplied by the index lessened by 1.

And the sum of

The assumed power multiplied by the index lessened by 1;

And the given number multiplied by the index added to 1

Then say, by the Rule of Proportion,

As the first of those sums,

Is to the second,

So is the assumed root,

To the required root, nearly. And if this root be taken for the assumed root, and the operation repeated, a nearer approximation will be obtained; and so on.

Examples of the Cube Root.

1. Required the 3d. or cube root of 184?

Assume 6 for the root, whose cube is 216, the assumed power. Then the index 3 added to 1, and lessened by 1, give 4 and 2.

Therefore,

As the sum of 216×4 and 184×2,

Is to the sum of 216×2 and 184×4;

So is the assumed root 6,

To the root, nearly.

Or dividing the two first terms of the proportion by 2 we have (90.)

As the sum of 216×2 and 184,

Is to the sum of 216 and 184×2;

So is 6,

To the root, nearly.

In words,

As twice the assumed cube added to the given number,

Is to the assumed cube added to twice the given number;

So is the assumed root,

To the required root, nearly.

The 4 decimals in the root are found by annexing 4 periods of three ciphers each.

3. Let the cube root of the decimal ·07 be required.

Here the periods or points are placed over every 3d. figure from the left hand.

$$\cdot 0\dot{7}000\dot{0} \text{ \&c. } (\cdot 412128 \text{ \&c. root.}$$

$$
\begin{array}{rl}
4^3 = & \underline{64} \\
4^2 \times 300 = 4800 \;) & \underline{6000} \;(\; 1 \\
 & 70000 \\
41^3 = & \underline{68921} \\
41^2 \times 300 = 504300 \;) & \underline{1079000} \;(\; 2 \\
 & 70000000 \\
412^3 = & \underline{69934528} \\
412^2 \times 300 = 50923200 \;) & \underline{65472000} \;(\; 1 \\
 & 70000000000 \\
4121^3 = & \underline{69985463561} \\
4121^2 \times 300 = 5094792300 \;) & \underline{14536439000} \;(\; 2 \\
 & 70000000000000 \\
41212^3 = & \underline{69995653640128} \\
41212^2 \times 300 = 509528683200 \;) & 4346359872000 \;(\; 8 \\
\text{\&c.} & \text{\&c.}
\end{array}
$$

The reason for pointing the number into periods of 3 figures each is manifest from the principles of common multiplication; for any number with one or more ciphers on the right hand, must have exactly 3 times as many ciphers in its cube.

118. But all the usual or common rules for extracting the cube and higher roots are extremely prolix. The following general method of approximation however, derived from the *rational formula* of Dr. Halley, (vol. 2, *art.* 111) is more expeditious, and easily remembered.

To extract the Root of any Power.

ASSUME the root (the nearer the true root the better), then raise this root to the power whose root is required, and call it the assumed power.

Thus the cube root of $\frac{1}{1000}$ is $\frac{1}{10}$.

And the cube root of $\frac{24}{81}$ is $\frac{2}{3}$; for $\frac{24}{81} = \frac{8}{27}$, whose root is $\frac{2}{3}$.

But when the terms of the fraction are not perfect cubes, let them be multiplied by the square of the denominator, then extract the root of the new fraction for the root required.

Thus, suppose the cube root of $\frac{3}{7}$ is required.

The fraction $\frac{3}{7}$ is $= \frac{3 \times 49}{7 \times 49} = \frac{147}{343}$ whose cube root is $\frac{5{\cdot}27763\ \&c.}{7} =$ ·75394 &c. the *root required.*

Or the fraction may be reduced to a decimal. And *mixed numbers* are prepared as in extracting the square root.

120. The *Biquadratic* or *4th* root is obtained by extracting the square root, and then extracting the square root of that root.

Thus the *4th root* of 6561 is 9. For the square root of 6561 is 81 whose square root is 9.

Let the *5th* root of 27 be required.

Assume 2 for the root; then its *5th* power is 32.

And the index 5 added to 1, and lessened by 1 give 6 and 4.

$$\begin{array}{lr} \text{Then } 32 \times 6 = & 192 \\ 27 \times 4 = & 108 \\ \text{Sum} & \overline{300} \end{array} \qquad \begin{array}{lr} 32 \times 4 = & 128 \\ 27 \times 6 = & 162 \\ \text{Sum} & \overline{290} \end{array}$$

As 300 : 290 :: 2 : 1·93, *root nearly*, the first approximation.

Now assume 1·93 for the root; then its *5th*. power, or the assumed power is 26·778 (retaining 3 places of decimals only).

$26{\cdot}778 \times 6 = 160{\cdot}668$

Assumed cube	216	Given number	184
	2		2
	432		368
Given number	184	Assumed cube	216
Sum	616	*Sum*	584

As 616 : 584 :: 6 : 5·7 *root nearly.*

Now taking 5·7 for the assumed root, its cube is 185·193 the assumed cube.

Assumed cube	185·193	Given number	184
	2		2
	370·386		368
Given number	184	Assumed cube	185·193
Sum	554·386	*Sum*	553·193

As 554·386 : 553·193 :: 5·7 : 5·687734 *root*, which is true in the last decimal.

2. Required the cube root of the decimal ·07?

Assume ·4 for the root, its cube being ·064

	·064		·07
	2		2
	·128		·14
	·07		·064
Sum	·198	*Sum*	·204

As ·198 : ·204 :: 4 : ·41 *root* nearly.

Now take ·068921 the cube of ·41 for the second assumed cube.

	·068921		·07
	2		2
	·137842		·14
	·07		·068921
Sum	·207842	*Sum*	·208921

As ·207842 : ·208921 :: ·41 : ·4121285 *root*, true to the last figure.

For $·4121285^3 = ·06999998 +$ (retaining 8 places of decimals only) which is less than ·00000002 short of the truth.

119. *To extract the cube root of a Vulgar Fraction.* Reduce it to its lowest terms: then the roots of the numerator and denominator will form the fractional root required.

Thus the cube root of $\frac{1}{3375}$ is $\frac{1}{15}$.

And the cube root of $\frac{16}{54}$ is $\frac{2}{3}$; for $\frac{16}{54} = \frac{8}{27}$, whose root is $\frac{2}{3}$.

But when the terms of the fraction are not perfect cubes, let them be multiplied by the square of the denominator, then extract the root of the new fraction for the root required.

Thus, suppose the cube root of $\frac{3}{7}$ is required.

The fraction $\frac{3}{7}$ is $= \frac{3 \times 49}{7 \times 49} = \frac{147}{343}$ whose cube root is $\frac{5{\cdot}27763 \text{ \&c.}}{7} =$ $\cdot 75394$ &c. the *root* required.

Or the fraction may be reduced to a decimal. And *mixt numbers* are prepared as in extracting the square root.

120. The *Biquadratic* or *4th* root is obtained by extracting the square root, and then extracting the square root of that root.

Thus the *4th* root of 6561 is 9. For the square root of 6561 is 81 whose square root is 9.

Let the *5th* root of 27 be required.

Assume 2 for the root; then its *5th* power is 32.

And the index 5 added to 1, and lessened by 1 give 6 and 4.

$$\begin{array}{rl} \text{Then } 32 \times 6 = & 192 \\ 27 \times 4 = & \underline{108} \\ \text{Sum} & \underline{300} \end{array} \qquad \begin{array}{rl} 32 \times 4 = & 128 \\ 27 \times 6 = & \underline{162} \\ \text{Sum} & \underline{290} \end{array}$$

As 300 : 290 :: 2 : 1·93, *root nearly*, the first approximation.

Now assume 1·93 for the root; then its *5th*. power, or the assumed power is 26·778 (retaining 3 places of decimals only).

$$\begin{array}{rl} 26{\cdot}778 \times 6 = & 160{\cdot}668 \\ 27 \times 4 = & \underline{108} \\ \text{Sum} & \underline{268{\cdot}668} \end{array} \qquad \begin{array}{rl} 26{\cdot}778 \times 4 = & 107{\cdot}112 \\ 27 \times 6 = & \underline{162} \\ \text{Sum} & \underline{269{\cdot}112} \end{array}$$

As 268·668 : 269·112 :: 1·93 : 1·933181 the *root* true to the last figure.

OF ARITHMETICAL PROPORTION AND PROGRESSION.

121. WHEN four numbers have a common difference they are said to be in continued arithmetical proportion. But if the difference of the first and second is equal to the difference of the third and fourth, but not to that between the second and third, it is called discontinued proportion.

> 2, 4, 6, 8, continued proportion.
> 2, 4, 7, 9, discontinued proportion.

122. A series or rank of the first kind form a progression:

$\left.\begin{matrix} 1, 2, 3, 4, 5, 6, \&c. \\ 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, \&c. \end{matrix}\right\}$ ascending series or progressions.

$\left.\begin{matrix} 32, 29, 26, 23, 20, 17, \&c. \\ 10\tfrac{1}{2}, 10\tfrac{1}{4}, 10, 9\tfrac{3}{4}, 9\tfrac{1}{2}, 9\tfrac{1}{4}, \&c. \end{matrix}\right\}$ descending progressions.

123. The first and last numbers or terms are called the extremes; and the others between them the means.

Thus 1 and 6 are the extremes; and, 2, 3, 4, 5, the means of the rank 1, 2, 3, 4, 5, 6.

124. It is evident from the nature of the progressions, that the double of any term is equal to the sum of the two adjacent terms, or to the sum of any two terms equidistant from it.

> Thus in the rank 1, 2, 3, 4, 5, 6, &c.
> twice $4 = 3 + 5 = 2 + 6$.

125. Hence if three numbers are in arithmetical proportion, twice the mean is equal to the sum of the two extremes.

> Thus, if the three numbers are $10\tfrac{1}{4}$, 10, $9\tfrac{3}{4}$,
> Then $10 \times 2 = 10\tfrac{1}{4} + 9\tfrac{3}{4}$.

126. And when 4 numbers are in arithmetical proportion, the sum of the two means is equal to that of the extremes.

Thus if 29, 32, 20, 17, are the 4 numbers,
Then 29 + 20 = 32 + 17.

127. Since the terms of an arithmetical progression are found by continually adding or subtracting the common difference; if the difference, twice the difference, three times the difference, &c. be added to the first term, the several sums will give an ascending series; or subtracted, a descending one.

Thus the terms of the progression 3, 5, 7, 9, 11, &c. having the common difference 2, will be

$$3,\ 3+2,\ 3+4,\ 3+6,\ 3+8,\ \&c.$$

And the terms of the series 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$, 4, &c. where the common difference is $\frac{1}{2}$,

$$\text{is } 6,\ 6-\tfrac{1}{2},\ 6-1,\ 6-1\tfrac{1}{2},\ 6-2,\ \&c.$$

128. Consequently when the first and last terms are given, if their difference be divided by the number of terms lessened by 1, the quotient will be the common difference of the terms.

For example, let the first term be 2, the last 20, and the number of terms 7:

Then 20 — 2 = 18 the difference, which divided by 6 (or 7 — 1) gives 3 for the common difference of the terms. And the progression will be

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20.$$

129. In this manner we can find any proposed number of arithmetical means between two given numbers; or interpose any number of terms between two given extremes.

For example, let 9 arithmetical means be found between 1 and 2.

Now the whole number of terms being 11, that number lessened by 1 is 10:

And 2 — 1 = 1 the difference of the extremes, which divided by 10 gives $\frac{1}{10}$ the common difference of the terms.

And the series will be

$$1,\ 1\tfrac{1}{10},\ 1\tfrac{2}{10},\ 1\tfrac{3}{10},\ 1\tfrac{4}{10},\ 1\tfrac{5}{10},\ 1\tfrac{6}{10},\ 1\tfrac{7}{10},\ 1\tfrac{8}{10},\ 1\tfrac{9}{10},\ 2.$$

130. Hence it appears that the difference of the extremes divided by the common difference of the terms, gives the number of terms less by 1.

For example, let the extremes be 2 and 20, and 3 the common difference.

Then $\frac{20-2}{3} = 6$; therefore $6 + 1 = 7$ the number of terms.

131. Therefore it is evident that the number of terms less by 1, multiplied by the common difference, is equal to the difference of the two extremes.

Thus if the number of terms be 7, and the common difference 3;

Then $7 - 1 = 6$ the number of terms less by 1;

And $6 \times 3 = 18$ the difference of the extremes; which added to the less extreme will give the greater; or subtracted from the greater will give the less.

132. The sum of all the terms, in a continued arithmetical series or progression, is equal to the sum of the two extremes, multiplied into half the number of terms.

Examples.

1. Required the sum of 2, 4, 6, 8, 10, 12?

2, 4, 6, 8, 10, 12 to these add the same series in an inverted order.
12, 10, 8, 6, 4, 2
14, 14, 14, 14, 14, 14. Now the sum of these numbers is evidently equal to twice the proposed series:

But their sum is 14×6 (or 84) or the sum of the first and last terms multiplied by the number of terms.

Therefore half that sum or the sum of the series is $14 \times 3 = 42$: viz. the sum of the two extremes into half the number of terms.

2. Suppose 1000 stones be placed on the ground in a direct line at the distance of a yard from each other; how far would a person travel in fetching them one at a time, to a basket placed a yard behind the first stone?

The distance for the first stone will be 2 yards, and that for the last 2000, which therefore, are the two extremes.

2002	sum of extremes.
500	half the number of terms.
1001000	yards, or $568\frac{3}{4}$ miles, the *answer*.

OF GEOMETRICAL PROPORTION AND PROGRESSION.

133. IN arithmetical proportion numbers are compared by means of their differences; but in geometrical proportion by the quotient arising from the division of one number by another. Thus, when the quotients are equal, the numbers which produce them are said to be in geometrical proportion. For example, the numbers 2, 4, 5, 10, are in geometrical proportion, because $\frac{2}{4} = \frac{5}{10}$; *see art.* 92, &c. What we have to add concerning proportion chiefly relates to the permutation, composition, &c. of the terms, and ratios.

134. In any number of proportionals taken two and two in order, the first, third, fifth, &c. terms are called antecedents; and the second, fourth, sixth, &c. their consequents.

Thus, if the terms are 2 : 4 :: 5 : 10 :: 9 : 18,
Then 2, 5, 9 are the antecedents; and 4, 10, 18 their consequents.

135. When 4 numbers are proportional, the terms admit of 8 variations or permutations.

Let the numbers be 3, 5, 9, 15.

Then $\frac{3}{5} = \frac{9}{15}$
$\frac{3}{9} = \frac{5}{15}$
$\frac{9}{3} = \frac{15}{5}$
$\frac{5}{3} = \frac{15}{9}$

Therefore (92.) 3 : 9 :: 5 : 15
9 : 3 :: 15 : 5
5 : 15 :: 3 : 9
15 : 5 :: 9 : 3
3 : 5 :: 9 : 15
5 : 3 :: 15 : 9
9 : 15 :: 3 : 5
15 : 9 :: 5 : 3

136. In a rank of proportionals standing in order, two and two.—As any antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let the proportionals be $3 : 5 :: 9 : 15 :: 36 : 60$.
Then $3 : 5$ (or $9 : 15$) $:: 3 + 9 + 36 : 5 + 15 + 60$
or $3 : 5 :: 48 : 80$.

For $3 : 3 :: 5 : 5$, hence $\frac{3}{3} = \frac{5}{5}$.
$3 : 9 :: 5 : 15$, hence $\frac{9}{3} = \frac{15}{5}$.
$3 : 36 :: 5 : 60$, hence $\frac{36}{3} = \frac{60}{5}$.
&c. &c.

Now the sums of the equal fractions must also be equal,

viz. $$\frac{3 + 9 + 36}{3} = \frac{5 + 15 + 60}{5};$$

Therefore (92) $3 : 5 :: 3 + 9 + 36 : 5 + 15 + 60$.
This is called *composition* of proportion.

137. If 4 numbers are proportional, then, as the difference of the first and second, is to the first (or second), so is the difference of the third and fourth, to the third (or fourth).

Suppose $3 : 5 :: 9 : 15$
Then $5 - 3 : 3 :: 15 - 9 : 9$
And $5 - 3 : 3 :: 15 - 9 : 15$.

For $\frac{5}{3} = \frac{15}{9}$; and if we take $\frac{3}{3}$ (or 1) from $\frac{5}{3}$ the remainder is $\frac{5-3}{3}$.

And $\frac{9}{9}$ (or 1) taken from $\frac{15}{9}$ leaves $\frac{15-9}{9}$.

And since equal numbers subtracted from equal numbers must give equal remainders, the fractions $\frac{5-3}{3}$, $\frac{15-9}{9}$ must be equal.

Therefore (92) $5 - 3 : 3 :: 15 - 9 : 9$.
This is called *division* of proportion.

138. Since $3 : 5 :: 9 : 15$, and (by composition) $5 + 3 : 3 :: 15 + 9 : 9$; therefore $5 + 3$ and $15 + 9$ have the same proportion as $5 - 3$ and $15 - 9$ (137). Hence when 4 numbers are proportional, As the sum of the first and second is to their difference, so is the sum of the third and fourth, to their difference.

$$3+9:5-3::15+9:15-9$$
$$\text{or } 8 : 2 :: 24 : 6.$$

139. If several numbers are proportionals, their squares, cubes, &c. are proportionals.

For example, suppose $3 : 5 :: 9 : 15$

Then $\frac{3}{5} = \frac{9}{15}$; now those fractions being equal, their like powers must be equal,

$$viz.\ \frac{3^2}{5^2} = \frac{9^2}{15^2}$$
$$\text{and } \frac{3^3}{5^3} = \frac{9^3}{15^3}, \text{ \&c.}$$

Therefore (92) $3^2 : 5^2 :: 9^2 : 15^2$
or $9 : 25 :: 81 : 225$

And $3^3 : 5^3 :: 9^3 : 15^3$
or $27 : 125 :: 729 : 3375$, &c.

Hence the square, cube, &c. roots of proportional numbers, are also proportional.

140. If there are several ranks of proportionals standing in order two and two, the products of the corresponding terms will be proportional.

For example, let $\left.\begin{array}{l} 3 : 5 :: 9 : 15 \\ 12 : 6 :: 8 : 4 \end{array}\right\}$ be two ranks.

Then $3 \times 12 : 5 \times 6 :: 9 \times 8 : 15 \times 4$
or $36 : 30 :: 72 : 60$.

For $\frac{3}{5} = \frac{9}{15}$; and $\frac{12}{6} = \frac{8}{4}$. And since equal numbers multiplied by equal numbers must give equal products, $\frac{3}{5} \times \frac{12}{6}$ must be equal to $\frac{9}{15} \times \frac{8}{4}$, or $\frac{3 \times 12}{5 \times 6} = \frac{9 \times 8}{15 \times 4}$; therefore (92) $3 \times 12 : 5 \times 6 :: 9 \times 8 : 15 \times 4$; and so of any other number of ranks.

141. Hence the ratio of the products is compounded of the ratios of the terms:

For $\frac{3}{5}$ denotes the ratio of 3 to 5; and $\frac{12}{6}$ that of 12 to 6;

[redacted]

And the product $\frac{3 \times 12}{5 \times 6}$ denotes the ratio of 3×12 to 5×6; and so of the other terms.

Therefore ratios are compounded by multiplying together the fractions denoting those ratios.

PROGRESSION.

142. THE terms of a geometrical progression result from successive multiplications, or divisions, by some number which is called the common ratio of the terms.

Thus, if 1 be the first term, and 2 the ratio;
Then 1, 2, 4, 8, 16, 32, &c. is an ascending progression.
And 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c. a descending progression.

143. The first and last terms are called the extremes; and the intermediate ones the geometrical means.

144. In any continued geometrical series, the product of the two extremes is equal to that of any two means equally distant from them.

Thus, if the series be 2, 4, 8, 16, 32, 64;
Then $2 \times 64 = 4 \times 32 = 8 \times 16$.

For the ratio of every two adjacent terms being the same, we have $2 : 4 :: 32 : 64$

Therefore $2 \times 64 = 4 \times 32$.

The terms $2 : 4 :: 32 : 64$ are said to be in discontinued proportion, because the ratio of the first and second terms (2, 4,) and that of the second and third (4, 32,) are unequal.

145. In any continued geometrical series, the ratio of the first term to the last is compounded of the ratios of all the antecedents to their consequents.

Thus, in the progression 1, 2, 4, 8, 16, 32, the fractions denoting the ratios are $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, $\frac{8}{16}$, $\frac{16}{32}$:

And (141) the compounded ratio is $\frac{1 \times 2 \times 4 \times 8 \times 16}{2 \times 4 \times 8 \times 16 \times 32}$, which fraction in its lowest terms is $\frac{1}{32}$, denoting the ratio of 1 to 32.

146. All the terms of a geometrical progression may be expressed by means of the common ratio and one of the extremes.

Thus, the series 3, 6, 12, 24, 48, &c. where the common ratio is 2, and first term 3, will be

$3,\ 3 \times 2,\ 3 \times 2 \times 2,\ 3 \times 2 \times 2 \times 2,\ 3 \times 2 \times 2 \times 2 \times 2$, &c.

or $3,\ 3 \times 2,\ 3 \times 2^2,\ 3 \times 2^3,\ 3 \times 2^4$, &c. (§11)

147. Therefore in any ascending progression, if the first term be multiplied by the ratio raised to the power whose index is the number of terms less by 1, the product will be the last term.

For example, suppose the first term is $\frac{1}{4}$, the common ratio 2, and the number of terms 10; what is the last term?

The number of terms less by 1 is 9:

And $2^9 = 512$, which multiplied by $\frac{1}{4}$ (the first term) gives 128 the last term.

148. But in a descending progression (where the terms result from division) the first term divided by the said power of the ratio gives the last term.

Thus, suppose the first term is 128, the common ratio 2, and the number of terms 10; what is the last term?

$2^9 = 512$; and 128 divided by 512 gives $\frac{128}{512}$ or (in its lowest terms) $\frac{1}{4}$ the last term.

149. Hence, if one extreme be divided by the other, the quotient will be that power of the ratio whose index is the number of terms less by 1; and consequently its root will be the ratio.

For example, if 7 be the first term, 189 the last, and 4 the number of terms; what is the ratio?

$\frac{189}{7} = 27$ the 3*d.* power of the ratio (the number of terms being 4), whose cube root is 3 the ratio required.

Therefore the 4 terms are $7,\ 7 \times 3,\ 7 \times 3^2,\ 189.$
or $7,\quad 21,\quad 63,\quad 189.$

150. In like manner we find a proposed number of geometrical mean proportionals between two given numbers.

For example, let it be required to find 3 geometrical means between 6 and 1536.

$\frac{1536}{6} = 256$ the 4*th.* power of the ratio (the number of terms being 5).

The square root of 256 is 16 whose square root is 4, the 4*th* root of 256 or the required ratio.

And the three means will be $6 \times 4,\ 6 \times 4^2,\ 6 \times 4^3$;
or $24,\quad 96,\quad 384$;

And the series 6, 24, 96, 384, 1536.

151. When only one mean proportional between two given numbers is required, the square root of their product will be the answer.

For example, to find a mean proportional between 8 and 18.

$8 \times 18 = 144$ whose square root is 12 the *answer.*

For $8 : 12 :: 12 : 18.$

And 18 is called a third proportional to 8 and 12.

152. To find the sum of all the terms in a given progression; suppose 2, 6, 18, 54, 162; where the common ratio is 3.

2, 6, 18, 54, 162	
3	
6, 18, 54, 162, 486	the series multiplied by the ratio 3.
2, 6, 18, 54, 162, ...	the series itself, subtract:
. . . . 486	lessened by 2 is the remainder.

This remainder is equal to twice the sum of the series, because it is the difference between the series and three times the series.

Therefore if 486 less by 2, be divided by 2 (*viz.* the ratio less by 1) the quotient will be the sum of the series.

But 486 less by 2 is the difference between the first term, and the product of the last by the ratio: hence the following

Rule. Multiply the last term by the ratio, and take the first term from the product, then divide the difference by the ratio lessened by 1, and the quotient is the sum of the progression.

In a descending progression take the first term for the last, and *vice versâ.*

Ex. 2. Required the sum of the series 65536, 16384, 4096, &c. continued to 12 terms?

The ratio or divisor is 4; and $4^{11} = 4194304$;

And 65536 divided by 4194304 gives $\frac{65536}{4194304}$ or (in its lowest terms) $\frac{1}{64}$ the 12*th.* or last term of the series, which being made the first term, and 65536 the last, the work will stand as below.

$$
\begin{array}{rrl}
 & 65536 & \\
 & 4 & \text{ratio.} \\
\hline
 & 262144 & \\
 & \frac{1}{64} & \text{subtract.} \\
\hline
4 \text{ the ratio less by } 1 = 3) & 262143\frac{63}{64} & \\
\hline
 & 87381\frac{21}{64} & \textit{sum of the series.}
\end{array}
$$

3. An officer with a detachment of 60 men having taken a very strong fort by surprize, desired as a reward for himself and the party, 1 musket bullet for the first man, 2 for the second, 4 for the third, 8 for the fourth, and so on, doubling to 60 times (the number of men) Now suppose each bullet to be an ounce, and the lead at 5 shillings the hundred weight; what would be the value of his request?

Here the first term is 1, the ratio 2, and the number of terms 60; therefore 2^{59}, or 2 raised to the 59*th.* power will be the last term of the series.

The 6*th.* power of 2 is 64, which cubed is 262144 the 18*th.* power (111,) and that cubed gives 18014398509481984 the 54*th.* power, which

$\frac{189}{7} = 27$ the 3d. power of the ratio (the number of terms being 4), whose cube root is 3 the ratio required.

Therefore the 4 terms are $7,\ 7 \times 3,\ 7 \times 3^2,\ 189.$
or $7,\ 21,\ 63,\ 189.$

150. In like manner we find a proposed number of geometrical mean proportionals between two given numbers.

For example, let it be required to find 3 geometrical means between 6 and 1536.

$\frac{1536}{6} = 256$ the 4*th*. power of the ratio (the number of terms being 5).

The square root of 256 is 16 whose square root is 4, the 4*th* root of 256 or the required ratio.

And the three means will be $6 \times 4,\ 6 \times 4^2,\ 6 \times 4^3;$
or $24,\ 96,\ 384;$

And the series 6, 24, 96, 384, 1536.

151. When only one mean proportional between two given numbers is required, the square root of their product will be the answer.

For example, to find a mean proportional between 8 and 18.

$8 \times 18 = 144$ whose square root is 12 the *answer*.

For $8 : 12 :: 12 : 18.$

And 18 is called a third proportional to 8 and 12.

152. To find the sum of all the terms in a given progression; suppose 2, 6, 18, 54, 162; where the common ratio is 3.

$$\begin{array}{r}
2,\ 6,\ 18,\ 54,\ 162 \\
3 \\
\hline
6,\ 18,\ 54,\ 162,\ 486 \\
2,\ 6,\ 18,\ 54,\ 162 \ldots\ldots \\
\hline
486
\end{array}$$

the series multiplied by the ratio 3:
the series itself, subtract:
lessened by 2 is the remainder.

... series, because it is the

ADDITIONAL EXAMPLES

IN THE

FOREGOING RULES OF ARITHMETIC.

Vulgar Fractions.

153. Required the greatest common measure

of 1728, 1458.................*Ans.* 54.
1400, 35000...........................700
1353, 1419, 187.. .... 11.
2678, 4056, 6708, 7917........ 13.
2057, 121.
219, 9101.
10307, 8433, 937.
5600, 6705, 1033.

Reduce to the lowest terms

[illegible], [illegible], [illegible]..........*Ans.* [illegible], [illegible], [illegible].
[illegible], [illegible], [illegible]..............................[illegible], [illegible].
[illegible], [illegible], [illegible].

Reduce to equivalent whole or mixt numbers

[illegible], [illegible], [illegible]...............*Ans.* 14, 2[illegible], 8.
[illegible], [illegible], [illegible]..........11[illegible], 115[illegible], 2940[illegible].

Reduce to improper fractions

11[illegible], 12[illegible], 1[illegible], ...*Ans.* [illegible], [illegible], [illegible].
510[illegible], 1000[illegible], 10[illegible]........[illegible], [illegible], [illegible].

Reduce to simple fractions

[illegible] of [illegible] of [illegible] of [illegible]...................... ..*Ans.* [illegible].
[illegible] of [illegible] of 4[illegible] .... [illegible].
[illegible] of [illegible] of [illegible] of 7... ....[illegible].
[illegible] of 11[illegible] of 13[illegible].................................[illegible].
[illegible] of [illegible] of 10 of [illegible] ....[illegible].
[illegible] of [illegible] of [illegible].
[illegible] of 1[illegible] of [illegible] of [illegible] of 1[illegible].

multiplied by 32 (the 5*th.* power of 2) is 576460752303423488 the 59*th.* power or last term of the series; this multiplied by 2 the ratio, and 1 (the first term) subtracted from the product, gives 1152921504606846975 *the sum of the series*, or *number of bullets*, or *ounces* (because the ratio lessened by 1 is 1), equal to 643371375338642$\frac{511}{1792}$ hundred weight, which at 5 shillings the hundred, amounts to £160842843834660$\frac{4095}{7168}$ the *answer*.

ADDITIONAL EXAMPLES

IN THE

FOREGOING RULES OF ARITHMETIC.

Vulgar Fractions.

153. Required the greatest common measure

of 1728, 1458.................*Ans.* 54.
1400, 35000.......................700
1353, 1419, 187.. .... 11.
2678, 4056, 6708, 7917........ 13.
2057, 121.
219, 9101.
10307, 8433, 937.
5600, 6705, 1033.

Reduce to the lowest terms

[illegible], [illegible], [illegible].........*Ans.* [illegible], [illegible], $\frac{11}{13}$.
[illegible], [illegible], [illegible]..........................[illegible], [illegible].
[illegible], [illegible], [illegible].

Reduce to equivalent whole or mixt numbers

[illegible], [illegible], $\frac{56}{7}$,..............*Ans.* $1\frac{4}{7}$, $2\frac{3}{10}$, 8.
$\frac{1000}{84}$, [illegible], [illegible]..........$11\frac{19}{21}$, $115\frac{6}{7}$, $2940\frac{2}{7}$.

Reduce to improper fractions

$11\frac{1}{5}$, $12\frac{7}{10}$, $1\frac{1}{21}$, ...*Ans.* $\frac{56}{5}$, $\frac{127}{10}$, $\frac{22}{21}$.
$510\frac{6}{7}$, $1000\frac{1}{10}$, $10\frac{10}{999}$........$\frac{3576}{7}$, $\frac{10001}{10}$, $\frac{10000}{999}$.

Reduce to simple fractions

$\frac{3}{4}$ *of* $\frac{4}{5}$ *of* $\frac{5}{6}$ *of* $\frac{6}{7}$....................... ..*Ans.* $\frac{3}{7}$.
$\frac{1}{6}$ *of* $\frac{6}{7}$ *of* $4\frac{1}{2}$ ....[illegible].
$\frac{4}{7}$ *of* $\frac{3}{5}$ *of* $\frac{1}{4}$ *of* 7... ....$\frac{3}{5}$.
$\frac{3}{11}$ *of* $11\frac{1}{2}$ *of* $13\frac{1}{4}$...........................[illegible].
$\frac{6}{7}$ *of* $\frac{1}{2}$ *of* 10 *of* $\frac{7}{1}$.......................[illegible].
[illegible] *of* [illegible] *of* [illegible].
[illegible] *of* [illegible] *of* [illegible] *of* [illegible] *of* [illegible].

Required the least common multiple of the nine digits, or the least whole number that is divisible by 1, 2, 3, 4, 5, 6, 7, 8, and 9, without leaving a remainder? *Ans.* 2520.

Of 10, 35, 15, and 12.......................*Ans.* 1260.
Of 50, 120, 76, and 39.
Of 162, 27, 729, 486.

Required the least common multiple of $10\frac{1}{2}$, $13\frac{1}{2}$, and $26\frac{1}{4}$? *Ans.* $2782\frac{1}{2}$

Reduce to the least common denominators

$\frac{2}{3}$, $\frac{5}{6}$, $\frac{7}{8}$..............................*Ans.* [illegible]
[illegible]....................................[illegible]
[illegible]....................................[illegible]
$2\frac{1}{3}$, $\frac{5}{6}$, 6....................................$\frac{42}{18}$, $\frac{15}{18}$, $\frac{108}{18}$.
[illegible] *and* [illegible] *of* $5\frac{1}{3}$....................................[illegible]
$\frac{4}{5}$ *and* 12....................................$\frac{4}{5}$, $\frac{60}{5}$.
[illegible] *and* [illegible].
[illegible]
[illegible]

Addition.

Required the sums of

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$....................................*Ans.* $1\frac{11}{12}$.
[illegible]....................................$\frac{4}{5}$.
[illegible]....................................[illegible]
[illegible]....................................1.
$\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$....................................2.
$\frac{1}{3}$, $\frac{1}{2}$....................................$\frac{5}{6}$.
$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$....................................$1\frac{17}{60}$.
[illegible]....................................2.
$\frac{1}{6}$ *and* $\frac{2}{3}$ *of* $18\frac{1}{4}$....................................[illegible]
[illegible] *of* 10, [illegible] *of* 13, *and* [illegible] *of* 11....................................[illegible]
$74\frac{4}{5}$, [illegible]....................................$349\frac{3}{5}$.
[illegible]....................................[illegible]
[illegible] *and* [illegible] *of* [illegible]....................................[illegible]
[illegible]....................................[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]

Subtraction.

[illegible]quired the differences

of $\frac{1}{2}$, $\frac{1}{3}$ .... *Ans.* $\frac{1}{6}$.
$\frac{1}{3}$, $\frac{1}{4}$ .... $\frac{1}{12}$.
$\frac{11}{12}$, $\frac{7}{12}$ .... $\frac{1}{3}$.
19, $9\frac{1}{11}$ .... $9\frac{10}{11}$.
$19\frac{2}{11}$, $9\frac{1}{11}$ .... $10\frac{1}{11}$.
$19\frac{1}{11}$, $9\frac{2}{11}$ .... $9\frac{10}{11}$.
$\frac{1}{2}$, $\frac{1}{3}$ .... $\frac{1}{6}$.
$\frac{1}{3}$, $\frac{1}{5}$ .... $\frac{1}{6}$.
$\frac{4}{5}$, $\frac{1}{2}$ .... $\frac{3}{10}$.
$\frac{1}{3}$, $\frac{1}{7}$ .... $\frac{1}{7}$.
$\frac{191}{1142}$, $\frac{7}{11}$.
$1000\frac{1}{3}$, $100\frac{4}{5}$ .... $899\frac{3}{5}$.
$10\frac{5}{8}$, *and* $\frac{1}{2}$ *of* $10\frac{5}{8}$ .... $1\frac{3}{8}$.
$\frac{2}{3}$ *of* 8, *and* $\frac{5}{8}$ *of* 7 .... $\frac{5}{8}$.
$\frac{1}{3}$ *of* $\frac{11}{12}$, *and* $\frac{1}{3}$ *of* $\frac{19}{12}$ .... $\frac{2}{11}$.
10, *and* $\frac{19}{20}$ *of* 10 .... $\frac{19}{20}$.
10000 *and* $999\frac{1}{999}$.
$\frac{1999}{2000}$ *and* 1.

Multiplication.

Required the products

of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ .... *Ans.* $\frac{1}{4}$.
$\frac{5}{6}$, $\frac{3}{10}$ .... $\frac{1}{4}$.
$2\frac{1}{7}$, $\frac{7}{15}$ .... 1.
$\frac{2}{3}$ *of* $\frac{1}{4}$, *and* $\frac{3}{5}$ *of* $\frac{5}{6}$ .... $\frac{1}{6}$.
$3\frac{1}{4}$, $3\frac{1}{7}$, $1\frac{1}{4}$ .... $10\frac{5}{8}$.
20, $10\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{2}$ .... 70.
$\frac{1}{156}$, 8 .... $\frac{1}{6}$.
22, $\frac{3}{11}$ .... 6.
22, $\frac{3}{10}$ .... $6\frac{3}{5}$.
$2\frac{1}{2}$, $4\frac{1}{2}$, $\frac{1}{3}$, $\frac{4}{5}$, 10, *and* $\frac{5}{8}$ *of* $\frac{4}{5}$ .... 25.
$\frac{3}{4}$ *of* 20, *and* $\frac{1}{2}$ *of* 30 .... 150.
$\frac{2}{3}$ *and* $74131\frac{1}{2}$ .... 49421.
$\frac{1}{6}$, $\frac{1}{4}$, 56427 .... $9404\frac{1}{2}$.
$646124\frac{1}{2}$, $64\frac{1}{2}$ .... $41675030\frac{1}{4}$.
84672, $1000\frac{1}{2}$ .... 84681408.
$8320\frac{1}{112}$, $\frac{11}{8}$ .... $10\frac{1}{2}$.

64180 [illegible], 101 [illegible] .... *Ans.* 6492217 [illegible]
[illegible], [illegible], [illegible], [illegible], *and* 1 [illegible]
[illegible], 5 [illegible], and 1000.
5734 [illegible], 100, 14921.
1 [illegible], 1 [illegible], [illegible], *and* [illegible]
[illegible], [illegible], [illegible], [illegible], [illegible], *and* 5040.

Division.

Divisors.	*Dividends.*	*Quotients.*
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	1 [illegible]
[illegible]	[illegible]	[illegible]
[illegible]	1	[illegible]
[illegible]	[illegible]	1.
[illegible]	[illegible]	[illegible]
2 [illegible]	1	[illegible]
[illegible]	[illegible]	[illegible]
3 [illegible]	12 [illegible]	3 [illegible]
[illegible]	10	14.
10	[illegible]	[illegible]
37065 [illegible]	24710 [illegible]	[illegible]
[illegible]	9404 [illegible]	56127.
421 [illegible]	24012	57.
[illegible] of 30	300	13 [illegible]
[illegible] of [illegible]	[illegible] of [illegible]	1 [illegible]
[illegible] of 9	[illegible] of 9	[illegible]
[illegible]	10000 [illegible]	15001.
7	4164217 [illegible]	594888 [illegible]

Divide the difference of 3 [illegible] and [illegible] by the sum ... 1 [illegible]

1263 [illegible] by 24012.
[illegible] by [illegible] of [illegible]
4 [illegible] by [illegible] of [illegible] of [illegible]
4879107 [illegible] by 9.
10000 [illegible] by [illegible] of [illegible]
31930 [illegible] by 1101 [illegible]

$\frac{1}{100}$ .... ·01.
$\frac{3}{400}$ .... ·0075.
$\frac{2197}{4096}$ .... ·536376953125.
$\frac{69}{8}$ .... 8·625.
[illegible]

Duodecimals.

	f.	*in.*		*f.*	*in.*		
Multiply	7	6	by	4	8	....	*Ans.* 35 *square feet.*
	9	10	by	8	11	....	87 *sq. f.* 98 *sq. in.*
	29	8½	by	17	4¼	....	511 *f.* 2*in.* 3½*l.*
							or 511 *sq. f.* 27½ *sq. in.*
							or 511$\frac{11}{72}$ *sq. feet.*

If the length and breadth of a board be 7 *f.* 8 *in.*, and 13¼ *in.*; what is the content in square feet?

Ans. 8½.

The parts of the section of a field-work being

f.	*in.*		*f.*	*in.*
13	4	×	2	6
3	3	×	3	10
16	8	×	10	4
4	5	×	6	8
3	7	×	2	9

Required the whole number of square feet?

Ans. 252$\frac{1}{48}$.

Reduction.

Reduce	£929 to pence.	*Ans.* 221230.
	1000000 farthings to pounds, &c.	£1041 13 4.
	£19 19 10½ to farthings.	*Ans.* 19195.
	£2$\frac{1}{2}$ to farthings.	*Ans.* 2310.
	805148 pence to pounds, &c.	£3658 1 11.
	26½ guineas to pence.	*Ans.* 6678.
	£$\frac{1}{6}$ to pence.	*Ans.* 42$\frac{6}{7}$.
	£$\frac{1}{960}$ to the denomination (or fraction) of a penny?	*Ans.* ¼.
	What is $\frac{11}{14}$ of a pound?	*Ans.* 15*s.* 3*d.* 2$\frac{2}{7}$*qrs.*
	½*d.* to the fraction of a pound.	*Ans.* $\frac{1}{480}$.
	3¼*d.* to the fraction of a pound.	*Ans.* $\frac{7}{480}$.

Division.

Divisors.	*Dividends.*	*Quotients.*
·04	·00448	·112.
4·01	1·9248	·48.
·0089	·1782	21.
8·8	38·72	4·4.
2400	100·8	·042.
5·426	54260	10000.
2500	·0412	·00001648.
10000	7410·01	·741001.
100	·62	·0062.
·125	100	800.
700	2·25	·003214285 &c.
3510	23·4	·00666 &c.
29100	46214·72	1·58813 &c.
1000	97400	
64	6111	
4200	56126	
288	·3456	
·288	3456	
·00288	345600	
2880	·003456	
·0288	·3456	
·125	10000	
10000	·125	

Divide the sum of ·375 and ·0625 by their difference.

Divide 1400 by ·001953125.

Reduce to decimals

the fractions $\frac{1}{3}$ .... *Ans.* ·3333 &c.

$\frac{2}{3}$ .... ·6666 &c.

$\frac{7}{8}$ .... ·875.

$\frac{13}{16}$ .... ·8125.

$\frac{1}{32}$ .... ·03125.

$\frac{1}{81}$ .... ·012357 &c.

$\frac{14}{15}$ .... ·9333 &c.

$\frac{19}{20}$ .... ·95.

$\frac{10}{99}$ .... ·1010101 &c.

$\frac{29}{30}$ .... ·9666 &c.

Reduce $\frac{1}{17}$*lb.* to grains . . . *Ans.* $338\frac{14}{17}$.
3·175*lb.* to pennyweights. . . . *Ans.* 762.
·55*lb.* to ounces, &c. . . . *Ans.* 6*oz.* 12*dwts.*
15·125*dwts.* to the decimal of a *lb.* . . . *Ans.* ·063020833 &c.

The full weight of a half-crown is 9*dwts.* $16\frac{1}{2}$*gr.* then how many are a *lb. troy?*

Ans. $24\frac{144}{186}$.

Reduce $\frac{2}{7}$*lb.* (apoth. weight) to ounces, &c. . . . *Ans.* 3*oz.* 3*dr.* $1\frac{2}{7}$*sc.*
1 ton to drams, avoirdupoise weight. . . . *Ans.* 573440.
65771*oz.* to tons, &c. . . . *Ans.* 1*t.* 16*cwt.* 78*lb.* 11*oz.*
184 drams to the fraction of a *lb.* . . . *Ans.* $\frac{23}{32}$.
10*lb.* 8*oz.* to the fraction of a *cwt.* . . . *Ans.* $\frac{3}{32}$.
·85*cwt.* to *lbs.* &c. . . . *Ans.* 95*lb.* 3·2*oz.*
5*lb.* 4*oz.* to the decimal of a *cwt.* . . . *Ans.* ·046875.

A cubic foot of cast iron being 464*lb.* *avoirdupoise*, then how many cubic feet are contained in a 32 pounder whose weight is 54*cwt?*

Ans. $13\frac{1}{29}$.

Suppose 20000 foot soldiers, each man having 20 rounds of cartridge with ball; now if the balls are an ounce each, and the weight of powder $\frac{1}{4}$ of the ball; what is the whole weight of lead, and of powder?

	t.	*c.*	*lb.*	
Ans.	11	3	24	lead.
	2	15	90	powder.

How many *ounce*, 3 *ounce*, $\frac{1}{2}$*lb.* and *lb.* balls, and of each an equal number, can be cast from a ton of lead?

Ans. 1280 of each.

Reduce $7\frac{1}{3}$ miles to yards, &c. . . . *Ans.* 12906*yds.* 2*f.*
56142 feet to miles, &c. . . . *Ans.* 10*m.* 1114*yds.*
10000 inches to yards. . . . *Ans.* $277\frac{7}{9}$.
7 inches to the denomination, or fraction of a yard.
Ans. $\frac{7}{36}$.
$\frac{4}{5}$ of a yard to feet, &c. . . . *Ans.* 2*f.* $4\frac{4}{5}$*in.*
$\frac{4}{5}$ of an inch to the fraction of a foot. . . . *Ans.* $\frac{1}{15}$.
$5\frac{1}{3}$ inches to the fraction of a foot. . . . *Ans.* $\frac{4}{9}$.
$2\frac{2}{5}$ feet to the fraction of a yard. . . . *Ans.* $\frac{4}{5}$.
$\frac{1}{7}$ of a mile to the fraction of yards. . . . *Ans.* $\frac{1760}{7}$.
100 yards to the fraction of a mile. . . . *Ans.* $\frac{5}{88}$.
$7\frac{3}{5}$ feet to inches. . . . *Ans.* $91\frac{1}{5}$.

1128$\frac{1}{4}$ feet to yards. Ans. 376[illegible]

What is the value of $\frac{11}{12}$ of a mile?

of $\frac{3}{4}$ of a fathom?

of $\frac{11}{12}$ of a foot?

Reduce 7$\frac{1}{2}$ fathoms to the fraction of a mile.

7$\frac{1}{2}$ feet to the fraction of a pole.

7$\frac{1}{2}$ poles or perches to feet.

7$\frac{1}{2}$ inches to the fraction of a fathom.

·64 of a mile to yards, &c. Ans. 1126yds. 1f. 2·4in.

·125 of a foot to inches. Ans. 1$\frac{1}{2}$.

10·56 miles to feet. Ans. 55756·8.

429·85 fathoms to feet. Ans. 2579·1.

·855 of a foot to the decimal of a yard. Ans. ·285.

2·84 feet to the decimal of a yard. Ans. ·9466 &c.

·0095 of a foot to the decimal of an inch. Ans. ·114.

10$\frac{3}{4}$ inches to the decimal of a foot.

Ans. ·895833 &c.

2f. 3$\frac{1}{2}$in. to the decimal of a yard. Ans. ·76388 &c.

·074418 of a fathom to the decimal of a foot.

·01356 of an inch to the decimal of a foot.

·074418 of a foot to the decimal of a fathom.

·015 of a mile to poles.

·5076 of an inch to the decimal of a yard.

What is the value of ·0625 of a mile?

·7862 of a pole?

·445 of a fathom?

·124 of a yard?

·83 of a foot?

Reduce 1000 *toises* to fathoms. Ans. 1065·75.

1000 fathoms to *toises*. Ans. 938·3 &c.

4$\frac{1}{2}$ English miles to *toises*. Ans. 3715·69 &c.

9000 Rhynland feet to yards. Ans. 3099.

10 German miles (15 to a degree) to English miles.

Ans. 46$\frac{1}{2}$, nearly.

The circumference of the earth being 360 degrees, and each degree 69$\frac{1}{2}$ miles; what is the number of yards?

Ans. 43824000.

If the average step of a horse is 2$\frac{3}{4}$ feet; then how many in a mile?

Ans. 1920.

Reduce $\frac{1}{17}$*lb.* to grains. *Ans.* $338\frac{14}{17}$.
$3\cdot175$*lb.* to pennyweights. *Ans* 762.
$\cdot55$*lb.* to ounces, &c. *Ans.* 6*oz.* 12*dwts.*
$15\cdot125$*dwts.* to the decimal of a *lb.* *Ans.* $\cdot063020833$ &c.

The full weight of a half-crown is 9*dwts.* $16\frac{1}{2}$*gr.* then how many are a *lb. troy?*

Ans. $24\frac{24}{31}$.

Reduce $\frac{2}{7}$*lb.* (apoth. weight) to ounces, &c. *Ans.* 3*oz.* 3*dr.* $1\frac{2}{7}$*sc.*
1 ton to drams, avoirdupoise weight. *Ans.* 573440.
65771*oz.* to tons, &c. *Ans.* 1*t.* 16*cwt.* 78*lb.* 11*oz.*
124 drams to the fraction of a *lb.* *Ans.* $\frac{31}{64}$.
10*lb.* 8*oz.* to the fraction of a *cwt.* *Ans.* $\frac{3}{32}$.
$\cdot85$*cwt.* to *lbs.* &c. *Ans.* 95*lb.* $3\cdot2$*oz.*
5*lb.* 4*oz.* to the decimal of a *cwt.* *Ans.* $\cdot046875$.

A cubic foot of cast iron being 464*lb.* *avoirdupoise*, then how many cubic feet are contained in a 32 pounder whose weight is 54*cwt?*

Ans. $13\frac{1}{29}$.

Suppose 20000 foot soldiers, each man having 20 rounds of cartridge with ball; now if the balls are an ounce each, and the weight of powder $\frac{1}{4}$ of the ball; what is the whole weight of lead, and of powder?

	t.	*c.*	*lb.*	
Ans.	11	3	24	lead.
	2	15	90	powder.

How many ounce, 3 ounce, $\frac{1}{2}$*lb.* and *lb.* balls, and of each an equal number, can be cast from a ton of lead?

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Reduce $7\frac{1}{3}$ miles to yards, &c. *Ans.* 12906*yds.* 2*f.*
56142 feet to miles, &c. *Ans.* 10*m.* 1114*yds.*
10000 inches to yards. *Ans.* $277\frac{7}{9}$.
7 inches to the denomination, or fraction of a yard.
Ans. $\frac{7}{36}$.
$\frac{4}{5}$ of a yard to feet, &c. *Ans.* 2*f.* $4\frac{4}{5}$*in.*
$\frac{2}{3}$ of an inch to the fraction of a foot. *Ans.* $\frac{1}{18}$.
$5\frac{1}{3}$ inches to the fraction of a foot. *Ans.* $\frac{4}{9}$.
$2\frac{2}{5}$ feet to the fraction of a yard. *Ans.* $\frac{4}{5}$.
$\frac{1}{17}$ of a mile to the fraction of yards. *Ans.* $\frac{1760}{17}$.
100 yards to the fraction of a mile. *Ans.* $\frac{5}{88}$.
$7\frac{5}{8}$ feet to inches. *Ans.* $91\frac{1}{2}$.

Reduce $7\frac{3}{4}$ hogsheads (*beer meas.*) to pints. *Ans.* 3348.
64237 gallons to barrels. *Ans.* $1784\frac{13}{36}$.

How many hogsheads of beer will serve a garrison of 1350 men for 78 weeks, allowing each man $1\frac{1}{2}$ pints *per* day?

Ans. 2559*h.* $20\frac{1}{4}$*gall.*

Reduce $2\frac{7}{10}$ hours to seconds. *Ans.* 9720.
$\frac{1}{15}$ of a minute to the fraction of an hour. *Ans.* $\frac{1}{900}$.
365*d.* 5*h.* 48*m.* 48*sec.* (the solar year) to seconds.
Ans. 31556928.

7·96 degrees of a circle to minutes. *Ans.* 477·6.
25 seconds to the decimal of a degree. *Ans.* ·006944 &c.
What is the value of ·0825 of a degree?
of ·625 of a minute of a degree?
of ·44 of an hour?

N. B. 60 *seconds* make a *minute*, and 60 *min.* a *degree*.

Compound Addition.

1. Suppose a debt is discharged in 6 weeks, after the following manner, namely, 3*l.* 17*s.* $7\frac{1}{2}$*d.* the *first* week, twice that sum the *second*, three times that sum the *third*, four times that sum the *fourth*, five times that sum the *fifth*, and six times that sum the *sixth*; what was the debt?

Ans. £81 10 $6\frac{1}{2}$.

2. What is the whole amount
of 41 guineas,
37 half guineas,
£21,
19 crowns,
33 half-crowns,
101 dollars, at 4*s.* $2\frac{1}{2}$*d.* each,
147 gold mohurs, at 1*l.* 13*s.* $2\frac{3}{4}$*d.* each,
191 sicca rupees, at 2*s.* $2\frac{1}{4}$*d.* each?

Ans. 381*l.* 0*s.* 1*d.* $2\frac{7}{102}$*qrs.*

3. What is the sum of 10*l.* $12\frac{3}{4}$*s.*—$1\frac{3}{8}$*l.*—6*l.* 6*s.* $4\frac{1}{4}$*d.*—and 17*s* $6\frac{3}{4}$*d.*?

Ans. 19*l.* 3*s.* $1\frac{1}{8}$*d.*

4. Required the sum of 8·76*l.*—21*l.* 16·44*s.*—and 19*s.* 10·32*d.*

Ans. 31*l.* 11*s.* 6*d.*

If a company of foot march 63 paces of 2½ feet each in a minute; what is the rate *per hour*?

Ans. 1m. 1490*yds.*

What is the extent of a front consisting of 100 men, allowing 22 inches per man?

Ans. 61*yds.* 4*in.*

How many palisades will surround a square fort whose side is 150 yards, the centres of the palisades being 10 inches asunder?

Ans. 2160.

If I observe the flash from a cannon, and 6 seconds after hear the report, what is its distance; the velocity of sound being 1100 feet *per second*?

Ans. 2200*yds.*

Reduce 74⅞ square feet to inches. *Ans.* 10782.
100 7/18 square yards to feet. *Ans.* 903½.
64212 square inches to feet. *Ans.* 445 11/12.
119*f.* 10½*in.* (square) to yards. *Ans.* 13 [illegible].
56 *sq. in.* to the fraction of a square foot. *Ans.* 7/18.
·85 of a foot square to inches. *Ans.* 122·4.
7·48 feet *sq.* to the decimal of a yard. *Ans.* ·8311 &c.
59290 square yards to acres. *Ans.* 12¼.
7846789 square links to acres. *Ans.* 78·46789.
What is the value of ⅞ of a square foot?
·85 of a square yard?
·753 of an acre?
·735 of a square pole?

Reduce 111½ square inches to the fraction of a yard square.
·1296 of an inch square, to the decimal of a yard square.

Reduce 140¾ cubic yards to feet. *Ans.* 3800¼.
·56 of a cubic foot to inches. *Ans.* 967·68.
9846*f.* 980*in.* (*cub.*) to yards. *Ans.* 364 [illegible].
100 bushels (*dry meas.*) to pints. *Ans.* 6400.
4½ quarters to gallons. *Ans* 277⅓.
2900 pecks to quarters. *Ans.* 90⅝.

If 1 horse is allowed 1½ pecks of corn in 2 days, how many quarters will serve 70 horses 39 weeks?

Ans. 373 [illegible].

Ans. 2560.

gsheads (beer meas.) to pints. Ans. 1784$\frac{4}{11}$

57 gallons to barrels.

hogsheads of beer will serve a garrison of 1350 men for

wing each man 1$\frac{1}{4}$ pints per day? Ans. 2559h. 20$\frac{1}{4}$gall.

Ans. 9720.

Ans. $\frac{1}{216}$

$\frac{7}{10}$ hours to seconds.

$\frac{1}{6}$ of a minute to the fraction of an hour.

365d. 5h. 48m. 48sec. (the solar year) to seconds. Ans. 31556928.

Ans. 477$\cdot$6.

7 96 degrees of a circle to minutes. Ans. $\cdot$006944 &c.

25 seconds to the decimal of a degree.

What is the value of $\cdot$0825 of a degree?

of $\cdot$695 of a minute of a degree?

of $\cdot$44 of an hour?

. 60 seconds make a minute, and 60 min. a degree.

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37 half guineas,

£21,

19 crowns,

33 half-crowns,

101 dollars, at 4s. 2$\frac{1}{2}$d. each,

147 gold mohurs, at 1l. 13s. 2$\frac{3}{4}$d. each,

191 sicca rupees, at 2s. 2$\frac{1}{2}$d. each?

Ans. 381l. 0s. 1d. 0$\frac{3}{100}$qrs.

3. What is the sum of 10l. 12$\frac{1}{2}$s.—1$\frac{1}{2}$l.—6l. 6s. 4$\frac{1}{2}$d.—and 17s 6$\frac{1}{4}$d.?

Ans. 19l. 3s. 1$\frac{1}{4}$d.

4. Required the sum of 8$\cdot$76l.—21l. 16$\cdot$44s.—and 19s. 10$\cdot$32d.

Ans. 31l. 11s. 6d.

6. At 1s. 2¾d. *per* lb. what is that *per* hundred weight?

Ans. 6l. 17s. 8d.

7. At 3s. 7¼d. *per* day what is that *per annum*, or for 365 days?

Ans. 65l. 15s. 6¼d.

8. What is the expense *per annum*, or for 365 days, of a regiment of cavalry, according to the following statement?

			£	s.	d.	
	Colonel		1	15	0	*daily pay.*
2	Lieutenant Colonels	*each*	1	4	6	
2	Majors	*each*	1	0	6	
7	Captains	*each*	0	15	6½	
	Captain Lieutenant		0	9	0	
10	Lieutenants	*each*	0	9	0	
10	Cornets	*each*	0	8	0	
	Adjutant		0	5	0	
	Chaplain		0	6	8½	
	Surgeon		0	6	0½	
2	Surgeon's Mates	*each*	0	3	6½	
	Paymaster		0	15	6½	
10	Quarter Masters	*each*	0	5	6	
	Serjeant Major		0	2	2½	
40	Serjeants	*each*	0	2	2	
	Trumpet Major		0	2	2½	
9	Trumpeters	*each*	0	1	7	
40	Corporals	*each*	0	1	7½	
709	Privates	*each*	0	1	3	

Clothing.

	Serjeant Major		0	0	6	*per day.*
40	Serjeants	*each*	0	0	6	
	Trumpet Major		0	0	6	
9	Trumpeters	*each*	0	0	4	
40	Corporals	*each*	0	0	4	
709	Privates	*each*	0	0	4	

Arms and Appointments.

	Serjeant Major		0	1	2½	*per day.*
40	Serjeants	*each*	0	1	2½	
	Trumpet Major		0	1	0	
9	Trumpeters	*each*	0	1	0	
40	Corporals	*each*	0	1	2½	
709	Privates	*each*	0	1	2½	

3. If the discount on 90*l.* is 1*l.* 4*s.* 6$\frac{1}{4}$*d.*—on 100*l.* 10*s.* is 1*l.* 10*s.* 6*d.* 1$\frac{3}{4}$*grs.*—on 200*l.* is 2*l.* 11*s.* 4$\frac{1}{2}$*d.*—and on 90*l.* is 17*s.* 11$\frac{1}{2}$*d.* What is the whole difference or sum to be received?

Ans. 463*l.* 16*s.* 7*d.* 2$\frac{1}{8}\frac{1}{2}$*grs.*

4. If the quantity of provisions in a garrison is 111*ton.* 12*cwt.* how much would be left at the expiration of 7 weeks, supposing the weekly consumption to be 12*ton.* 13*cwt.* 1*qr.* 21*lb.* 7*oz.*?

Ans. 22*ton.* 17*cwt.* 3*qr.* 17*lb.* 15*oz.*

5. If three pieces whose lengths are 4*f.* 10·6*in.*—2*f.* 7·7*in.*—and 1*f.* 5·5*in.* be cut from a plank whose length is 4*yds.* 1*f.* 9$\frac{1}{4}$*in.* how long is the remainder?

Ans. 1*yd.* 1*f.* 9·7*in.*

6. From a piece of ground containing 3*ac.* 4$\frac{1}{4}$*pol.* a part equal to 1050 square yards was marked off for a surrounding ditch. Required the content of the inner space?

Ans. 2*ac.* 129$\frac{1}{2}\frac{1}{2}$*pol.*

7. Three hogsheads and an half of liquor, wine measure, being poured into a vessel whose cubic capacity was 1*yd.* 7*f.* 13*in.*; what remained empty?

Ans. 4*f.* 917$\frac{1}{4}$*in.*

Compound Multiplication and Division.

1. When oats are at 3*s.* 11$\frac{1}{4}$*d.* *per* bushel, what is that *per* quarter?

Ans. 1*l.* 11*s.* 6*d.*

2. What must be given for 10 sacks of barley at 1*l.* 7*s.* 7$\frac{1}{2}$*d.* *per* sack?

Ans. 13*l.* 16*s.* 5$\frac{1}{2}$*d.*

3. At 9*s.* 10$\frac{1}{2}$*d.* *per* bushel, what is that *per* load of 40 bushels?

Ans. 19*l.* 15*s.*

4. At 1*s.* 0$\frac{3}{4}$*d.* *per lb.* what cost 16 barrels of gunpowder, each weighing 90*lb.*?

Ans. 76*l.* 10*s.*

5. What cost 29 yards of cloth at 4*s.* 5$\frac{1}{4}$*d.* *per* yard?

Ans. 6*l.* 8. 8$\frac{1}{4}$*d.*

12. At 2*l*. 11*s*. 7¾*d*. *per* hundred weight, what cost 10¾*cwt*.?

Ans. 2

13. What cost 93½*lb*. of powder at 1*s*. 0¾*d*. *per lb*.?

Ans. 4*l*. 15*s*.

14. What is the neat weight of 38 barrels of gunpowder weight of each being 96*lb*. 14*oz*, and that of each empty barre

Ans. 3

15. What is the weight of 44 guineas, each being 5*dwts*. 9½

Ans. 11*oz*. 17

16. What is the whole length of 26 planks, each being 5*yds*.

Ans. 150*yds*

17. How many square yards are contained in 17 boards, 12*f*. 37·8*in*?

Ans. 23*y*. 2

18. If 1 man can dig 6*yds*. 13*f*. cubic measure in a day, would 37 men dig in 3 days?

Ans.

19. How many hogsheads of beer in 47 barrels, each barrel 34 *gall*. 7 *pints*?

Ans. 30*hds*. 1

20. If oats are 39*s*. 5*d*. *per* quarter, what is that *per* bushel

Ans.

21. When coals are 44*s*. 6*d*. *per* chaldron, what is the price o

Ans. 1*s*.

22. If the whole pay of 100 men be 41*l*. 11*s*. 3*d*. for a wee the daily pay of each?

Ans

23. If I give 4*l*. 17*s*. for 23 yards of cloth, what is that *per*

Ans. 3*s*. 10

24. If 7¾*lb*. of gunpowder cost 8*s*. 1½*d*. what is that *per lb*.

A

25. If 3¾*cwt*. cost 22*l*. 0*s*. 3*d*. what is that per *lb*.?

Ans

Forage.

		£	s.	d.	
87	Officer's Horses ...... each	0	1	5¼	*per day.*
800	Troop Horses .... each	0	2	1½	

Ans. £84762 17 8½.

9. What is the annual expense, or for 365 days, of a regiment of foot, consisting of 10 companies; according to the following statement?

		£	s.	d.	
	Colonel	1	2	6	*daily pay.*
2	Lieutenant Colonels ... each	0	15	11	
2	Majors .... each	0	14	1	
7	Captains .... each	0	9	5	
	Surgeon	0	9	5	
	Assistant Surgeon	0	5	6	
10	Lieutenants .... each	0	5	8	
	Quarter Master	0	5	8	
10	Ensigns... .... each	0	4	8	
	Adjutant	0	5	0	
	Paymaster	0	15	0	
40	Serjeants .... each	0	1	6¾	
40	Corporals.................. each	0	1	2¼	
10	Drummers .... each	0	1	1¾	
910	Privates ... . each	0	1	0	

Clothing.

		£	s.	d.	
40	Serjeants ... . each	0	0	5	*per day.*
40	Corporals .... each	0	0	4	
10	Drummers .... each	0	0	4	
910	Privates .... each	0	0	4	

Arms and Appointments.

		£	s.	d.	
40	Serjeants .... each	0	0	0¾	*per day.*
40	Corporals .... each	0	0	1½	
10	Drummers .... each	0	0	1	
910	Privates .... each	0	0	1½	

Ans. £39161 1 3.

10. What cost 25½ quarters of oats, at 1*l.* 11*s.* 4½*d.* *per* quarter?

Ans. 40*l.* 0*s.* 0¾*d.*

11. At 5*s.* 1½*d.* *per* yard, what cost 57¾ yards?

Ans. 14*l.* 15*s.* 11*d.* 2½*qrs.*

Rules of Proportion.

1. Required a 3d. proportional to 21 and 39?

Ans. $72\frac{3}{7}$.

2. .... to ·16 and ·071?

Ans.

3. .... to [illegible] and $15\frac{1}{2}$?

Ans.

4. Required a 4th. proportional to $2\frac{1}{2}$, $19\frac{1}{3}$, and ·0111?

Ans. ·08760.

5. .... to $\frac{2}{17}$, [illegible], and [illegible]?

Ans.

6. .... to 1·75, 0·11, and ·005?

Ans.

7. Divide 1 into two parts having the ratio of $\frac{1}{2}$ to $\frac{1}{3}$.

Ans.

8. Let 10 be divided into three parts that shall have the same proportions as the three decimals ·8, ·01, and ·0098.

Ans.

9. If gunpowder is 4*l.* 16*s.* 6*d.* *per cwt.* what cost 17*cwt.* 2*qr.* 11*lb.*?

Ans.

10. When oats are 1*l.* 17*s.* 8*d.* *per* quarter, what cost 17*qr.* 5 *bush.* 3 *pecks*?

Ans.

11. What will $2\frac{5}{8}$ *cwt.* of gunpowder come to at the rate of 7*lb.* for 8*s.*?

Ans. 16 *guineas.*

12. If 16*cwt.* 3*qr.* 16*lb.* of lead cost 13*l.* 15*s.* 11*d.* how much will 2*ton*, 17¼*cwt.* come to?

Ans. 46*l.* 19*s.* 9*d.*

13. If the clothing of 600 men cost 1288*l.* 15*s.* what will be the expense of clothing a regiment consisting of 911 men?

Ans. 1956*l.* 15*s.* 0½*d.*

14. If a bankrupt owes 740*l.* 18*s.* and his whole property amounts to no more than 310*l.* 12*s.* what can he pay *per* £ to his creditors?

Ans. 8*s.* 4*d.* 2[illegible]*qrs.*

15. When a person's annual income is 343*l*. 10*s*. 5*d*. what should be his daily expenses in order to lay by 50*l*. a year?

Ans. 16*s*. 1*d*.

16. What will the tax on 529*l*. 10*s*. amount to at 2*s*. 5¾*d*. in the pound?

Ans. 65*l*. 12*s*. 8⅝*d*.

17. If the average step of a horse be 2¾ feet, and that of a man 2½ feet, then how many men's paces are equal to 40 of a horse?

Ans. 44.

18. If a garrison of 860 men have provisions for 270 days, how long will those provisions last if the garrison be reduced to 644 men?

Ans. 360$\frac{90}{161}$ *days*.

19. Two hundred and forty men having raised a certain work in 8 days: how many men would be necessary to finish a like quantity of work in 20 days?

Ans. 96.

20. If 720 men when put in column of march with 8 men in front, extend 216 paces; what will be the extent if they march 9 men in front?

Ans. 192 *paces*.

21. If a certain number of workmen can throw up an entrenchment in 10 days when the day is 6 hours long; in what time would they do it when the day is 8 hours long?

Ans. 7½ *days*.

22. If the garrison of a besieged place have provisions for 12 weeks, at the rate of 18 ounces *per* day for each man; what must be the allowance if they intend to hold out 16 weeks?

Ans. 13½ *oz*.

23. What length must be cut off a board that is 14⅔ inches wide to make a foot square?

Ans. 9$\frac{9}{11}$ *inches*.

24. How many yards of paper which is 2 feet wide, will hang a room that is 6 yards long, 5¼ broad, and 8½ feet high?

Ans. 95⅝ *yards*.

25. If the penny loaf weighs 6¾ *oz*. when wheat is 12*s*. 6*d*. *per* bushel; what should it weigh when the wheat is 15*s*. 10*d*. the bushel?

Ans. 5$\frac{25}{76}$ *oz*.

26. If a garrison of 800 men have provisions for 12 weeks at the rate of

ounces a day for each man: what must be the allowance to m
visions last 20 weeks if the garrison is reduced to 700 men?

A

27. If the quantity of provisions in a garrison serve 1200
eks, at the rate of 20 ounces a day for each man; how many
same provisions maintain 18 weeks, allowing each man 16
y?

28. If 840 men require 5880 rations of bread for a week, h
ions will 2520 men require for 7 weeks?

Ans

29. In the latitude of London, the distance round the ear
arallel of latitude is nearly 15560 miles; now as the earth tu
ce in 23*h*. 56*m*. 4*sec*. at what rate *per* minute is the City o
carried from west to east by this motion?

Ans. 10$\frac{1}{1}$

30. Suppose a General imposes a contribution of 2000*l*. on 4
be paid in proportion to the number of inhabitants contained
now if the first contains 1200, the second 1400, the third 1600
fourth 1800; what part must each town pay?

A

31. Four companies consisting of 42, 57, 66, and 78 men,
ly, being sent into a garrison where the duty requires 81 men a
many must each company furnish in proportion to its strength?

Ans. 14, 19, 2

32. Suppose the forage on 2½ acres of land will supply a b
horse for 3 days; how many such acres will serve 750 horse for

33. If the charge of keeping 10 horses 52 weeks is 457*l*.; wh
keep of 68 horses amount to in 21 weeks at the same rate?

Ans. 12

34. Three troops of horse rent a field for which they pay 9

sent 56 horses for 12 days; the second sent 64 horses for 15 days; and the third sent 80 horses for 18 days; what must each troop pay?

Ans. 1st. 17*l.* 10*s.*
2*d.* 25*l.* 0*s.*
3*d.* 37*l.* 10*s.*

35. If the carriage of 30*cwt.* of baggage cost 1*l.* 4*s.* for 20 miles; what will the carriage of 76*cwt.* for 84 miles amount to at the same rate?

Ans. 12*l.* 15$\frac{9}{25}$*s.*

36. If a piece of canvas 18 Flemish ells long, and $\frac{3}{4}$ *yd.* wide, cost 18*s.* 6$\frac{1}{2}$*d.*; what cost another piece of the same quality which is 63 English ells in length, and a yard wide?

Ans. 7*l.* 4*s.* 4$\frac{1}{2}$*d.*

37. Bought a silver tankard weighing 36$\frac{1}{2}$*oz.* *avoirdupois* at 5*s.* the ounce, and sold it at 5*s.* 5$\frac{1}{2}$*d.* the ounce *troy*; what was gained or lost?

Ans 10$\frac{715}{768}$*d.* *lost.*

38. Suppose 1$\frac{1}{2}$*cwt.* of gunpowder at 5*l.* 12*s.* *per cwt.*; 2$\frac{1}{2}$*cwt.* at 4*l.* 13*s.* 4*d.* *per cwt.*; and 2$\frac{1}{2}$*cwt.* at 6*l.* 1*s.* 4*d* *per cwt.* to be mixed together; what is a hundred weight of the compound worth?

Ans. 5*l.* 8$\frac{3}{13}$*s.*

39. A General having detached $\frac{3}{5}$ of his army to take possession of two strong posts, and 750 men to watch the motions of the enemy, found that he had only $\frac{1}{5}$ his army left; what was his whole force?

Ans. 3500 men.

40. The ordinary Grecian army consisted of 28672 men: the *psiles* or light armed foot were twice the number of the cavalry; and the *oplites* or heavy armed foot were twice the number of the light armed. Query the number of each?

Ans. Cavalry 4096.
Light armed 8192.
Heavy armed 16384.

41. Three soldiers A, B, C, divide 3850 cartridges in the following manner, *viz.* A took 2 as often as B took 3; and C got 5 for every 4 which B had; what number did each get?

Ans. A 880.
B 1320.
C 1650.

20 ounces a day for each man: what must be the allowance to make those provisions last 20 weeks if the garrison is reduced to 700 men?

Ans. $13\frac{5}{7}$ oz.

27. If the quantity of provisions in a garrison serve 1200 men 24 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain 18 weeks, allowing each man 16 ounces a day?

Ans. 2000.

28. If 840 men require 5880 rations of bread for a week, how many rations will 2520 men require for 7 weeks?

Ans. 123480.

29. In the latitude of London, the distance round the earth on the parallel of latitude is nearly 15560 miles; now as the earth turns round once in 23*h.* 56*m.* 4*sec.* at what rate *per* minute is the City of London carried from west to east by this motion?

Ans. $10\frac{17990}{21541}$ *miles.*

30. Suppose a General imposes a contribution of 2000*l.* on 4 towns, to be paid in proportion to the number of inhabitants contained in each; now if the first contains 1200, the second 1400, the third 1600, and the fourth 1800; what part must each town pay?

$$\textit{Ans.}\ \left\{\begin{array}{l} \pounds \\ 400. \\ 466\frac{2}{3}. \\ 533\frac{1}{3}. \\ 600. \end{array}\right.$$

31. Four companies consisting of 42, 57, 66, and 78 men, respectively, being sent into a garrison where the duty requires 81 men a day; how many must each company furnish in proportion to its strength?

Ans. 14, 19, 22, and 26.

32. Suppose the forage on $2\frac{1}{2}$ acres of land will supply a body of 400 horse for 3 days; how many such acres will serve 750 horse for 7 days?

Ans. $10\frac{15}{16}$.

33. If the charge of keeping 10 horses 52 weeks is 457*l.*; what will the keep of 68 horses amount to in 21 weeks at the same rate?

Ans. 1254*l.* $19\frac{11}{13}$*s.*

34. Three troops of horse rent a field for which they pay 80*l.*; the first

sent 56 horses for 12 days; the second sent 64 horses for 15 days; and the third sent 80 horses for 18 days; what must each troop pay?

Ans. 1st. 17*l.* 10*s.*
2d. 25*l.* 0*s.*
3d. 37*l.* 10*s.*

35. If the carriage of 30*cwt.* of baggage cost 1*l.* 4*s.* for 20 miles; what will the carriage of 76*cwt.* for 84 miles amount to at the same rate?

Ans. 12*l.* 15$\frac{9}{25}$*s.*

36. If a piece of canvas 18 Flemish ells long, and $\frac{3}{4}$ *yd.* wide, cost 18*s.* 6$\frac{3}{4}$*d.*; what cost another piece of the same quality which is 63 English ells in length, and a yard wide?

Ans. 7*l.* 4*s.* 4$\frac{1}{2}$*d.*

37. Bought a silver tankard weighing 36$\frac{1}{2}$*oz.* *avoirdupois* at 5*s.* the ounce, and sold it at 5*s.* 5$\frac{1}{2}$*d.* the ounce *troy*; what was gained or lost?

Ans 10$\frac{715}{768}$*d.* *lost.*

38. Suppose 1$\frac{3}{4}$*cwt.* of gunpowder at 5*l.* 12*s.* *per cwt.*; 2$\frac{1}{2}$*cwt.* at 4*l.* 13*s.* 4*d.* *per cwt.*; and 2$\frac{1}{4}$*cwt.* at 6*l.* 1*s.* 4*d* *per cwt.* to be mixed together; what is a hundred weight of the compound worth?

Ans. 5*l.* 8$\frac{2}{39}$*s.*

39. A General having detached $\frac{3}{4}$ of his army to take possession of two strong posts, and 750 men to watch the motions of the enemy, found that he had only $\frac{1}{28}$ his army left; what was his whole force?

Ans. 3500 men.

40. The ordinary Grecian army consisted of 28672 men: the *psiles* or light armed foot were twice the number of the cavalry; and the *oplites* or heavy armed foot were twice the number of the light armed. Query the number of each?

Ans. Cavalry 4096.
Light armed 8192.
Heavy armed 16384.

41. Three soldiers A, B, C, divide 3850 cartridges in the following manner, viz. A took 2 as often as B took 3; and C got 5 for every 4 which B had; what number did each get?

Ans. A 880.
B 1320.
C 1650.

42. A body of 2520 troops is composed of 4 battalions; what is t strength of each, if $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, and $\frac{1}{5}$ the fourth are equal?

Ans. 360, 540, 720, 9

43. A party of foot begin their march at 8 in the morning; two ho afterwards a troop of horse follow them (from the same place); the fo march 80 paces *per* minute, and the horse 90; now if a man's step be feet, and that of a horse $2\frac{1}{4}$ feet; in what time will the horse overtake t foot; and what distance will they have marched?

Ans. 8h. 25 [illegible] *min.*
Dist. 23m. 3612 [illegible] *f*

44. At what time between 10 and 11 o'clock are the hour and minu hands of a watch together?

Ans. $54\frac{6}{11}$ *min. past* 1

45. A party of horse leave London for Oxford at 7 in the morning; a another party leave Oxford for London at 9 the same morning; the form march $3\frac{1}{2}$ miles an hour, and the latter $4\frac{1}{2}$; how far will each have tr velled when they meet, the distance from Oxford to London being miles?

m.
Ans. 30 [illegible] from Londo
28 [illegible] from Oxfor

46. A bank of earth 330 yards long was to have been raised by 40 me in 7 days, but at the end of 5 days only 220 yards were completed; no how many men should be added to finish the bank in the proposed tim at the same rate of working?

Ans. 1

47. A General after detaching [illegible] of his army to take possession of height, and [illegible] of the remainder to reconnoitre the enemy, had 1280 me left; what was his whole force?

Ans. 3380 me

48. If a garrison of 1200 men have provisions for 12 months, but the end of 3 months are reinforced with 500 men, and 2 months after th with 400 more; how long will the provisions last, supposing no alteratio in the daily allowance of each man?

Ans. 8 [illegible] *months in the whol*

49. Two labourers A and B if they work together can dig a tren

in 20 days; A can dig it himself in 34 days; in what time would B do it if he worked alone?

Ans. $48\frac{4}{7}$ *days.*

50. A can dig 32 yards of a trench in 6 days; B can dig 29 yards in 5 days; and C can dig 54 yards in 10 days; in what time would they finish 100 yards if they work together?

Ans. $6\frac{1}{21}$ *days.*

51. If A can finish a certain number of yards of an entrenchment in 6 days of 7 hours each, and B can do 4 times as much in 15 days of 9 hours each; what is their comparative strength?

Ans. the strength of B is to that of A as 56 to 45.

52. Suppose 20 men in 15 days of 8 hours each, can dig 45 cubic yards; how many cubic yards can 25 men dig in 40 days of 10 hours long, supposing the hardness of the ground in the former case, is to that in the latter, as 9 to 11, and the strength of each of the 20 men is to that of each of the 25, as 6 to 7?

Ans. $178\frac{43}{44}$ *yards.*

53. If 30 men in 40 hours can dig 80 cubic yards; how many men, which are stronger in the proportion of 5 to 4, would it require to dig 120 yards in 90 hours, supposing the ground in the latter case is harder than that in the former, in the ratio of 9 to 8?

Ans. 18.

54. Suppose two labouring parties, one consisting of 40, the other of 50 men, and let the strength of each man of the former party be to that of each of the latter as 3 to 4; now if the 40 men can dig 100 cubic yards in 10 hours; in what time would the other party dig 480 yards, if the ground in the former case is twice as hard as that in the latter?

Ans. $14\frac{2}{5}$ *hours.*

N. B. In the three last questions, the labour in digging a like number of yards, is supposed to be directly proportional to the hardness of the ground.

55. A plan of raising the siege of Brunswick, by Prince Ferdinand in 1761, has a *scale* of 300 *Rhynland roods*; the *scale* is just 2·62 *inches* in length: the plan is $18\frac{1}{4}$ *inches* long, and $15\frac{1}{4}$ broad; now if it be enlarged to 6 *inches* the English mile, what will be its length and breadth?

Ans. 29·8*in.* long.
25·4*in.* broad.

56. A, B, and C, can dig a trench in 4 days; A can do it by himself in 7 days, and B in 14; in what time would C finish it if he worked alone?

Ans. 28 *days.*

57. A, B, and C, can do a piece of work in 10 days; B, C, and D, in 12 days; C, D, and A, in 14 days; and D, A, and B, in 16 days; in what time would each do it by himself?

	A	B	C	D
Ans.	$44\frac{68}{113}$,	$29\frac{23}{173}$,	$23\frac{13}{109}$,	$173\frac{23}{29}$ *days.*

58. Suppose a clock has three hands, and that one moves round once in a day, another once in 30 days, and the third once in 365 days; now if they are all together at any particular time, how long is it before they come together again?

Ans. 2190 *days.*

59. Divide 10 into three such parts, that when the 1*st.* is multiplied by 2, the 2*d.* by 3, and the 3*d.* by 4, the three products may be equal?

Ans. $4\frac{8}{13}$, $3\frac{1}{13}$, $2\frac{4}{13}$.

60. Let 10 be divided into 4 parts such, that when they are respectively divided by 2, 3, 4, and 5, the quotients shall be in the same proportion as 6, 7, 8, and 9?

Ans. $1\frac{1}{11}$, $1\frac{10}{11}$, $2\frac{10}{11}$, $4\frac{1}{11}$.

Questions respecting the march of Troops.

1. If the force of a battalion be 490 men, in three ranks; what is the extent of its front, the allowance for each man in front being 22 *inches* or $1\frac{5}{6}$ *feet*? (See quest. 27, art. 104.)

2. Suppose the same battalion in line of two ranks; what is the extent of its front?

Ans. $449\frac{1}{6}$ *feet.*

3. In what time would a column consisting of 7 battalions, the extent of each being 317 feet, march its own length at the ordinary rate of 75 *paces* of $2\frac{1}{2}$ *feet* each *per* minute?

Ans. $11\frac{5}{6}$ *min.*

4. In what time would a column of 11 such battalions march through a defilé $1\frac{1}{2}$ miles long at the same rate?

Ans. 60[illegible] *min.*

5. Supposing the march is according to quick time or 108 *paces per* minute; in what time would the column pass through the defilé?

Ans. $42\frac{1}{2}$ *min.*

6. In what time would a column of horse whose extent is 896 *feet*, march through a defilé $\frac{1}{2}$ a mile in length, at the rate of 90 *paces per* minute, supposing the average step of a horse to be $2\frac{1}{2}$ *feet*?

Ans. 14[illegible] *min.*

7. Suppose 12 battalions, the extent of each including 2 field pieces, being 510 *feet*, have to pass a defilé $1\frac{1}{4}$ miles in length; now if the column can move at the rate of 75 *paces* ($2\frac{1}{2}$ *feet* each) in the first mile, but the last $\frac{1}{4}$ mile being a bad road in which the horses attached to the cannon can march only 40 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute; in what time will the column pass the defilé?

Ans. 111[illegible] *min.*

8. If in the last question the first mile is a bad road, and the $\frac{1}{4}$ mile a good one; in what time would the column march through the defilé; the other circumstances remaining the same?

Ans. 120[illegible] *min.*

9. Suppose a column whose extent is 2000 *paces* of $2\frac{1}{2}$ *feet* each, has to pass a defilé $3\frac{1}{2}$ miles in length, and that it can march 80 *paces per* minute in the first mile, 50 in the next $\frac{1}{2}$ mile, 65 in the following $1\frac{3}{4}$ miles, and only 45 in the last $\frac{1}{4}$ mile; in what time will it clear the defilé?

Ans. 2*h.* 53[illegible] *min.*

10. Admit the column A has a good road 6600 *paces* in length; the column B a middling road 4000 *paces* in length; and the column C a bad road 3310 *paces* in length; now if the first column march 108, the second column 75, and the third column only 50 *paces per* minute; how must the march be regulated that the heads of the columns may arrive at the same parallel together?

Ans. A *must halt* 5[illegible] *min.*
B *must halt* 12[illegible] *min.*

11. If in the last question it is required that the heads of the columns shall arrive at the same parallel at the expiration of $1\frac{1}{2}$ hours; how must the march be regulated?

Ans. A *must halt* $13\frac{1}{3}$ *min.*
B *must halt* $21\frac{1}{3}$ *min.*
C *must halt* $8\frac{1}{3}$ *min.*

12. Suppose 17 battalions, the extent of each being 520 *feet*, have pass 3 defilés; the first 1 mile, the second $1\frac{1}{4}$ miles, and the third $1\frac{3}{8}$ es in length; how many battalions must pass through each defilé that whole march through them may be made in the least time; and what [illegible] that time be if the rate of marching is 75 *paces* ($2\frac{1}{2}$ *feet* each) *per* nute?

Ans. 10 *through the shortest.* } the nearest
5 *through the next.* } whole battalions.
2 *through the longest* }

And the respective times will be 53[illegible], 56[illegible], 54[illegible] *min.*

13. Suppose the same 17 battalions have to pass 2 defilés, the first being $1\frac{1}{4}$ miles, and the second 1 mile in length; now if the troops can march 108 *paces per* minute in the first defilé, and 75 in the other; how must the battalions be divided that the whole march through the defilés may be made in the least time?

Ans. 10 *battalions must march through the longest.*
7 *through the shortest.*

14. Suppose 22 battalions have to pass 3 defilés of equal extent; the first admitting of 4 men to march in front, the second of 6, and the third of 8; now if the length of a battalion (including 2 field pieces) when in column of march with 4 men in front is 660, with 6 men in front is 490, and with 8 men in front is 410 *feet*, respectively; how many battalions must pass each defilé that the whole march through them may be made in the least time; and what will that time be if the defilés are each 2 miles in extent, and the rate of marching 75 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute?

Ans. 3 *battalions through the first.* *time* 73[illegible] *min.*
8 .... *through the second.* $77\frac{4}{5}$.
9 .... *through the third.* 76.

15. If 12 battalions have to pass 2 defilés, one 2 miles, the other 1 mile in length, the former admitting 7, and the latter 4 men to march abreast, respectively; now if the length of a battalion (including 2 field pieces) is 280 *paces* of $2\frac{1}{2}$ *feet* each when 7 men march in front, and 407 *paces* when 4 men march in front; how many battalions must pass each defilé that the whole march through them may be made in the least time; and what will that time be, supposing the march is 70 *paces per* minute

Ans. 4 battalions *through the broadest.* *time* 76[illegible] *min*
8 .... *through the other.* .... 70[illegible] *min*

16. Suppose in the last example, the march through the shortest defilé is at the rate of 50 paces per minute, and that through the other 65; how must the battalions be divided, the other circumstances remaining the same?

time.
Ans. 6 battalions *must march through the longest.* $90\frac{5}{63}$ *min.*
6 .... *through the other.* .... $91\frac{1}{70}$.

17. Admit 15 battalions of unequal strength have to pass 2 defilés, one a mile, the other $1\frac{1}{2}$ miles in length, each admitting of a like number of men to march in front; now if the extent of each of 9 battalions when in column of march is 480 *feet*, and the extent of each of the other 6 is 620 *feet*; what number of battalions must pass each defilé that the whole march through them may be performed in the least time, at the rate of 75 *paces* ($2\frac{1}{2}$ *feet* each) *per* minute?

Ans. 6 *of the less battalions, and* 4 *of the greater, must march through the shortest defilé.*
3 *of the less and* 2 *of the greater through the other.*
And the time of marching through the former $56\frac{4}{5}$ *min.*
through the latter $56\frac{4}{5}$ *min.*

N. B. In the foregoing questions, the fronts of the columns are supposed to enter the defilés nearly at the same time. And in reducing feet to paces, the nearest integer is usually taken.

Interest.

1. What is the simple interest of 319*l.* 12*s.* for 4 years at 4 *per cent. per annum*?

Ans. 51*l.* 2*s.* 8·64*d.*

2. What is the simple interest of 217*l.* 15*s.* 8*d.* for $4\frac{3}{4}$ years, at $3\frac{1}{2}$ *per cent. per ann*?

Ans. 36*l.* 4*s.* $1\frac{56}{100}$*d.*

3. What is the simple interest of 279*l.* 10*s.* for 190 days at $4\frac{1}{2}$ *per cent. per ann.*?

Ans. 6*l.* 10*s.* $11\frac{24}{73}$*d.*

4. What will be the amount of 251*l.* 10*s.* in 5 years at 4 *per cent. per ann.* simple interest?

Ans. 301*l.* 16*s.*

5. What is the discount on 200*l.* at 4 *per cent*?

Ans. 8*l.*

6. What is the discount of 200*l.* due a year hence at 4 *per cent. per ann.* simple interest?

Ans. 7*l.* 13*s.* $10\frac{2}{13}$*d.*

7. If 150*l.* become due to me at the end of $1\frac{1}{2}$ years, what should I receive immediately, discounting at the rate of 4 *per cent. per ann.* simple interest?

Ans. 141*l.* 10*s.* $2\frac{14}{53}$*d.*

8. If I receive 275*l.* for 300*l.* due $2\frac{1}{2}$ years hence, what am I charged *per cent. per ann.* discount, reckoning simple interest?

Ans. $3\frac{7}{11}$*l.*

9. What is the purchase of 1000*l.* *bank annuities* at $91\frac{1}{8}$ *per cent.*?

Ans. 911*l.* 5*s.*

10 What is the purchase of 1000*l.* *India stock* at $112\frac{3}{8}$ *per cent.*?

Ans. 1123*l.* 15*s.*

11. What is the amount of 56*l.* 10*s.* in 4 years at $4\frac{1}{2}$ *per cent. per ann.* compound interest?

Ans. 67*l.* 7*s.* $6\frac{1}{2}$*d.*

12. What is the compound interest of 120*l.* for 5 years at 5 *per cent. per ann.*?

Ans. 33*l.* 3*s.* 0·9*d.*

Double Position.

1. What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 10?

Ans. $\frac{1}{11}$ and $\frac{10}{11}$.

2. A general having detached 620 men to take possession of a strong post, and $\frac{4}{5}$ of the remainder of his troops to watch the motions of the enemy, finds that he has only $\frac{1}{13}$ of his army left; what was his whole force?

Ans. 1040 men.

3. Three battalions of unequal force are in column of march; the extent of the first battalion is 216 *paces*, the extent of the second is equal to that of the first and third together, and the extent of the third is equal to that of the first and half the second; what is the extent of the column?

Ans. 1728 *paces.*

4. What number is that which being added to its square shall make the sum 70?

Ans. 7·881527 &c.

5. Required that number which added to its cube shall make the sum 70?

Ans. 4·040415, *nearly.*

Involution.

1. What is the square of 8765?

Ans. 76825225.

2. What is the cube of 8765?

Ans. 673373097125.

3. Required the cube of ·07001?

Ans. ·000343147021001.

4. What is the 4*th.* power of 9·3?

Ans. 7480·5201.

5. What is the 13*th* power of 3?

Ans. 1594323.

6. Required the square of $\frac{11}{17}$?

Ans. $\frac{121}{289}$.

7. What is the 11*th* power of $\frac{4}{5}$?

Ans. $\frac{4194304}{48828125}$.

8. What is the 5*th* power of $5\frac{3}{7}$?

Ans. $4714\frac{6772}{16807}$.

Extraction of Roots.

1. How many ranks are in a column consisting of 5625 men, when the number of men in front are equal to the number of ranks?

Ans. 75.

2. What is the square root of 3418801?

Ans. 1849.

3. What is the square root of 250401160801?

Ans. 500401.

4. Required the square root of 4609·0521?

Ans. 67·89.

5. What is the square root of ·0003118801?

Ans. ·01819.

6. What is the square root of 11?

Ans. 3·3166248 *nearly.*

7. Required the square root of 3?

Ans. 1·7320508 *nearly.*

8. Required the square root of $\frac{100}{196}$?

Ans. $\frac{10}{14}$.

9. What is the square root of $\frac{441}{625}$?

Ans. $\frac{21}{25}$.

10. Required the square root of [illegible]?

Ans.

11. What is the square root of [illegible]?

Ans.

12. What is the square root of 20339 [illegible]?

Ans. 142 [illegible].

13. Required the square root of $3\frac{3}{7}$?

Ans. 1·8516402 *nearly.*

14. What is the square root of the decimal ·0183?

Ans. ·1352775 *nearly.*

15. Required the 4*th* root of 37015056?

Ans. 78.

16. Required the cube root of 961504803.

Ans. 987.

17. What is the cube root of 193·100552?

Ans. 5·78.

18. Required the cube root of 51230158344?

Ans. 3714.

19. What is the cube root of $\frac{27}{64}$?

Ans. $\frac{3}{4}$.

20. What is the cube root of [illegible]?

Ans. [illegible]

21. Required the cube root of [illegible]?

Ans.

22. What is the cube root of [illegible]?

Ans.

23. What is the cube root of 11706[illegible]?

Ans. 22[illegible].

24. Required the cube root of 16?

Ans. 2·519842 *nearly.*

25. Required the cube root of 197?

Ans. 5·818648 *nearly.*

26. What is the cube root of the decimal ·014?

Ans. ·2410142 *nearly.*

27. Required the cube root of ·000001?

Ans.

28. What is the cube root of $\frac{8}{9}$?

Ans. ·961499 *nearly.*

29. The diameter of a 9*lb.* iron shot being 4 inches, what is the weight of a shot 6 inches in diameter?

Ans. 30$\frac{3}{8}$*lb.*

N. B. It is proved by geometry, that the cubic contents (and consequently the weights) of similar solids are directly proportional to the cubes of their like sides or diameters.

30. What is the diameter of a 48*lb.* iron shot?

Ans. 6·99 *inches.*

31. What is the diameter of a 24*lb.* shot?

Ans. 5·55 *inches.*

32. A lead ball whose diameter is 4$\frac{1}{2}$ inches weighs 17*lb.* nearly; hence it is required to find the diameter of a musket ball whose weight is an ounce?

Ans. ·656 *of an inch.*

33. If the depth of a barrel which holds 80*lb.* of gunpowder be 20 inches, what is the depth of another barrel of similar dimensions which holds three times that quantity?

Ans. 28·84 *inches.*

34. If a musket barrel which carries an ounce ball (·656*in.* in diam.) is 3 feet in length; what would be the diameter of the bore, and

Geometrical Progression.

1. If the first term be 1½, the ratio or multiplier 3, and the number of terms 10, what is the last term?

Ans. 29524½.

2. Let the first term be 9, the ratio or divisor 1½, and number of terms 8, what is the last term?

Ans. $1\frac{1}{8}$.

3. Suppose the first term is 100, the ratio or multiplier 1·05, and the number of terms 8, what is the last term? In other words—What is the amount of 100*l.* in 7 years, at 5 *per cent. per annum* compound interest?

Ans. £140·710042265625.

4. Let the extremes be 6 and 24, and number of terms 3; required the middle term? Or, what is the mean proportional between 6 and 24?

Ans. 12.

5. Required a geometrical mean between 10 and 20?

Ans. 14·1421356 *nearly*.

6. If the first term is 22, last term 1305018, and the number of terms 4; what is the ratio, and the two middle terms?—Or let it be required to find 2 geometrical means between 22 and 1305018?

Ans. 39 *ratio.*

And the middle terms 858 *and* 33462.

7. Required two geometrical mean proportionals between 10 and 100?

Ans. $\left.\begin{array}{l} 21{\cdot}54435 \\ 46{\cdot}4159 \end{array}\right\}$ *nearly.*

8. Suppose the musket cartridges necessary for an army to be counted at 16 times; the first count being 3, the next 6, the third 12, the fourth 24, and so on; what is the whole number of cartridges?

Ans. 196605.

9. What would be the produce (or last crop) in 10 years from a grain of wheat, the increase or crop being constantly sown, and each grain producing yearly an ear of 40 grains, supposing 7000 grains to weigh a pound, and 60*lb.* to the bushel?

Ans. 3120761904*grs.* 6$\frac{2}{7}$*bush.*

1, 2, 4, 8, 16, 32, 64, 128, &c. numbers.
0, 1, 2, 3, 4, 5, 6, 7, &c. logarithms.

156. Now the sums and differences of the indices or logarithms answer to the products and quotients of the corresponding terms or numbers.

Thus 2 + 3 make 5 the index or logarithm answering to 32.

(111.) And the product of 4 and 8 (the terms corresponding to 2 and 3) make 32.

Again, the difference of the indices or logarithms 7 and 4 is 3, the index or logarithm of the term or number 8.

And the quotient of the corresponding terms, or 128 divided by 16 is 8.

Therefore the products and quotients of the numbers in the geometrical progression are found by taking the sums or differences of the corresponding indices or logarithms.

157. But the indices 0, 1, 2, 3, 4, 5, 6, 7, &c. may denote the powers of any other number or ratio; consequently different ratios or geometric progressions give different systems of logarithms.

Thus if 1 be the first term, and 10 the ratio of a geometrical progression, the terms will be

1, 10, 100, 1000, 10000, 100000, &c.
or 1^0, 10^1, 10^2, 10^3, 10^4, 10^5, &c.

And the indices 0, 1, 2, 3, 4, 5, &c. are the logarithms of the corresponding terms or numbers, as before.

1, 10, 100, 1000, 10000, 100000, &c. *numbers.*
0, 1, 2, 3, 4, 5, &c. *logarithms.*

And according to this system or *scale*, the common logarithmic tables now in use, are calculated*.

158. Now 0 being the logarithm of 1; 1 the logarithm of 10; 2 the logarithm of 100; &c. it follows that the logarithm of any number between 1 and 10 will be 0 with a fraction; between 10 and 100, 1 with a fraction; between 100 and 1000, 2 with a fraction, &c.

159. It is also evident from the nature of the progressions, that if any number of geometrical mean proportionals be interposed between any two terms of the geometrical series 1, 10, 100, 1000, &c. and the like number of arithmetical means between the corresponding indices 0, 1, 2, 3, &c. that the latter will be the indices or logarithms of the former.

Thus one geometrical mean proportional between 100, and 10000 is 1000 (151.)

And the arithmetical mean between the indices 2 and 4 is 3 (129), the logarithm of 1000.

In like manner the geometrical mean between 10 and 100 is $\sqrt{1000}$† or 31·6227 &c.

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* The invention of Logarithms is due to Lord Neper, Baron of Merchiston, in Scotland, who in 1614, published the first table of these numbers in a small treatise, entitled *Mirifici Logarithmorum Canonis Descriptio.* His logarithms, however, are of that form which has since been called *hyperbolic logarithms.* The present scale or system of logarithms we owe to Mr. Henry Briggs, at that time (1614) Professor of Geometry at Gresham College.

The modern Logarithmic tables, in most esteem at present for general use are, Gardener's, 4*to.* 1742. Taylor's, large 4*to.* 1792. Tables Portatives, par Callet, 8*vo.* (the stereotype edition). Dr. Hutton's Mathematical Tables, 8*vo.* 1801; this also contains a very complete History of Logarithms.

† $\sqrt{}$ signifies the *square root*; thus $\sqrt{18 \times 2}$ or $\sqrt{36}$ is 6.

And the corresponding arithmetical mean between the indices 1 and 2 is 1·5, which is the logarithm or index of the term 31·6227 &c.

Therefore the business of computing the logarithm of a given number principally consists in finding a geometrical mean or term of the series equal to, or nearly equal to, the number proposed; then its corresponding arithmetical mean or index will be the logarithm sought.

Now, by repeated extractions of the square root, such an approximate mean proportional may be found, as in the following example:

160. Let it be required to find the logarithm of 2?

First. The number 2 lies between 1 and 10;
(151) and the geometrical mean between 1 and 10 is $\sqrt{1 \times 10}$ $= 3·162278$.

And the arithmetical mean between the indices 0 and 1 (the logarithms of 1 and 10) is 0·5:
therefore the index or logarithm of 3·162278 is 0·5.

Secondly. The number 2 now lies between 1 and 3·162278:
and the geometrical mean between those numbers is $\sqrt{1 \times 3·162278}$ $= 1·778279$.

And the arithmetical mean or half the sum of the indices 0 and 0·5 (the logarithms of 1 and 3·162278) is 0·25:
therefore the logarithm of 1·778279 is 0·25.

Thirdly. The number 2 lies between 1·778279 and 3·162278;
and the geometrical mean is $\sqrt{1·778279 \times 3·162278} = 2·371374$

And the arithmetical mean between the indices 0·25 and 0·5 is 0·375:
therefore the logarithm or index of 2·371374 is 0·375.

Fourthly. The terms next less and next greater than 2 are 1·778279 and 2·371374;
and the geom. mean is $\sqrt{1·778279 \times 2·371374} = 2·053525$.

And half the sum of the corresponding indices or logarithms 0·25 and 0·375 is 0·3125:
therefore the log. or index of 2·053525 is 0·3125.

And in this manner by constantly making use of the resulting geometrical means next less and next greater than 2, after 22 extractions we get the term 1·999999, and the corresponding arithmetical mean or logarithm 0·3010299 for its index. Therefore as 1·999999 differs but 0·000001 from 2, we may take 0·3010299 or 0·301030 (the nearest 6 decimals) for the logarithm of 2.

This is one of the methods by which logarithms were first computed. But more direct and expeditious rules have since been derived from algebraic formulæ, and the fluxional calculus.

161. Now from the logarithm of 2, the logarithms of 4, 8, 16, &c. the powers of 2, are obtained by multiplication.

Thus, 0·301030 × 2 = 0·602060 the log. of 2^2 or 4.
0·301030 × 3 = 0·903090 the log. of 2^3 or 8.
0·301030 × 4 = 1·204120 the log. of 2^4 or 16.
&c. &c.

162. And since 10 divided by 2 gives 5, if the logarithm of 2 be subtracted from the logarithm of 10, the remainder will be the logarithm of 5 (156).

Thus 1·000000 log. of 10.
0·301030 log. of 2.
0·698970 log. of 5.

163. And if the logarithm of 5 be multiplied by 2, 3, 4, &c. the products will be the logarithms of its powers; thus 0·698970 × 4 = 2·795880 the log of 5^4 or 625.

164. Hence in the common scale or system of logarithms, every number is supposed to be that power of 10 whose index is the logarithm of the number.

Thus by the foregoing operation $10^{0·301030}$ is equal to 2, nearly.
$10^{0·903090}$ equal to 8,
$10^{0·698970}$ equal to 5,
$10^{2·795880}$ equal to 625.
&c. &c.

165. The integral part of a logarithm is called its index or characteristic; thus in the logarithms 0·301030, 1·204120

2·795880, the indices are 0, 1, 2; the other figures being decimals. And as the indices are easily supplied by the computer himself, they are commonly omitted in the tables.

166. Since the logarithm of the divisor taken from that of the dividend gives the logarithm of the quotient (162), it follows that the index of the logarithm of a proper fraction will be negative.

Thus suppose the logarithm of $\frac{10}{16}$, or the decimal ·625 is required:

10, its log.	1·000000	
16, its log.	1·204120	sub.
	$\bar{1}$·795880	log. of $\frac{10}{16}$ or ·625.

In this subtraction 1 is carried to the index 1, which together make 2, then 1 *minus* 2 gives 1 *negative*, marked with the negative sign (—) in the remainder.

167. But the logarithm of an improper fraction will have a positive index, because its value is greater than 1.

Thus to find the logarithm of $\frac{25}{4}$ or 6·25.

25, its log.	1·397940	(twice the log. of 5.)
4, its log.	0·602060	subtract.
	0·795880	log. of 6·25.

168. Because 625 × 10 = 6250; and 625 × 100 = 62500, if we add the logarithms of 10, and 100 to that of 625, we get 3·795880 the log. of 6250, and 4·795880 the log. of 62500.

169. Hence it appears, that the logarithm of a whole number and that of a mixed number, or a fraction, consisting of the same significant figures, differ in nothing but the index, which varies according to the place of the first figure.

Thus,

Numbers.		Logarithms.
62500	....	4·795880
6250	....	3·795880
625	....	2·795880
62·5	....	1·795880
6·25	....	0·795880
·625	.... —	1·795880
·0625	.... —	2·795880
·00625	.... —	3·795880

Therefore the index or characteristic of any logarithm is always 1 *less* than the number of figures in the integral part of the natural number.

Explanation and use of the Table of Logarithms.

170. THE table contains the logarithms of the natural numbers from 1 to 10000, to 6 places of figures. The logarithms of the first 100 numbers are printed with the indices. Thus the logarithm of 8 is 0·903090: and the log. of 97 is 1·986772. The indices or characteristics of the other logarithms are to be annexed according to the value of the integral part of the number, as in *art.* 169.

171. To find the logarithm of a number consisting of 3 figures: suppose 123.

Look in the left-hand column for the number 123; then ·089905 in the next or 2*d*. column is the decimal part of its logarithm; and as the number 123 consists of 3 integers, the index will be 2 (169); therefore 2·089905 is the logarithm of 123.

172. To find the logarithm of a number consisting of 4 figures: suppose 2157.

The two first figures of the logarithm of 215 are ·33; then under 7 at the top of the table, and in the horizontal row answering to 215 is 3850 which are the right hand figures of the

logarithm required: therefore the logarithm with its index will be 3·333850.

173. When the 4 right-hand figures of a logarithm are less than the 4 figures next preceding, it shows that the two first figures of the logarithm in the 2d. column are changed or augmented: thus the logarithm of 2344 (without the index) is ·369958; but the logarithm of 2345 is ·370143.

174. To find the logarithm of a number consisting of 5 figures.

Take the logarithm of the four left-hand figures of the proposed number from the logarithm next greater; then say,

As 10, is to the difference, so is the *5th.* figure of the number, to a *4th.* number, which added to the least of the two logarithms gives the log. sought.

Let the number be 24676.

2467	log. ·392169	
next greater	·392345	
	176	dif.

As 10 : 176 :: 6 : 105·6 the *4th.* number.

2467	log.	·392169
		106
24676	log.	4·392275

Here we suppose the differences of the logarithms to be nearly proportional to the differences of the corresponding natural numbers:

Thus the log. of 24670	is 4·392169	
of 24680	is 4·392345	
diff. of numbers 100	176	diff. of logs.

Then, as 10 : 176 :: 6 : 105·6 the proportional part for 6, the whole for 10 being 176.

175. When the logarithm of a number consisting of 6 figures is required, the difference is taken for 100:

Thus to find the logarithm of 54·6347.

	54·6300	log.	1·737431
	54·6400	log.	1·737511
diff.	100		80 diff.

Then, as 100 : 80 :: 47 : 37·6 the proportional part for 47.

1·737431
38
1·737469 log. of 54·6347.

But if the logarithms next less and next greater are in the latter part of the table, the required logarithm may err in the last figure when the natural number consists of 6 figures.

176. The logarithm of a vulgar fraction is found by subtracting the logarithm of the denominator from that of the numerator:

Thus to find the log. of $\frac{117}{147}$:

117 log. 2·068186
147 log. 2·167317
−1·900869 log. of $\frac{117}{147}$.

Or the fraction may be reduced to a decimal.

177. A mixt number may be reduced to an improper fraction:

Thus to find the logarithm of $20\frac{3}{4}$.

$20\frac{3}{4} = \frac{83}{4}$.

83 log. 1·919078
4 log. 0·602060
1·317018 log. of $20\frac{3}{4}$.

Or the fraction may be reduced to a decimal.

$20\frac{3}{4}$ = 20·75, and its log. is 1·317018 as before.

To find the number answering to a given logarithm.

178. This is only the reverse of finding the logarithm of a given number. Therefore look for the two left-hand figures of the proposed logarithm in the 2d. column, and for the

right, and take out the corresponding

ering to the log. 2·397155 is 249·6.
ering to the log. 4·350054 is 22396.
ering to the log.—3·360404 is ·002293.

ed logarithm is not found exactly in the
ce of the logarithms next greater and next
ference between the given logarithm and

of those differences,
cond;

figure of the required number.

umber is required to 6 places of figures,
erm of the proportion. And the figures
xed to the number answering to the next
ve the number sought.

equired to find the number answering to the

2·2[illegible]686
2·[illegible]61 the log. of 184·6
[illegible] diff.
[illegible]
·26[illegible]761
2[illegible]5 diff.

the 5th. figure; therefore the required number

n of the proportion 100 instead of 10,
33 the 5th. and 6th. figures; and the number to

exactly the reverse of that in art. 175.
ry to remark, that when the logarithms
eater fall in the latter part of the table
re small, the number answering to the
nnot be depended upon to more than 5

Multiplication by Logarithms.

181. Add the logarithms of the factors together, and the sum will be the logarithm of the product. (168)

Examples.

1. Required the product of 26 by 74?

26	log.	1·414973
74	log.	1·869232
product 1924	log.	3·284205

2. What is the product of 1·447 and 1·375?

1·447	log.	0·160169
1·375	log.	0·138303
product 1·98963	log.	0·298772

3. What is the product of ·0054 and ·95?

·0054	log.	— 3·732394
·95	log.	— 1·977724
product ·00513	log.	— 3·710118

In this addition I carried to the indices cancells *negative* 1; and the index in the sum is — 3.

182. But to avoid the use of negative indices when one or more of the factors are decimals, multiply such factor or factors by 10, 100, or 1000, &c. so as to make the product or products whole or mixt numbers, then having added the logarithms of those products together, divide the corresponding number by the like 10, 100, or 1000, &c. for the answer.

Thus, taking the last example:

·0054 × 1000 = 5·4	log.	0·732394
·95 × 10 = 9·5	log.	0·977724
		1·710118 the log. of 51·3 which

is evidently 1000 × 10 times too great; therefore 51·3 divided by 10000 gives ·00513 the product as before.

Division by Logarithms.

183. Subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder is the logarithm of the quotient. (169)

Examples.

1. Divide 1416 by 59.

1416	log.	3·151063
59	log.	1·770852
quotient 24	log.	1·380211

2. Divide 25100 by 1997.

25100	log.	4·399674
1997	log.	3·300378
quotient 12·5688	log.	1·099296

3. Divide ·04271 by ·8799.

·04271	log. —	2·630530
·8799	log. —	1·944433
quotient ·0485396	log. —	2·686097

184. But if we proceed as in the 3d. example of multiplication, remembering always to make the dividend greater than the divisor, the operation may be performed without the negative indices:

Thus, taking the last example;

·04271 × 1000	=	42·71	log.	1·630530
·8799 × 10	=	8·799	log.	0·944433
		4·85396	log.	0·686097

But this quotient 4·85396 is 1000 times *too great* on account of the dividend, and 10 times *too little* because the divisor was multiplied by 10, therefore it must be 100 times *too great*; consequently the quotient is ·0485396.

And in like manner we may avoid the negative index in all cases when the divisor is greater than the dividend.

To work a proportion by Logarithms.

185. Subtract the logarithm of the divisor from the sum of the logarithms of the other two terms, and the remainder will evidently be the logarithm of the 4*th.* term or number sought.

Example 1. Required a 4*th*. proportional to 4628, 978, and 1798?

As 4628	log.	3·665393
is to 978	log.	2·990339
so is 1798	log.	3·254790
		6·245129
		3·665393
to 379·958	log.	2·579736

186. But instead of subtracting the log. of the first term, it will be found more expeditious to add its *arithmetical complement*:

Thus, 4628	log.	3·665393	
		6·334607	the *arithmetical complement*:
978	log.	2·990339	
1798	log.	3·254790	
379·958	log.	2·579736	as before.

The arithmetical complement of any number is the difference between that number and 1 with as many ciphers annexed as there are figures in the number; thus the arithmetical complement of 57 is 43, which is the difference of 57 and 100; and therefore adding 43 to any number, and subtracting 100 from the sum, must give the same difference as when 57 is taken from that number; for by adding 43 instead of subtracting 57, we get 100 too much.

Thus the log. 3·665393 is taken from 10·000000, whence the sum becomes 12·579736, but as this is 10·000000 too much, the 10 is omitted in the index.

The easiest method of subtracting for the arithmetical complement is to begin at the left-hand and take each figure from 9, except the last figure on the right, which must be subtracted from 10.

Therefore in Division, instead of subtracting the logarithms of the divisors, add their arithmetical complements, and reject 10 in the sum of the indices for each arithmetical complement, and the result will be the logarithm of the quotient.

2. Required a 4th. proportional to the fractions $\frac{596}{1192}$, $\frac{749}{3745}$, and $\frac{8022}{1146}$?

As $\frac{596}{1192} : \frac{749}{3745} :: \frac{8022}{1146} : \frac{1192 \times 749 \times 8022}{596 \times 3745 \times 1146}$ the 4th. term in a compound fraction.

1192	.... log.	3·076276
749	.... log.	2·874482
8022	.... log.	3·904283
596	*arith. comp.* of the log.	7·224754
3745	*arith. comp.* ...... log.	6·426548
1146	*arith. comp.* ...... log.	6·940815
4th. term required 2·8	log.	0·447158

Here 3 tens or 30 is rejected in the sum of the indices for the 3 arithmetical complements; and the result is the log. of 2·8, or of $\frac{14}{5}$ which is the compound fraction reduced to its lowest terms.

For the log. of 14 is 1·146128
of 5 is 0·698970
0·447158 log. of $\frac{14}{5}$.

3. Suppose the result of a proportion is the compound fraction $\frac{19455}{847} \times \frac{1}{9474}$; what is its value?

847, 9474	*arith. comp.* logs.	7·072117, 6·023467
19455	.... log.	4·289031
		17·384615

Here 2 tens should be cancelled in the sum of the indices for the two arithmetical complements, but 17 is 3 short of 2 tens, therefore the index will be 3 with a negative sign,

thus — 3·384615.

The number, to 5 places, answering to the logarithm (without the index) is 24245; but the index — 3 shews that it must be 3 places below 1, (1·9), therefore ·0024245 is the value required, true to the last decimal.

4. Required a 4th. proportional to the three decimals ·14275, ·07468, and ·001278?

·14275 × 10	=	1·4275	*arith. comp.* log.	9·845424
·07468 × 100	=	7·468	.... log.	0·873204
·001278 × 1000	=	1·278	.... log.	0·106531
		6·6859	log.	0·825159

But the result 6·6859 is 100 × 1000 times *too great* on account of the multipliers, and 10 times *too little* because the divisor was increased 10 times (184), consequently it must be 100 × 100 or 10000 times *too great*; therefore 6·6859 divided by 10000 gives ·00066859 the 4*th*. proportional required.

Or, making use of the negative indices:

·14273 log.	—	$\bar{1}$·154576	
		10·845424	*arith. comp.*
·07468 log.	—	$\bar{2}$·873204	
·001278 log.	—	$\bar{3}$·106531	
Ans. nearly ·00066859 log.	—	$\bar{4}$·825159	

In taking the arithmetical complement of the 1*st*. term, the *negative* index 1 must be added to 9 instead of subtracted.—And the sum of the indices (with the *positive* 1 carried) make 6 *positive*, but 10 should be rejected in the sum on account of the arithmetical complement, therefore the index in the sum will be *negative* 4.

Involution by Logarithms.

187. MULTIPLY the logarithm of the number whose power is required by the index of the power, and the product is the logarithm of the power required. (161)

Examples.

1. What is the cube or 3*d*. power of 170?

170 log.	2·230449	
	3	*index.*
Ans. 4913000 log.	6·691347	

2. What is the 4*th*. power of the decimal ·7867?

To avoid the negative index, multiply the decimal by 10 and divide the 4*th*. power of the product by the 4*th*. power of 10.

·7867 × 10 = 7·867 log.	0·895809	
	4	
	3·583236	log. of 3830·3.

Which divided by 10000 (the 4*th*. power of 10) gives ·38303 the required power, true to 5 decimals.

Or thus:

$$\begin{array}{rl} \cdot 7867 \text{ log.} - & \bar{1}\cdot 895809 \\ & 4 \\ \textit{Ans.}\ \cdot 38303 \text{ log.} - & \overline{\bar{1}\cdot 583236} \end{array}$$

Here 3 carried to *negative* 4 make 1 *negative* the index.

3. What is the amount of £60 in 50 years at 5 *per cent. per ann.* compound interest?

It is evident from *Ex.* 1, *art.* 107, that $60 \times 1\cdot05 \times 1\cdot05 \times 1\cdot05$ &c. or $60 \times 1\cdot05^{50}$ is the amount.

$$\begin{array}{rl} 1\cdot05 \text{ log.} & 0\cdot021189 \\ & 50 \ \textit{index.} \\ & \overline{1\cdot059450} \text{ log. of } 1\cdot05^{50} \\ 60 \ldots\ldots \text{ log.} & 1\cdot778151 \\ \text{Amount } £688\cdot02 \text{ log.} & \overline{2\cdot837601} \end{array}$$

4. If in the last example, the interest is payable half-yearly, what would be the amount in the same time?

Here the amount of £1 in half a year will be £1·025.

Therefore $60 \times 1\cdot025^{100}$ is the amount.

$$\begin{array}{rl} 1\cdot025 \text{ log.} & 0\cdot010724 \\ & 100 \\ & \overline{1\cdot072400} \\ 60 \ldots\ldots \text{ log.} & 1\cdot778151 \\ \text{Amount } £708\cdot87 \text{ log.} & \overline{2\cdot850551} \end{array}$$

Evolution or Extraction of Roots by Logarithms.

188. Divide the logarithm of the number whose root is required by the index denoting the root, and the quotient will be the logarithm of the root. (187)

Examples.

1. What is the square root of 7569.

$$\begin{array}{rl} \text{Index } 2\,)\ 3\cdot879039 & \ldots\ldots\ldots \text{ log. of } 7569. \\ \overline{1\cdot939519} & \ldots\ldots\ldots \text{ log. of } 87 \text{ the } \textit{root.} \end{array}$$

2. Required the cube root of 10.

$$\begin{array}{rl} 3\,)\ 1\cdot000000 & \ldots\ldots\ldots \text{ log. of } 10. \\ \overline{0\cdot333333} & \ldots\ldots\ldots \text{ log. of } 2\cdot15443 \ \textit{root} \text{ nearly.} \end{array}$$

3. What is the 4th root of ·38303, (see examp. 2, preceding art.).

4) — 1·583236 .... log. of ·38303
— 1·895509 .... log. of ·7867 root nearly.

Here the operation is the reverse of that in the example referred to, and therefore in making the division by the exponent 4, we add 3 (the number carried in raising the power) to the index 1 so as to make the sum just divisible by 4, and the 3 is considered as so many tens added to the next figure on the right; hence the dividend will be — 4·3583236 which divided by 4 gives the log. of the root.—But if the cube root were required, 2 must be added to make the sum just divisible by the exponent 3, and the dividend becomes — 3· 2583236, the 3*d*. of which is — 1· 861079 the log. of the 3*d*. root, &c.

Or thus, (without the negative index).

·38303 × 10^4 = 3830·3 ...log. 4) 3. 583236
0. 895809 log. of 7·867 which divided by 10 the 4th root of 10^4 gives ·7867 the *root* as before.

4. What is the square root of the compound fraction $\frac{6421}{6547} \times \frac{9177}{9088}$?

6421..........................log. 3.807603
9177..........................log. 3·962701
6547 } *arith. comp. logs.* { 6·183958
9088 } { 6·041532
2)—1·995794
—1·997897 log. of ·99517 *root nearly.*

5. The diameter of a 9*lb*. iron shot being 4 *inches*; then what is the diameter of a 48*lb*. ball; the weights being as the cubes of the diameters?

As 9*lb*. : 4^3 :: 48*lb*. : $\frac{64 \times 48}{9}$ the cube of the diameter.

64 .... log. 1·806180
48 .... log. 1·681241
9 *arith. comp.* log. 9·045757
3)2·533178
Diam. nearly 6·99*in*. log. 0·844393

6. What is the diameter of a lead musket ball whose weight is 1 ounce? (See Examp. 4, *art*. 419, vol. 2.)

$$1 \times \cdot 2914 = \cdot 2914 \ldots\ldots \text{log.} \quad 3)\,\bar{1}\cdot 464490$$

$$\bar{1}\cdot 821497 \text{ log. of } \cdot 663 \text{ of an } inch.$$

the diameter nearly.

7. Required the geometrical mean proportional between 81 and 6561? (151.)

6561	log.	3·816970
81	log.	1·908485
		2)5·725455
Ans. 729	log.	2·862727.

And the three terms are 81, 729, 6561.

For 81 : 729 :: 729 : 6561. But the square roots are also proportional (139); viz. 9 : 27 :: 27 : 81, whence 27 × 27 = 9 × 81. Therefore the mean proportional is the product of the square roots of the two extremes.

8. Required 3 mean proportionals between 81 and 6561? (150.)

6561	log.	3·816970	
81	log.	1·908485	
		4)1·908485	
		0·477121	log. of 3 the *ratio or multiplier*.

Therefore the 3 means are 81 × 3 = 243
81 × 9 = 729
81 × 27 = 2187

And the 5 terms are 81, 243, 729, 2187, 6561.

9. To find 4 geometrical means between 2 and 10.

10	log.	1·000000	
2	log.	0·301030	
	5)	0·698970	
		0·139794	log. of the ratio or multiplier.
2	log.	0·301030	
		0·440824	log. 2·7595 the 1st.
		0·139794	
		0·580618	log. 3·8073 the 2d.
		0·139794	
		0·720412	log. 5·2531 the 3d.
		0·139794	
		0·860206	log. 7·2478 the 4th.

Examples of Fractional Powers and Roots.

1. What is the ⅔ power of 4096, or the cube root of the square of 4096, or the number answering to $4096^{\frac{2}{3}}$?

4096 log 3·612360
2
3) 7·224720 log. of the square of 4096.
Ans. 256 log. 2·408240 log. of the cube root of that square.

2. What is the ·4 power of 1000?

1000....log. 3·000000
·4
Ans. nearly 15.85 log. 1·200000

3. Required the $4\frac{1}{2}$ power of 0·98?

In this and similar cases, it is best to take the *power*, or the *root*, of the reciprocal of the proposed fraction, and then the reciprocal of that power, or root, will be the answer:

Thus the reciprocal of ·98 or of $\frac{98}{100}$ is $\frac{100}{98}$.

$\frac{100}{98}$....log. 0·008774
4·5
43870
35096
0·0394830 log. of the $4\frac{1}{2}$ power of the reciprocal.
$-$1·9605170 log. of ·9131 the required *power*.

The log. — 1· 960517 is found by subtracting the log. 0·039483 from the log. of 1.

4. What is the ·079 power of ·079?

$\frac{1000}{79}$....log. 1·102373
·079
9921357
7716611
0·087087467
$-$1·912912533 log. of ·8183 nearly, *Ans.*

5. What is the 0·75 root of 2?

·75) 0·301030....log. of 2.
0·401373 ...log. of 2.5198 *Ans.*

This however, is exactly the same thing as finding the 3*d.* root of the 4*th.* power of 2 (because ·75 = $\frac{3}{4}$), and therefore 2·5198 is the number denoted by $2^{\frac{4}{3}}$.

6. Required the $31\frac{1}{2}$ root of 0·8?

96910.

log. of the 31⅓ root of the reciprocal.

log. of ·9929 *root required.*

0754?

1·122623.

?

log. of ·018743 *root nearly.*

r Examples.

oduct of [illegible], [illegible], and [illegible]?

Ans. ·000000224034 &c.

oduct of [illegible], [illegible], and [illegible]?

Ans.

roduct of ·76141, ·01779, and 1·999?

Ans. 0·27077 *nearly.*

roduct of 1·593, ·007655, and 01198?

Ans.

..............................*Quot.* ·023413 &c.

..............................*Quotient.*

..............................*Quotient* ·65072 *nearly.*

..............................*Quot.*

..............................*Quot.*

nal to [illegible] and [illegible]?

Ans. ·400185 &c.

al to [illegible] and [illegible]?

Ans.

al to ·3777 and ·9777?

Ans.

nal to ·9876 and ·9786?

14. Required a 4*th.* proportional to ·07655, ·1531, and ·15791?

Ans. ·31582

15..............a 4*th.* proportional to ·3777, ·2987, and ·09876?

Ans.

16. Required a 4*th.* proportional to [illegible], [illegible], and [illegible]?

Ans.

17. What is the geometrical mean between ·07414 and ·7414?

Ans. ·23445 &c.

18..between [illegible] and [illegible]?

Ans.

19. Required 2 geometrical means between 3 and 3000?

Ans.

20. Required the cube root of [illegible]?

Ans. ·046417 *nearly.*

21. Required a 4*th.* proportional to the cube roots of 17, 19, and 21?

Ans.

22. What is the 100*th.* root of 10?

Ans. 1,02329 *nearly.*

23. To what power must 10 be raised to produce 700?

Ans.

24. Required the $\frac{1}{3}$ root of $\frac{1}{3}$?..............................*Ans.* ·037037 *nearly.*

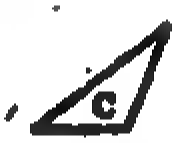
25. What is the [illegible] root of [illegible]?..*Ans.*

. When the number of sides are three, the figure is angle.

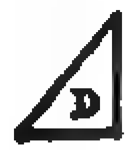
. An equilateral triangle is that whose sides all equal, as A.

. An isosceles triangle is that which has only sides equal, as B.

. A scalene triangle is when all the three sides unequal, as C.

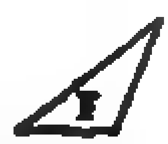


. A right angled triangle is that which has one t angle, as D.



. An acute angled triangle has all its angles e, as E.

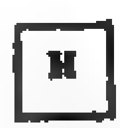
. An obtuse angled triangle has one obtuse le, as F.



. Every plane figure bounded by four right lines is ed a quadrangle or quadrilateral. And when the opposite s are respectively parallel, the quadrilateral is called a paral- gram.

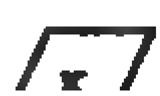
. A rectangle is a parallelogram having all its les right ones, as G.

. A square is a parallelogram having all its sides al, and all its angles right ones, as H.

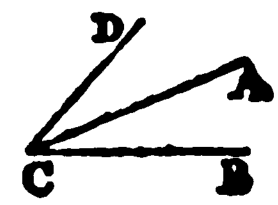


. A rhomboid is an oblique angled parallelo- m, as I.

. A rhombus is an equilateral rhomboid

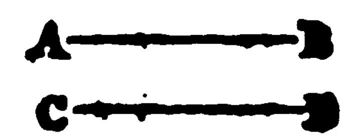


An angle is usually denoted by three letters, the middle letter being that at the angular point. Thus, the angle formed by the lines AC, BC is the angle ACB. And the angle formed by the lines DC, BC, is the angle DCB. Therefore the magnitude or opening of an angle is not dependant on the lengths of the lines which include or make the angle: thus, DC is less than AC, but the angle DCB is greater than the angle ACB; or the inclination of the line DC to BC is greater than the inclination of AC to BC.

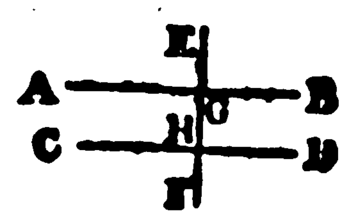


Scholium. From the foregoing definition of an angle, it follows, that if two straight lines in the same plane are not inclined to each other, they cannot form an angle, and consequently can never be produced so as to meet, in which case the lines are said to be parallel: Therefore,

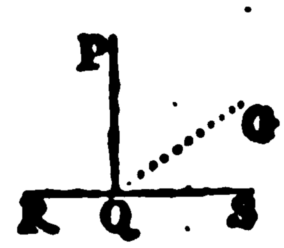
9. Parallel straight lines are such as are in the same plane but not inclined to each other, or when indefinitely produced both ways do never meet, as AB, CD.



So if two straight lines AB, CD intersect a third straight line EF (all in the same plane) and are equally inclined to that line, or make the angles AGE, CHE equal, the two lines have no inclination to one another, but are parallel or equidistant; and when all the angles at G and H are equal to each other, the line GH is the distance of those parallels.



10. A right angle is formed by two lines which are perpendicular to each other. Thus if PQ is perpendicular to RS, each of the angles PQR, PQS is a right angle.



11. An acute angle is less than a right angle; as the angle GQS.

12. An obtuse angle is greater than a right angle; as the angle GQR. Those are called oblique angles.

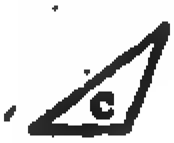
13. The sides of a right lined plane figure are straight lines.

ber of sides are three, the figure is

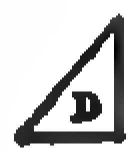
triangle is that whose sides

ngle is that which has only

le is when all the three sides

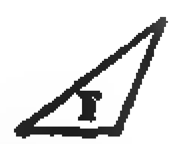


riangle is that which has one



d triangle has all its angles

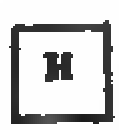
ed triangle has one obtuse



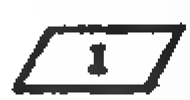
gure bounded by four right lines is
quadrilateral. And when the opposite
allel, the quadrilateral is called a paral-

parallelogram having all its
.

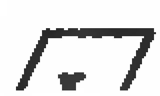
arallelogram having all its sides
right ones, as H.



an oblique angled parallelo-



s an equilateral rhomboid



26. A trapezoid is a quadrila
parallel sides, as L.

27. A trapezium is a quadril
of the sides are parallel, as M.

28. A right line joining
angles of a quadrilateral is call
NO.

29. The side PQ upon whi
lelogram PQRS, or triangle P
supposed to stand, is called the
perpendicular ST, falling ther
opposite angle at S, is called the
titude of the parallelogram or trian

The perpendicular ST is
point S from the line PQ,
S and PQ.

30. All right lined plane fig
are generally called polygons.
the angles as well as sides ar

Ax

31. THINGS which are eq
things, are equal to each other

32. If equals are added to

33. If equals are subtract
equal.

34. Every whole is equal t

35. Things which are the
equal.

[illegible] is
[illegible] ly
[illegible] be
[illegible] a
[illegible] ing
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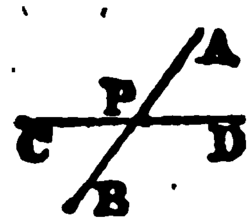
[illegible] to
[illegible] we
[illegible] we
[illegible] b
[illegible] *all*
[illegible] a
[illegible] pe-

j [illegible]
a, [illegible]
the [illegible]
an [illegible]
ou [illegible]
p, [illegible]
e, [illegible]
an [illegible]
gl [illegible]
an [illegible]
poi [illegible]
ep [illegible]
the [illegible]
rig [illegible]

Corol. 2. And if two angles DPB, DPA on both sides of the line DP are together equal to two right angles, then the sides PB, PA make one continued line.

39. *If two right lines intersect each other, the opposite angles will be equal.*

Let AB intersect CD in the point P. Then will the angle APD be equal to the angle BPC; and the angle APC equal to the angle BPD.

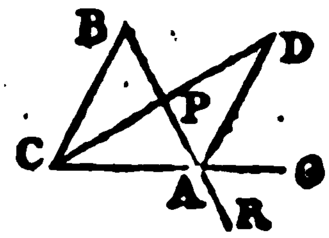


This might have been admitted as an axiom. For since all the parts of a straight line lie in the same direction, the segments PD, PB must have the same inclination to one another as the segments PC, PA on the other side of the point of intersection; consequently those parts form equal angles.—It is however, usually demonstrated thus:

Because the angles APC, APD are together equal to two right angles, and also the angles APC, BPC together equal to two right ones, (38), if the common angle APC be taken from each of those equal sums, there will remain the angle APD equal to the angle BPC, (33). In the same manner it is proved that the angles APC, BPD are also equal.

39*. *If one side* (CA) *of a triangle* (CBA) *be produced, the exterior or outward angle* (BAG) *will be greater than either of the interior opposite angles* (ACB, ABC).

Suppose CD is drawn to bisect AB, and that PD = PC, and the points A, D, are joined.



Because the sides PB, PC, of the triangle PBC, are respectively equal to the sides PA, PD of the triangle PAD, and the included angles at P also equal (39), the two triangles are identical (38), and therefore the angles opposite the equal sides PD, PC are equal, that is, the angle PAD = PBC; but the angle PAG or BAG is greater than the angle PAD, and therefore greater than its equal PBC

BC. In the same manner,
it may be proved that the
greater than ACB.

If two straight lines in th
ight line, and make the alt
s are parallel.

et the lines AB, CD, inte
e the alternate angles APS,
other; then AB is parallel

or if it be not parallel, the lin
her, and will meet when pro
ourse; then RPO is a tri
or QRD is greater than th
),but it is also equal to it (by
; therefore the lines when
of QS: and in the same ma
not meet when produced on t
are parallel.

nd the converse is equally ob
(QS) *intersect two parallel*
les (APS, QRD) *will be equ*

rol. 1. Two parallel lines ca
t.

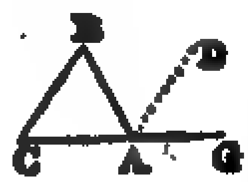
rol. 2. Because the angles A
right angles, and QRD, I
es, the two angles BRP, DR
es; therefore if two straight
e, meet another straight line
es (BPR, DRP) together eq
lines are parallel.

rol. 3. Hence also, if a

arallel straight lines, in given angles, it will intersect r lines in the same angles.

If one side of a triangle be produced, the exterior or d angle, wi' be equal to both the interior opposite and the three interior angles of the triangle are r equal to two right ones.

he side CA of the triangle CBA be pro- G. Then the exterior angle GAB will l' to both the interior opposite angles ACB; and the angles ABC, ACB, together make two right angles.



r AD parallel to CB.

because AD is parallel to CB, the angle DAG is equal ngle ACB (40).

the angle DAB is equal to the angle ABC (40).

he two angles DAG, DAB together constitute the out- gle GAB.

efore (31) the exterior angle GAB is equal to both the ABC, ACB.

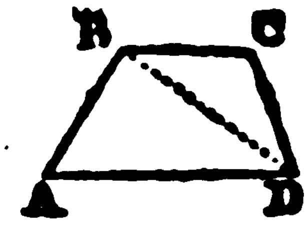
since the three angles DAG, DAB, BAC, together o right angles (38), and are respectively the same as the gles of the triangle CBA; therefore the sum of the gles of a plane triangle is equal to two right angles.

. 1. Hence the difference between an exterior angle of le and either of the interior opposite angles, is equal to the terior opposite angle.

. 2. Hence also, if one angle of a triangle be a right the sum of the other two make a right one.

The four inward angles of every right lined quadri- are together equal to four right angles.

Let ABCD be a quadrilateral. Then the sum of the angles at A, B, C, D, will be equal to four right angles.



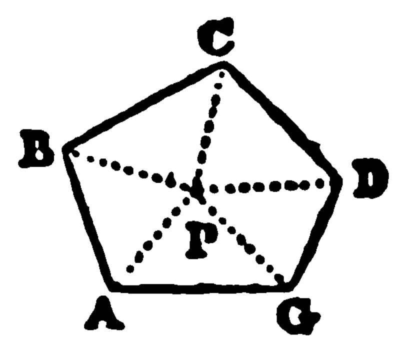
Draw the diagonal BD, which will divide the quadrilateral into two triangles BCD, BAD.

Then because the angles of those two triangles make up the four angles of the quadrilateral, and the sum of the angles of both the triangles are equal to four right angles (41), therefore the angles of the quadrilateral are together equal to four right angles.

Corol. Hence if two angles of a quadrilateral make two right angles, the sum of the other two will also be equal to two right angles.

43. *The sum of all the interior angles of any polygon is equal to twice as many right angles, wanting four, as the figure has sides.*

Let ABCDG be a polygon of 5 sides.



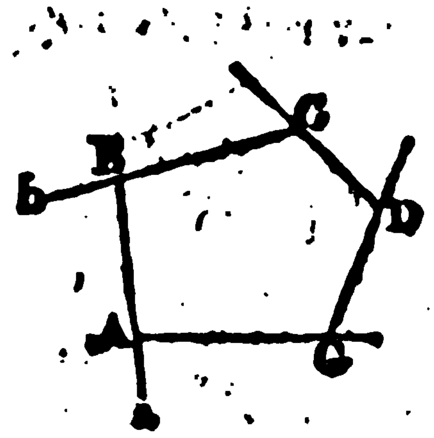
Then the sum of the angles at A, B, C, D, G, will be equal to six right angles.

From any point P in the polygon let right lines be drawn to the angles of the figure, which will divide it into as many triangles as the figure has sides.

Now all the angles of the triangles are together equal to twice as many right angles as there are triangles, or as the polygon has sides.

But the angles of the triangles, exclusive of the angles at P, which make four right angles (41), constitute the interior angles of the polygon, and therefore those angles together are equal to twice as many right angles, wanting four, as the polygon has sides.

44. *The sum of the exterior angles (aAG, bBA, &c.) of any polygon, are equal to four right angles.*

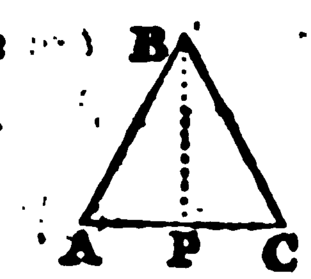


Since the interior and exterior angles at each angular point of the polygon make two right angles (38, *corol.* 1), all the interior and exterior angles must together make twice as many right angles as the figure has angles or sides.

But the sum of all the interior angles are equal to twice as many right angles, wanting four, as the figure has sides (43).

Therefore the difference of those sums, or four right angles, is the sum of the exterior angles.

46. *The angles opposite the equal sides of an isosceles triangle are also equal.*



If ABC be an isosceles triangle, having the side BA equal to the side BC. Then the angles at A and C are equal.

Suppose the angle ABC to be bisected by the line BP. Then because BA = BC, and the angle ABP = the angle CBP, and the side BP common to both the triangles APB, CBP, those triangles will therefore be identical or equal in all respects (38), and consequently will have the angles at A and C equal.

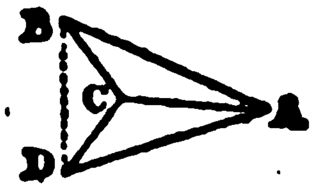
Corol. 1. Hence the line (BP) which bisects the vertical angle (ABC) of an isosceles triangle, bisects the base (AC), and is also perpendicular to it.

Corol. 2. And if two angles of a triangle be equal, the sides subtending those angles will also be equal.

Corol. 3. Hence also, every equilateral triangle is likewise equiangular.

46*. *If the sides of one triangle* (ACB) *be equal to the sides of another triangle* (ACD), *each to each; the angles opposite the like sides are also respectively equal.*

The truth of this seems sufficiently evident from Art. 38. It is however, demonstrated thus:



Let a side AC of one triangle coincide with the equal side AC of the other: then AB = AD, and CB = CD.

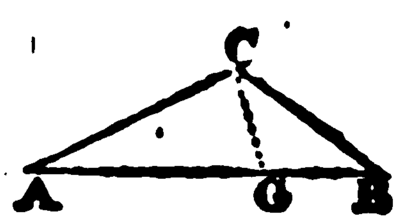
Draw BD. Then because AB = AD, and CB = CD, the triangles ABD and CBD are isosceles, and the angle ABD = ADB, and the angle CBD = CDB (46):

Now if the equal angles CBD, CDB are taken from the equal angles ABD, ADB, the two remainders or the angles ABC, ADC, must also be equal (33):

Therefore the sides CB, BA, and the included angle of one triangle, being respectively equal to the sides CD, DA, and the included angle of the other, the two triangles are identical (38); therefore the angle DCA = BCA, and the angle BAC = DAC.

47. *In any triangle* (ABC) *the greatest angle* (ACB) *is subtended by, or is opposite the longest side* (AB).

Make AG = AC, and draw CG. Then because AG = AC, the triangle GAC is isosceles, and the angles ACG, AGC are equal (46):



But the exterior angle AGC of the triangle GBC is equal to both the angles GBC, GCB (41):

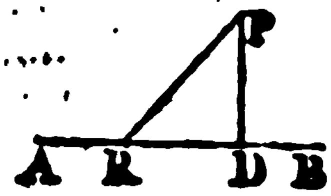
Therefore the angle ACG (equal to AGC) which is only a part of the angle ACB, exceeds the angle B; consequently the whole angle ACB is greater than B.

Corol. Hence the longest side of a triangle is opposite the

greatest angle; *for* it is proved that ACB cannot be greater than B, except AB is longer than AC.

48. *The shortest line which can be drawn from a given point* (P) *to an indefinite line* (AB) *is that right line* (PD) *which is perpendicular to it.*

Suppose PD is perpendicular to AB: then any other line, as PR, drawn from P to meet AB will be longer than PD.



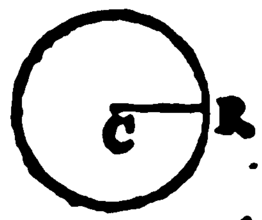
For the right angle RDP of the triangle RDP is greater than the angle PRD, because the latter with the angle P are together equal to a right angle (41, *corol.* 2), therefore PD is less than PR (47, *corol.*).

OF THE CIRCLE.

DEFINITIONS.

49. A CIRCLE is a plane figure bounded by one curve line called its circumference, which is every where equally distant from a point within it called the centre.

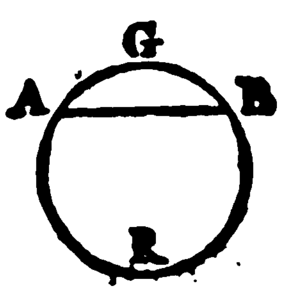
50. The radius of a circle is the distance of the centre from the circumference. Thus if C be the centre, CR is the radius.



51. The diameter of a circle is a right line drawn through the centre, and terminated by the circumference both ways, and therefore it is twice the radius.

52. An arc of a circle is any part of the circumference.

53. The chord or subtense of an arc AGB, or ARB, is a right line AB joining the extremities of that arc.



54. A segment is any part
chord; as the segment AB

55. A semicircle is half th
diameter. Half the circumf
cle.

56. A sector is any part of
radii drawn to its extremit

Thus if C be the centre, AC

When the angle at C is rig
etimes the arc AB) is calle

57. When two right lines
emity of a chord AB, and
e arc ADB, the angle ACB
ence) is said to be in the seg
and on the chord AB, or

58. A right line is said to
ugh a point in the circum
of the circle.

This line is also called a tang

59. A secant is a right line
of a circle.

60. Two circles are said to
ferences of both pass throug
ing each other.

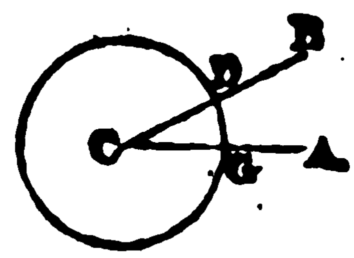
61. When all the angular p
he circumference of a circle,
le; and the circle is said to c

62. A right lined figure circ

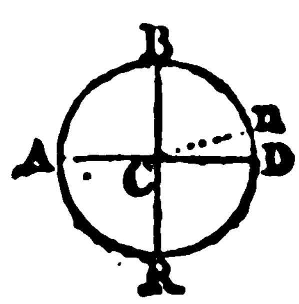
sides touch the circumference of the circle; and the circle is said to be inscribed in the figure.

63. The perimeter of a figure is the sum of all its sides taken together.

64. When two right lines AC, BC, form an angle ACB, and a circle is described about the angular point C as the centre, the arc GD intercepted by those lines is the measure of the angle ACB, the whole circumference of the circle being the measure of four right angles.



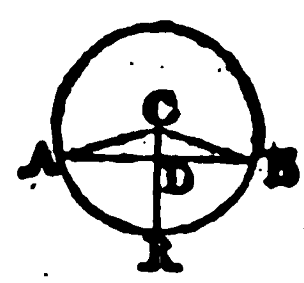
To estimate the opening or magnitude of an angle, the circumference of the circle is supposed to be divided into 360 equal parts called degrees, and each of those degrees into 60 equal parts called minutes, and each minute into 60 seconds, &c. This is called the sexagesimal division. Thus if the circumference ABDR is divided into 360 equal parts or degrees, and the diameters AD, BR intersect each other at right angles, the points A, B, D, R, will divide the circumference into 4 equal arcs of 90 degrees each; and each of the 4 angles at the centre C is said to be an angle of 90 degrees.



If the arc D*n* is $\frac{1}{16}$ of the whole circumference or $\frac{1}{4}$ of the arc DB, the angle DC*n* will be $22\frac{1}{2}$ degrees.

THEOREMS.

65. *If the radius of a circle bisects any chord, it will be at right angles to it, and the arc of that chord will also be bisected by the same radius.*



Let C be the centre of the circle, and AB a chord; then if the radius CR bisects the chord in the point D, CD will be perpendicular to AB; and the arc AR equal to the arc RB.

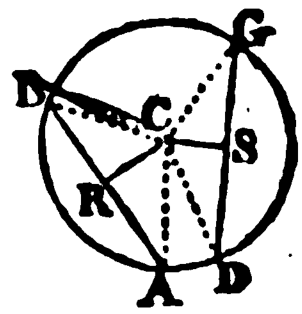
Draw CA and CB. Then because CA is equal to CB, the

triangle ACB is isosceles, and therefore (46, *corol.* 1.) CD bisects the angle ACB, and is perpendicular to AB.

And because the arcs AR, BR are the measures of the equal angles ACR, BCR (64), they must therefore be equal to each other.

Corol. Hence a right line which bisects any chord at right angles, will pass through the centre of the circle.

66. *In a circle, equal chords are equally distant from the centre.*



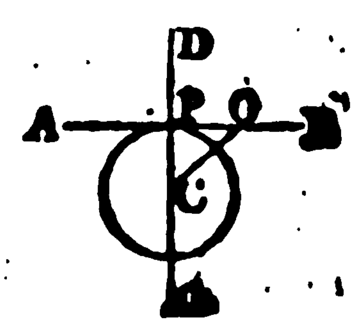
Let AB, GD be two equal chords in the circle whose centre is C; then the perpendiculars CR, CS drawn from the centre C will be equal.

Draw the radii CB, CA, CD, CG: then those radii being equal, and BA equal to GD; the triangles BCA, GCD will be identical, or equal in all respects (46[a]); and because they are isosceles, the perpendiculars CR, CS will bisect BA, GD (46, *cor.* 1); hence the triangles RCB, RCA, SCD, SCG are identical, therefore CR = CS.

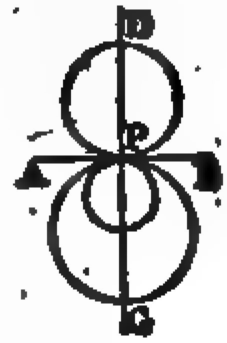
Corol. Chords in a circle equally distant from the centre are equal to each other.

67. *If two right lines* AB, DG *intersect each other at right angles in* P; *then if any circle, whose centre* C *is in the line* DG, *be described through the point of intersection* P, *it will touch the other line* AB *in that point.*

Draw CO to any point in PB. Then CO being greater than CP (48), the point O must necessarily fall without the circle; and as the same reasoning holds good with respect to every other point in PB or PA, it is evident that AB cuts off no part of the circle, but touches it at P.



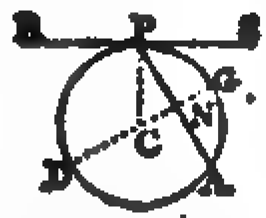
Corol. 1. Hence the angle formed by a tangent to a circle and the radius drawn to the point of contact, is a right angle.



Corol. 2. Hence also, it appears that any number of circles described through P, will touch each other in that point if their centres are in the line DG. And that AB is a tangent to them all.

Corol 3. Therefore if two circles touch inwardly or outwardly, their centres and the point of contact are in the same right line.

68. *The angle formed by a tangent and a chord drawn from the point of contact, is measured by half the arc of that chord.*



Let RS be a tangent to the circle whose centre is C; and PA a chord drawn from the point of contact P. Then the measure of the angle SPA is half the arc PGA; and the measure of the angle RPA is half the arc PDA: that is, if a circle were described about the centre P with the radius CP or CG, the arc intercepted by PS and PA would be equal to half the arc PGA, and the arc intercepted by PR and PA equal to half the arc PDA.

For let the diameter DCG be drawn to bisect the chord PA, and join CP.

Then CNP is a right angle (65); and the angles RPC, SPC are also both right angles (67, *corol.* 1).

Now in the right angled triangle CNP, the sum of the two acute angles NCP, CPN, is equal to a right angle. (41, *corol.* 2).

But the latter angle CPN together with the angle APS also make a right angle CPS.

Therefore the angle APS is equal to NCP (33). And since the arc PG (half of PGA) is the measure of the angle PCN, it must also be the measure of its equal APS.

Again, the external angle DCP of the triangle CNP is equal to both the inward opposite angles, or to the angle CPN and a right angle CNP (41).

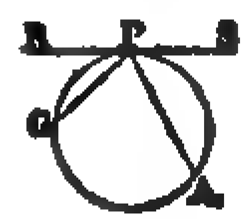
And the angle RPA is also equal to the same angle CPN and a right angle RPC.

Therefore the angles RPN, DCP are equal. And since the arc PD (half of PDA) is the measure of the angle DCP, it is also the measure of its equal RPA.

Corol. Because the arcs GP, PD together make half the circumference, and the sum of the two angles RPA, SPA equal to two right angles, therefore the sum of two right angles is measured by half the circumference.

69. *The angle at the circumference of a circle is measured by half the arc that subtends it.*

Let GPA be an angle at the circumference. Then half the arc GA is the measure of that angle.



Suppose RS is a tangent to the circle at P.

Then the sum of the three angles at P, or two right angles, is measured by half the circumference of the circle (68, *corol.*).

But half the circumference is half the arcs PG, GA, AP added together.

Now the angle RPG is measured by half the arc PG: and the angle SPA by half the arc AP (68):

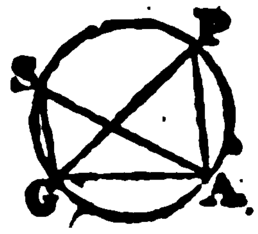
Take those two angles from the three angles at P, and there remains the angle GPA:

And take the measures of those two angles, or half the arcs

P, from half the circumference, and there remains half GA for the measure of the remaining angle GPA.

All angles in the same segment of a circle, or stand- the same arc, are equal to each other.

SA, GPA be two angles standing on the c GA. Then will those angles be equal other.

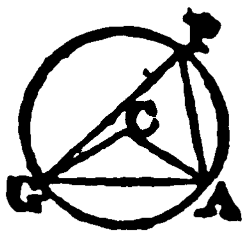


ach of those angles is measured by half the arc GA (69), sequently they must be equal.

. Hence equal chords in a circle, subtend equal angles rcumference.

The angle at the centre of a circle is double the it the circumference when both of them stand on the c.

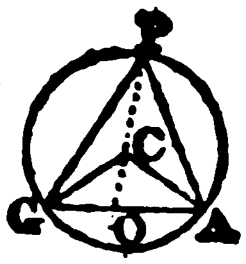
AC be an angle at the centre, and GPA at the circumference. Then the angle double the angle GPA.



CA is measured by the arc GA; and the angle GPA ured by half that arc (69), therefore the angle GCA double GPA.

Otherwise thus:

O be drawn through the centre C.

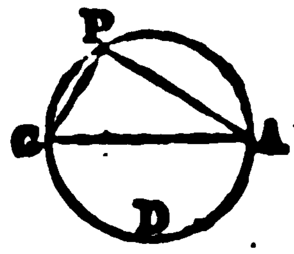


the triangles GCP, ACP being isosceles, CGP will be equal to the angle CPG; angle CAP equal to CPA (46).

ecause the external angle GCO is equal to both the pposite angles CGP, CGP (41), it is therefore equal le the angle CPG. And for the same reason, the angle ACO is double the angle CPA: therefore the ngle GCA is double the whole angle GPA.

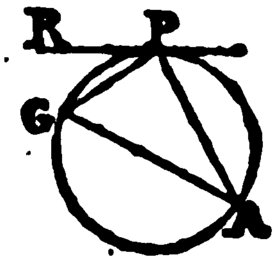
72. *The angle* GPA *in a semicircle is a right angle.*

For it is measured by half the arc GDA or half a semicircle (69), but half a semicircle is the measure of a right angle (64).



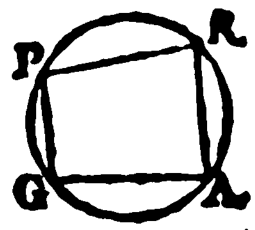
73. *The angle* RPG *formed by the tangent* RP *and the chord* PG, *is equal to the alternate angle* PAG *standing on the sume chord* PG.

For the angle RPG is measured by half the arc PG (68); and the angle PAG is measured by half the same arc (69); therefore those angles must be equal.



74. *The opposite angles of any quadrangle inscribed in a circle are together equal to two right angles.*

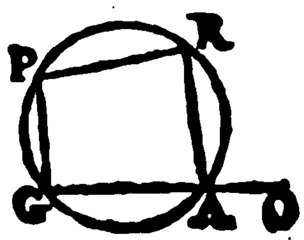
For the angle P is measured by half the arc GAR; and the angle A by half the arc GPR; therefore the sum of both angles must be measured by half the sum of both arcs, or by half the circumference.



But half the circumference is the measure of two right angles; consequently the opposite angles together are equal to two right angles.

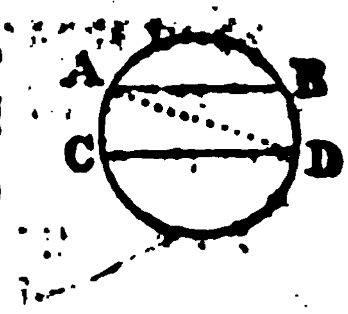
75. *If a side* GA *of a quadrangle inscribed in a circle be produced, the exterior angle* OAR *will be equal to the inward opposite angle* GPR.

For the angle GAR with its opposite angle GPR together make two right angles (74); and the same angle GAR with the exterior angle OAR make two right angles; therefore by equal subtraction, the angle OAR is equal to the angle GPR.



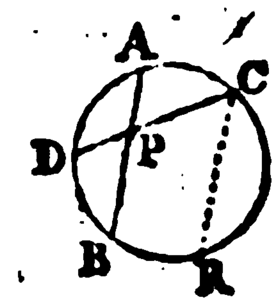
76. *In a circle, two parallel chords* AB, CD *intercept equal arcs* AC, BD,

Join AD. Then because AB is parallel to CD, the alternate angles BAD, CDA, are equal to each other (40); and therefore the arc BD is equal to the arc AC (70. *corol.*).



77. *The angle formed by two chords* AB, CD, *intersecting each other in a circle, is measured by half the sum of the intercepted arcs* AC, DB.

Let CR be parallel to AB.

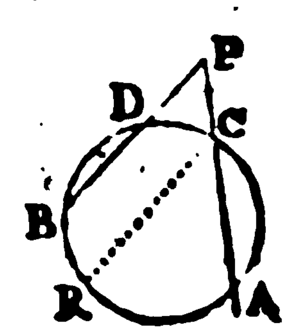


Then the angle of intersection DPB is equal to the angle DCR (40), which is measured by half the arc DBR (69).

But the arc BR is equal to the arc AC (76); therefore the arc DBR is equal to both the intercepted arcs DB, AC; consequently the angle DCR, or its equal DPB, is measured by half the sum of those arcs.

78. *The angle* P *without a circle, formed by two secants* PB, PA, *is measured by half the difference of the intercepted arcs* DC, BA.

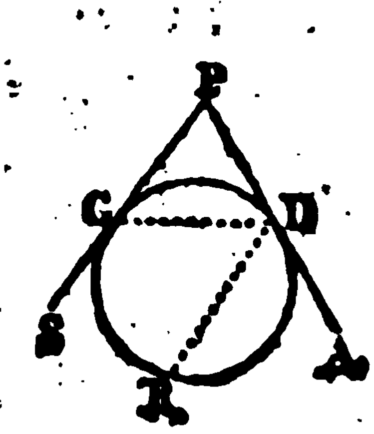
Let CR be parallel to PB.



Then because DC is equal to BR (76), the difference of the intercepted arcs DC, BRA is the arc RA, half of which is the measure of the angle RCA, or of its equal BPA.

79. *The angle* P *formed by the two tangents* PS, PA, *is measured by half the difference of the two intercepted arcs* GD, GRD.

Join the points of contact G, D; and let DR be parallel to PS.



Then because DR is parallel to GP, the angle GDR is equal to DGP (40).

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rol. 2. Because each of th
by half the arc GD (68),
PG = PD; hence the tange
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OF THE

UALITY OF PARALLELOGR.

THEOREMS

. *The diagonal* DB *of a paral*
to two equal parts or triangles D

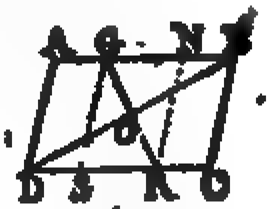
or the angles of the two triangles D
g respectively equal, each to each
side DB common to both triangles,
es will therefore be identical or equal
ts (38).

arol. Hence the opposite sides of a p
ectively equal to each other.

81. *If a right line* GR *bisects or divid*
the parallelogram DABC *into two equal*

rallelogram into two equal parts in trape- CRG.

R be parallel to the sides AD,



he triangles GBO, RDO are and the side OD equal to the angles will be equal in all respects (38), con- RD; and therefore GA = RC; hence the DS is equal to the parallelogram NBCR.

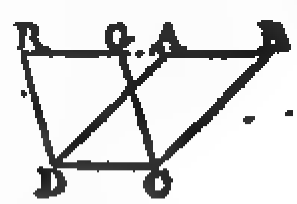
es the parallelogram SGNR into two equal GNR (60); therefore as the parallelograms are equal, the trapezoids GADR, BCRG, two equal figures, must also be equal.

cause DC is equal to DR and AG together, R) is half a parallelogram whose base is the el sides of the trapezoid, and whose height hose parallel sides.

nce all right lines that bisect the diagonal of a l are terminated by the sides, are also bisected

grams standing upon the same base, and parallels (or having equal altitudes), are er.

llel to DC. Then the pa- C is equal to the parallelo-



l to CG, and DA to CB; and RG, DC, other (60, *corol.*), hence if GA be added to spectively, RA will be equal to GB; and s of the triangles DRA, CGB are respec- consequently the triangles themselves must

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BC, the rema
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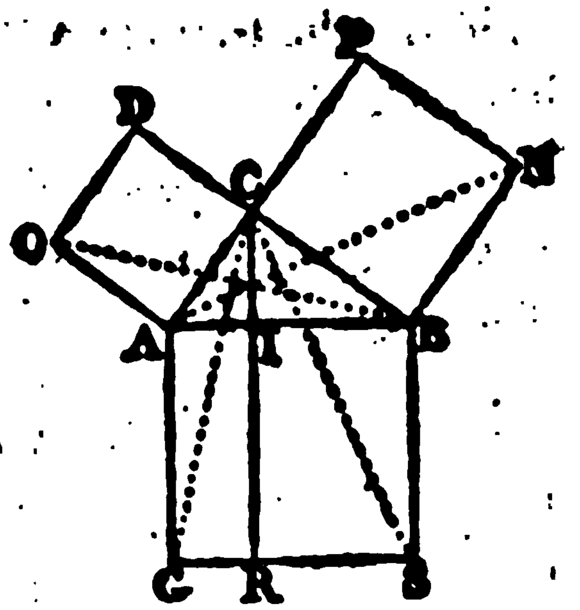


Draw CR parallel to AG; and join OB and CG.

Because the angles OAC, BAG are right angles, if to each be added the angle CAB, the angles OAB, CAG of the triangles OAB, CAG will be equal to each other.

And since the sides AO, AB; AC, AG including those equal angles, are respectively equal, the triangle OAB is equal to the triangle CAG (38).

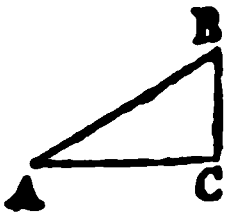
And because BD is parallel to AO, and CR to AG, the triangle AOB is equal to half the parallelogram AODC; and the triangle ACG equal to half the parallelogram AGRI (82^e. *corol.* 1); therefore the halves being equal, the wholes must also be equal, or the parallelogram or square AODC equal to the parallelogram AGRI.

And exactly in the same manner it is proved that the triangle BNA is equal to the triangle BCS; and the square BNPC equal to the remaining parallelogram BSRI.

Corol. Hence the difference between the square of the hypotenuse and the square of either of the other sides, is equal to the square of the remaining side.

Therefore when the lengths of two sides of a right angled triangle are given, the third side may be found by extracting the square root.

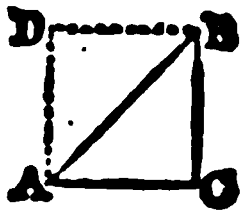
Let AC = 4, and BC = 3: Then the square of AC is 16; and the square of BC is 9; and the sum of those squares is 25 the square of AB, and the square root of that square is 5, the length of AB.



Again, suppose AB = 10, and BC = $4\frac{7}{10}$; then the square of AB is 100, and the square of BC is $22\frac{9}{100}$, and the difference of those squares is $77\frac{91}{100}$ the square of AC, and the square root of $77\frac{91}{100}$ is $8\frac{5}{6}$ the length of AC.

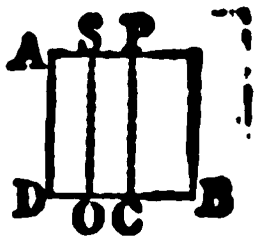
If AC = 1, and BC = 2, the sum of their squares is 20, and the square

root of 20 is the length of AB: but the square root of 20 is a surd: therefore AB and the other sides are *incommensurable.*



When AC and BC are equal, the hypothenuse AB is the diagonal of the square ACBD; and the square of AB is double the square of the side AC or CB: but twice a square number is not a rational square, or a square whose root can be exactly obtained; therefore the diagonal of a square and its side are *incommensurable*: In other words, whatever number of equal parts the side of a square is, or may be divided into, the diagonal cannot contain an exact number of those parts.

84. *If a right line* (DB) *be divided into any number of parts* (DO, OC, CB), *the rectangles made by the whole line and each part, are together equal to the square on the whole line.*



Let AB be the square on the line DB; and from the points of division O, C, draw OS, CP, perpendicular to DB. Then because those lines are equal to DA or DB the side of the square, AO, SC, PB are the rectangles made by the whole line and each part respectively, and these rectangles together evidently constitute the square, because the whole is equal to all its parts taken together. Or if we denote the rectangles after the manner of products, AO is equal to DB $\times$ DO, SC equal to DB $\times$ OC, and PB equal to DB $\times$ CB, and the three products together equal to DB^2.

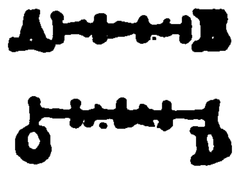
Of Ratios and Proportions which respect Magnitudes.

Definitions.

85. The following Definition of *Ratio* is usually given in the *5th*. Book of Euclid's Elements.

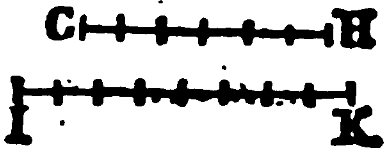
"Ratio is a mutual relation of two magntiudes of the same kind to one another in respect of quantity."

This definition is frequently objected to as imperfect and obscure. And it seems difficult to acquire a correct idea of the ratio of two magnitudes from the definition, if we are limited to the consideration of *magnitudes* abstractedly. By the help of numbers however, it becomes perfectly intelligible.



Thus, if we divide the line or magnitude AB into 3 equal parts, and the magnitude CD contains 4 of those parts, the relation of AB to CD is the same as that of 3 to 4, which in numbers, is the ratio of the magnitudes AB and CD in respect of quantity.

Let GH be any other line or magnitude divided into 6 equal parts, and suppose IK contains 8 of those parts.

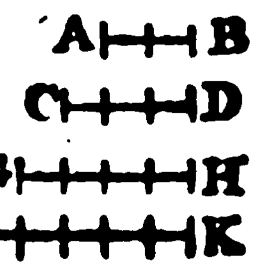


Then the relation or ratio of GH to IK is the same as that of AB to CD, because GH is contained or can be taken in IK as often as AB is contained or can be taken in CD, for the same reason that 6 is contained in 8 as often as 3 is contained in 4, that is, because $\frac{8}{6} = \frac{4}{3}$.

Those four lines or magnitudes are proportional; *viz.* AB is to CD, as GH is to IK; and are set down in the manner of proportional numbers, thus AB : CD :: GH : IK. And the proportion must evidently hold good whether AB and CD

are commensurable or incommensurable when compared with GH and IK.

86. Quantities of the same kind which are commensurable or can be divided into like parts, or parts of the same magnitude, may be compared in the same manner as we compare numbers in geometrical proportion (133, 134, *arith.*). Thus if AB contains 2; CD, 3; GH, 4; and IK, 6 equal parts, those lines or magnitudes will evidently have the same proportion as the number of equal parts into which they are respectively divided;



$$AB : CD :: GH : IK,$$
$$2 : 3 :: 4 : 6.$$
$$\text{Or } AB : GH :: CD : IK,$$
$$2 : 4 :: 3 : 6.$$

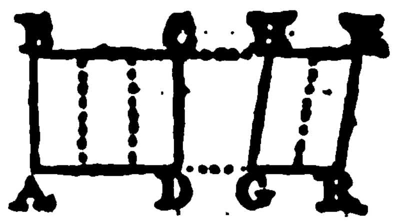
Or suppose the equal parts are again divided into a like number of equal parts, as 10 for example; then AB will contain 20; CD, 30; GH, 40; and IK, 60; therefore the quantities or lines will be in the proportion of 20, 30, 40, and 60; or as 2, 3, 4, and 6, the same as before.

Hence it is evident (if we make use of a common measure, as in Practical Geometry that commensurable magnitudes may be represented by numbers, and their properties, as far as relates to proportion, demonstrated arithmetically. In the following theorems therefore, we shall sometimes refer to the articles in arithmetic which treat of proportion, in order to abridge the operations.

THEOREMS.

87. *Parallelograms* AC, GK *between the same parallels, or having the same altitude, are to one another in the same ratio as their bases* AD, GR.

For suppose AD is divided into 3 equal parts, and that GR contains 2 of those parts. Then if lines are drawn from the points of division parallel to the sides, the parallelogram AC will be divided into 3, and the parallelogram GK into 2 equal parallelograms, because they stand upon equal bases (82° *corol.*)

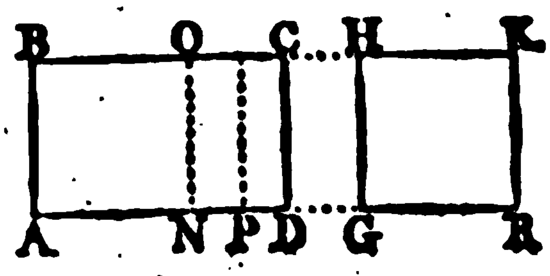


Therefore

3 is to 2, as the paral. AC is to the paral. GK.

Or AD : GR :: paral. C : paral. GK.

And if the bases AD, GR are incommensurable, the like proportion must evidently hold good. Suppose the base GR is the side of a square, and the base AD its diagonal (83, *corol.*). Let AN = GR, and draw NO parallel to DC: and take NP so that AN and NP are commensurable.



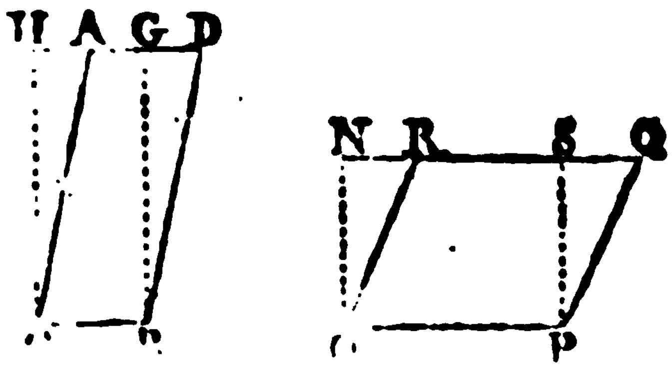
Then, paral. BN : paral. BP :: AN : AP.

And by continually taking commensurable parts in the remainder PD, we may at last, approximate nearer to D than any assignable distance. Consequently the parallelogram BD will ultimately be to the parallelogram BN (or HR) as AD to GR.

Corol. 1. Since triangles are the halves of their parallelograms (82°. *corol.* 1.), therefore triangles having the same, or equal altitudes, are to one another as their bases.

Corol. 2. If RK, and DC be taken for the equal bases of the parallelograms RH and DB, then RG and DA will be their altitudes: Therefore parallelograms, or triangles, on equal bases, are respectively in the same ratio as their heights.

88. *Parallelograms* CADB, ORQP, *having unequal bases and altitudes, are as the rectangles of the bases and altitudes.*



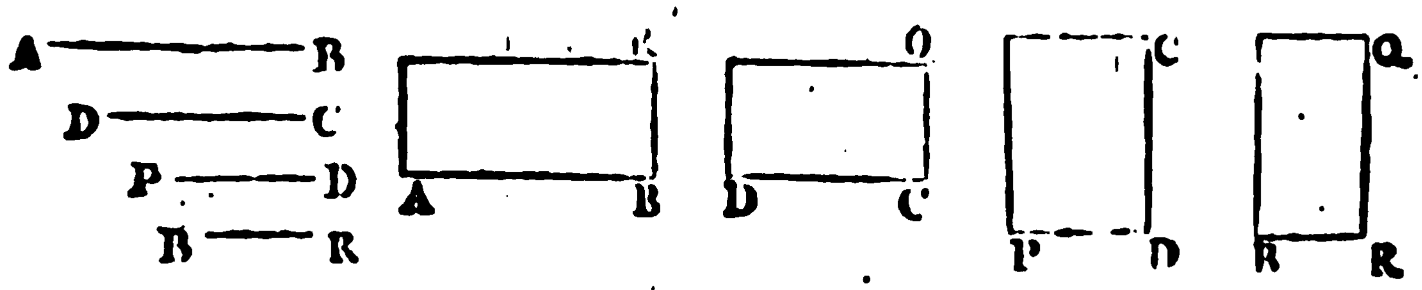
Make BG, CH, and PS, ON, perpendicular to CB, OP, respectively; then the rectangle HB is equal to the parallelogram AB, and the rectangle NP equal to the parallelogram RP (82).

Then, because equals must have equal ratios; As rectangle to rectangle, so is parallelogram to parallelogram.

Scholium. The parallelograms are also said to be in the compound ratio of their bases and altitudes. For if CB : OP, and BG : PS denote the ratio of the bases, and altitudes, respectively, the rectangles of the corresponding terms or CB × BG : OP × PS will denote the compound ratio or the ratio of their rectangles. (141, *Arith.*)

Suppose CB = 2, BG = 5, OP = 4, PS = 3; then $\frac{2}{4}$ denotes the ratio of CB to OP; and $\frac{5}{3}$ that of BG to PS; and their product $\frac{2}{4} \times \frac{5}{3}$ (or $\frac{10}{12}$) is the compounded ratio or that of the parallelograms, namely, as 10 to 12.

89. *If four right lines* AB, DC, PD, BR *are proportional* (AB : DC :: PD : BR, *or* AB : PD :: DC : BR); *the rectangle* PC *made with the two means* DC, PD, *is equal to the rectangle* AR *made with the two extremes* AB, BR.



Let CO = BR, and RQ = DC. Then the rectangles AR, DO having equal altitudes, will be as their bases (87); and for the same reason the rectangles PC, BQ will also be as their bases;

AB : DC :: *rectang.* AR : *rectang.* DO;
PD : BR :: *rectang.* PC : *rectang.* BQ;

But AB : DC :: PD : BR, therefore by equality of ratios *rectang.* AR : *rectang.* DO :: *rectang.* PC : *rectang.* BQ:

Now the surfaces or rectangles DO, BQ contained under the same or equal lines (DC, BR) must be equal; therefore the consequents being equal, the antecedents or rectangles AR, PC will also be equal.

Or thus: Since the rectangle of two lines is analogous to the product of two numbers, if AB : DC :: PD : BR, then AB × BR = DC × PD *. (93, *Arith.*)

Corol. 1. When DC and PD are equal, the rectangle PC becomes a square; and its side is a mean proportional between the other two lines AB and BR (151, *Arith.*).

Corol. 2. Hence also, the product of the base and perpendicular gives the area or surface of a parallelogram.

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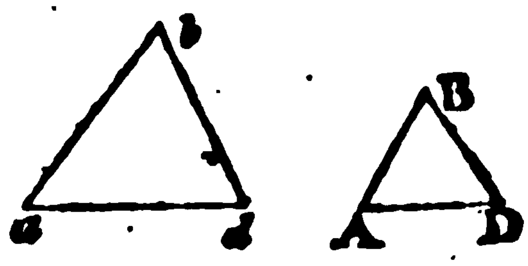
* Here the surfaces of the rectangles or parallelograms AR and PC are denoted by AB × BR, and DC × PD. And if AB = 8, BR = 3, DC = 6, and PD = 4 (inches, for example); then 8 × 3 and 6 × 4 are the surfaces or *areas* of those rectangles in square inches.

Of Similar Plane Figures.

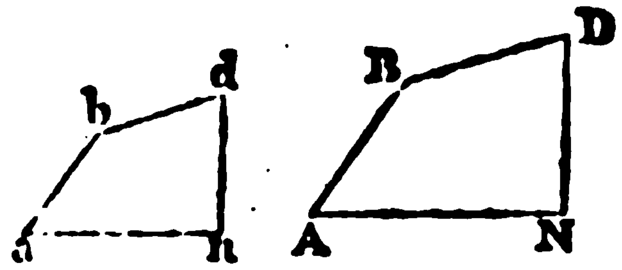
Definitions.

90. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportional.



Thus, if the angles of the triangles ABD, *abd* are respectively equal, and AB : BD :: *ab* : *bd*; and AB : AD :: *ab* : *ad*, &c. the triangles are said to be similar.

The sides opposite the equal angles are called homologous: thus AB, *ab* are homologous sides.



91. And if ABDN, *abdn* are equiangular, and AB : AN :: *ab* : *an*, &c. the two figures are similar.

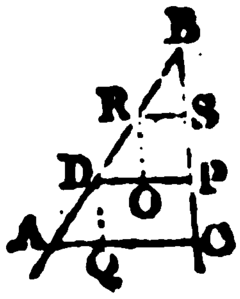
Corol. Hence all squares are similar.

92. All circles are similar.

Theorems.

93. *If one side of a triangle be divided into any number of equal parts, and from the points of division lines are drawn through the triangle parallel to one of the other sides, those lines will divide the third side into the same number of equal parts.*

Suppose BR, RD, DA are equal, and RS, DP parallel to AC. Then will BS, SP, PC, be equal to each other.



Draw RO, DQ, parallel to BC.

Then because the triangles RBS, DRO, ADQ are equiangular, and the like sides BR, RD, DA equal, those triangles will be identical or equal in all respects (38): consequently BS, RO, DQ, are equal.

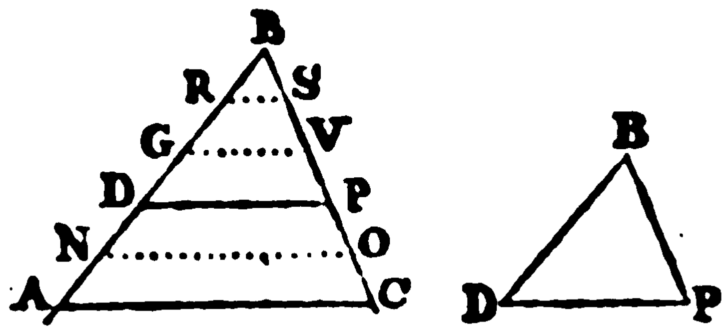
But RP, DC are parallelograms, therefore PC = DQ, and SP = RO, and each equal to BS.

Corol. Hence, if right lines AC, DP, RS, &c. cutting the sides of a triangle, make the segments DA = DR, PC = PS, &c. those lines will be parallel.

94. *If a line be drawn in a triangle parallel to one of the sides, it will divide the other two sides proportionally.*

Let DP be parallel to AC.

Then BD : BA :: BP : BC.
And BA : DA :: BC : PC.



Suppose BD is divided into 3 equal parts, and that DA contains two of those parts; and let lines be drawn from the points of division in BA parallel to AC, meeting BC. Then BC will also be divided into 5 equal parts (93).

Now, whatever part BD is of BA, the like part will PB be of BC, let the *actual* lengths of the equal parts in BA and BC be what they will: thus BD is $\frac{3}{5}$ of BA, and BP is $\frac{3}{5}$ of BC; therefore the relation or ratio of BD and BA is the same as that of BP and BC; consequently those four lines are proportional,

BD : BA :: BP : BC.

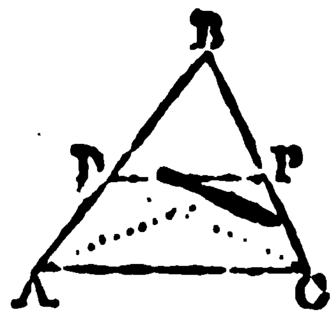
And because DA is $\frac{1}{3}$ of BA, and PC $\frac{1}{3}$ of BC, these are also proportionals,

$$BA : DA :: BC : PC.$$

And it is also evident that BD, DA, BP, PC are proportionals; for DA is the same part of BD, as PC is of BP (each being $\frac{1}{3}$);

$$\text{Hence } BD : DA :: BP : PC.$$

Otherwise thus:



Join AP, DC. Then the triangles DAP, DCP, standing upon the same base DP, and between the parallels DP, AC are equal (82[a]):

To each of these add the triangle DBP, then the triangles BDC, BPA will be equal (32):

But the triangles BDP, BDC standing on the bases BP, BC and having the same vertex D, have the same altitude:

Also, the triangles BPD, BPA on the bases BD, BA, and having the vertex P, have the same altitude;

Therefore BD : BA :: triang. BPD : triang. BPA (67, *corol.* 1.)
And BP : BC :: triang. BDP : triang. BDC.

But the ratio of the triangle BPD to the triangle BPA is the same as that of BDP to BDC, because they are respectively equal:

Therefore the ratio of BD to BA is the same as that of BP to BC (31); or BD : BA :: BP : BC.

Corol. 1. Because the parallels AC, DP make the angles BAC, BDP equal, and the angle BCA = BPD (40), the triangles BAC, BDP are equiangular or similar: Therefore similar triangles have the sides about the equal angles proportional;

Thus the angle ABC of the triangle ABC, is equal to the angle DBP of the triangle DBP; and BD : BA :: BP : BC. And if the triangle BDP were applied to the triangle BAC so that the angles DPB, ACB are made to coincide, it may be proved in the same manner, that the including sides BP, DP, and BC, AC are proportionals.

And conversely, if the angles DBP, ABC of two triangles DBP, ABC are equal, and the sides about those angles proportional, the triangles will be mutually equi-angular.

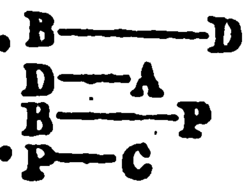
Corol. 2. Hence also, if lines (NO, DP, &c.) drawn through a triangle, are parallel to the base (AC), the intercepted segments of the sides (AN, CO; ND, OP, &c.) will be proportional:

For BA : AN :: BC : CO;
And BA : BC :: AN : CO (86 or 89):

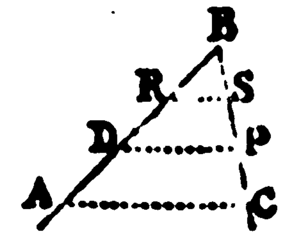
In like manner BN : BO :: ND : OP:
But BA:BC :: BN : BO; hence AN:CO::ND:OP, by equality.

Scholium. Hence all that relates to the composition and division of ratios when these respect the comparison of right lines, will easily be comprehended: thus, *If 4 right lines are proportional,* BD : DA :: BP : PC, *they will also be proportional by composition and division.*

That is, BD+DA:BD(or DA)::BP+PC:BP(or PC).
And BD−DA:BD(or DA)::BP−PC:BP(or PC).



On two indefinite lines BA, BC meeting in B, take BD = BD, BP = BP, DA and DR each = DA, and PC, PS each = PC: then as the corresponding segments in BA and BC have the same ratio as



those sides, and the sides of the the triangles ABC, DBP, RBS are also proportional, we have

$$BA : BD :: BC : BP,$$

That is $BD + DA : BD :: BP + PC : BP$.

But DA and PC have the same ratio as BD and BP,
Therefore $BD + DA : DA :: BP + PC : PC$.

Again, $BD - DA = BR$, and $BP - PC = BS$, and the sides of the triangles BRS, BDP being proportional,
BR (or $BD - DA$) $: BD :: BS$ (or $BP - PC$) $: BP$.
But BR and RD or DA, and BS and SP or PC are proportional,

Whence $BD - DA : DA :: BP - PC : PC$.

Also, because the sides of the triangles BAC, BRS are proportional,

$$BD + DA : BD - DA :: BP + PC : BP - PC.$$

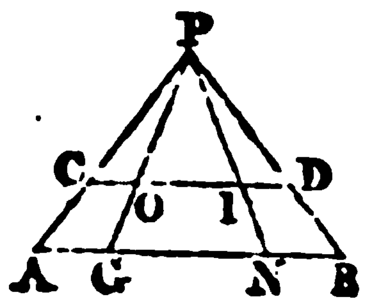
And if any number of right lines are proportional, $BR : BS :: RD : SP :: DA : PC$; then, *as any antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.* For BA is the sum of the antecedents, and BC that of the consequents, and the corresponding segments in BA, BC, in the same ratio as those sides, it will be

$$BR : BS :: BR + RD + DA \text{ (or } BA) : BS + SP + PC \text{ (or } BC).$$

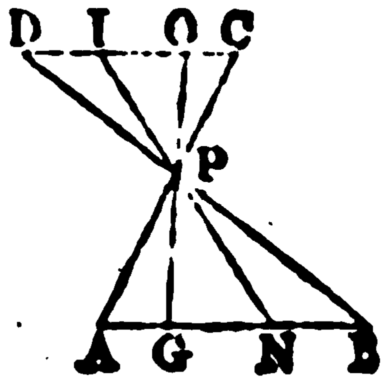
And the same will hold good with proportional quantities of any kind; for such magnitudes may be represented by lines, or by numbers. (Arith. *art.* 136).

95. *If several right lines meeting, or intersecting each other in a point* P, *are cut by two parallel lines* AB, CD; *the intercepted segments will be respectively proportional:*

AG : CO :: GN : OI :: NB : ID, &c.

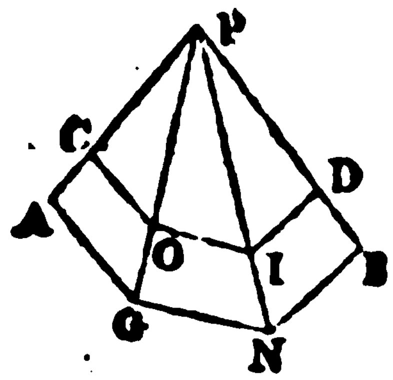


For the triangles APG, CPO; GPN, OPI; NPB, IPD are respectively equi-angular, and therefore similar;



Hence (94, *corol.* 1), AG : CO :: GP : OP :: GN : OI :: NP : IP :: NB : ID, &c.

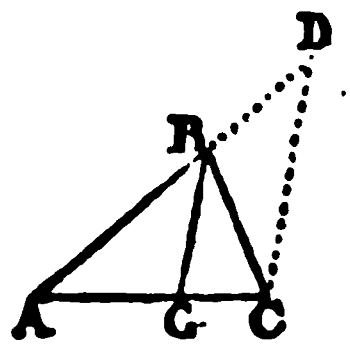
Therefore (31) AG : CO :: GN : OI :: NB : ID, &c.



Corol. Hence it is evident, if AG, GN, &c. and CO, OI, &c. are not in the same continued right lines, but respectively parallel as before, that CO, OI, ID, &c. will be in the same proportion as AG, GN, NB, &c.

96. *The line* BG *bisecting the vertical angle* ABC *of the triangle* ABC, *divides the base* AC *into two parts having the ratio of the sides* AB, BC:

AB : AG :: BC : GC.



Draw CD parallel to BG meeting AB produced in D.

Then because CD is parallel to BG, the angles BCD, GBC are equal (40).

And the external angle CBA (or double the angle GBC) of the triangle CBD, is equal to both the angles BCD, BDC (41).

Therefore the angles BDC, BCD are equal, and consequently BD is equal to BC (46, *corol.* 2).

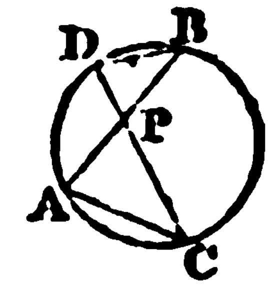
But the triangles ABG, ADC are similar;

Hence AB : AG :: BD (BC) : GC (94, *corol.* 1).

Corol. Hence, if a line bisects the vertical angle of a triangle, the rectangle of either side and the alternate segment of the base, is equal to the rectangle of the other side and the remaining segment:

$$AB \times GC = AG \times BC.$$

97. *In a circle, if two chords* AB, CD *intersect each other, and their extremities are joined, the triangles* PCA, PBD *will be similar; and the rectangle of the segments* PA × PB *equal to the rectangle of the segments* PC × PD.

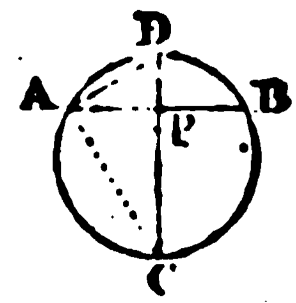


For the angles PBD, PCA, standing on the same arc DA, are equal to each other (70).

And the angles PDB, PAC, standing on the arc CB, are also equal.

And the equal angles at P being common to both triangles, those triangles are therefore equi-angular, and consequently similar;

Hence PA : PC :: PD : PB (94, *corol.* 1):
Therefore PA × PB = PC × PD (69).



Corol. 1. If one chord DC bisects the other AB at right angles, then DC is the diameter of the circle (63), and AP or PB is a mean proportional between DP and PC.

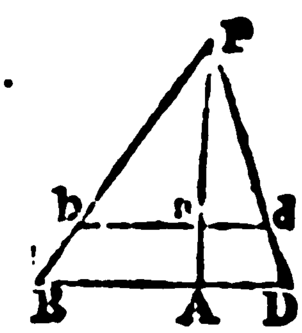
Corol. 2. And if AD, AC are joined, the angle CAD is a right one (72); therefore the perpendicular AP let fall from the right angle on the hypotenuse DC, is a mean proportional between the segments DP, CP.

Therefore the angles ABC, BDC, being equal, their supplements or the angles CBP, BDP must be equal.

Consequently the triangles PDB, PBC are equi-angular:

Hence PC : PB :: PB : PD.
Therefore PC $\times$ PD = PB^2.

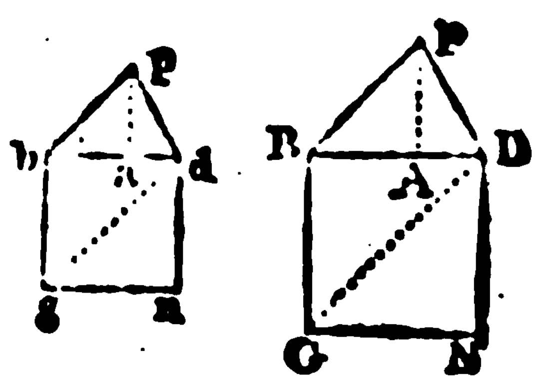
100. *If two triangles* BPD, bPd *are similar; the bases* BD, bd, *and perpendiculars* PA, Pa, *are proportional:*
BD : *bd* :: PA : P*a*.



Because the angles BAP, *ba*P are right ones, the triangles BAP, *ba*P are also similar;

Hence PB : P*b* :: PA : P*a* (94, *corol.* 1),
And since PB : P*b* :: BD : *bd*,
Therefore BD : *bd* :: PA : P*a* (by equality).

101. *The surfaces or areas of similar triangles are in the duplicate ratio (or as the squares) of their homologous sides.*



Let the triangles BPD, *bpd* be similar; and BN, *bn*, the squares on the sides BD, *bd*:

Then, triang. BPD : triang. *bpd* :: square BN : square *bn*.

Suppose the perpendiculars PA, *pa*, are let fall on BD, *bd*, respectively; and join DG, *dg*.

Then because the triangles BPD, BGD are on the same base BD, we have (87, *corol.* 2).

Triang. BPD : triang. BGD :: PA : BG (BD).

And, triang. *bpd* : triang. *bgd* :: *pa* : *bg* (*bd*).

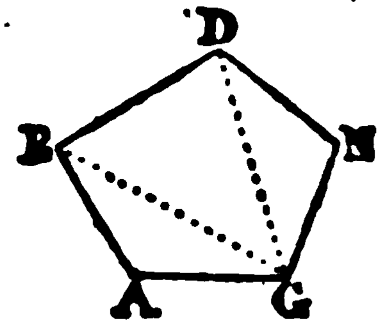
But PA : BD :: *pa* : *bd* (100):

Therefore (81),

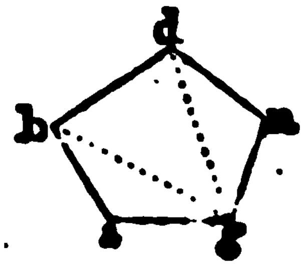
triang. BPD : triang. BGD :: triang. *bpd* : triang. *bgd*:
or triang. BPD : square BN :: triang. *bpd* : square *bn*;
because the two squares must evidently have the same ratio as their halves.

102. *All similar right lined plane figures* (ABDNG, abdng) *are to one another in the duplicate ratio, or, as the squares of their homologous sides* (AG, ag).

Draw GB, GD, *gb*, *gd*.



Then the figures being similar, the angle A is equal to the angle *a*; and the including sides AB, AG; *ab*, *ag*, are proportional (90); therefore the triangles ABG, *abg* are equi-angular and similar (94, *corol.* 1).



And if the equal angles ABG, *abg* are taken from the equal angles ABD, *abd*, the remaining angles GBD, *gbd*, must be equal.

Hence AB : *ab* :: BG : *bg*;

AB : *ab* :: BD : *bd*;

Therefore (81) BG : *bg* :: BD : *bd*: consequently (94, *corol.* 1) the triangles GBD, *gbd*, are similar. And in the same manner it may be proved that the triangles GDN, *gdn*, are similar.

Hence (101), triang. GAB : triang. *gab* :: $GB^2 : gb^2$ :: GBD : *gbd* :: $GD^2 : gd^2$ :: GDN : *gdn*; or GAB : *gab* :: GBD : *gbd* :: GDN : *gdn*:

And (94, *schol.*) GAB : *gab* :: GAB + GBD + GDN : *gab* + *gbd* + *gdn*.

But the antecedents together is the figure ABDNG, and the consequents the figure *abdng*;

Therefore $AG^2 : ag^2$ (GAB : *gab*) :: ABDNG : *abdng*.

To illustrate this by an example in numbers, suppose AG = 10 feet,

$ag = 8$ feet; and the area or surface of the figure ABDNG $= 650$ square feet;

Then $10^2 : 8^2 :: 650 : \frac{650 \times 64}{100} = 416$ square feet, the area or surface of *abdng*.

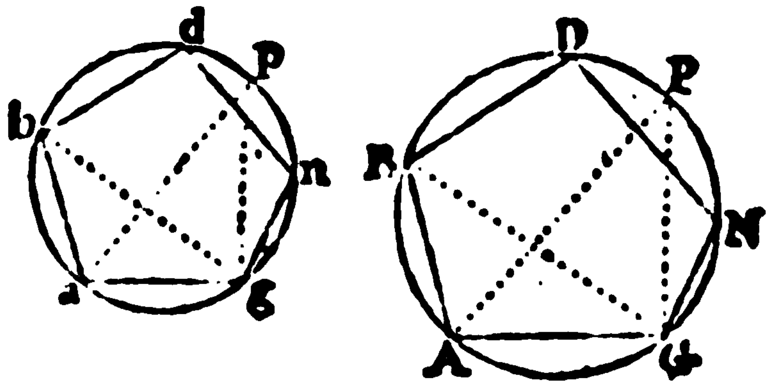
103. *The Perimeters of similar right lined plane figures are in the same ratio as their homologous sides.* (See the figures to the preceding Theorem.)

For the angles being equal, each to each, and the sides about the equal angles respectively proportional; we have

AG : *ag* :: GN : *gn* :: ND : *nd* :: DB : *db* :: BA : *ba*; therefore AG : *ag* :: sum of all the antecedents AG + GN + ND + DB + BA (the perimeter) : sum of all the consequents *ag* + *gn* + *nd* + *db* + *ba* (the perimeter).

104. *The perimeters of similar Polygons* (ABDNG, abdng) *inscribed in circles, have the same ratio as the diameters* (AP, ap,) *of those circles.*

Draw GB, GP, *gb*, *gp*.



Then the polygons being similar, the triangles ABG, *abg*, will be equi-angular, and the angle ABG equal to the angle *abg* (102).

But the angle APG is equal to the angle ABG; and the angle *apg* equal to the angle *abg* (70). And the angles AGP, *agp*, being right ones (72), the triangles APG, *apg*, are therefore equi-angular.

Hence AP : *ap* :: AG : *ag* :: perim. of polyg. ABDNG : perim. of polyg. *abdng* (103).

Corol. Hence it appears that the circumferences of circles have the same ratio as their diameters. For conceive regular

polygons of the like number of sides to be inscribed in both circles; then it follows that those polygons will be similar, and that their perimeters are in the same ratio as the diameters of the circles, let the number of sides be what they will. If now we suppose the number of sides to be continually augmented and their lengths diminished, it is manifest that at last, the differences between the perimeters and the circumferences of the circles, will be less than any assignable quantities; consequently the ultimate ratio of the perimeters and that of the circumferences must be equal.

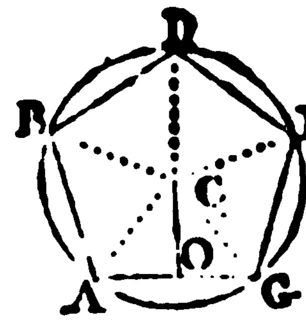
105. *The areas or surfaces of similar polygons inscribed in circles are in the duplicate ratio, or as the squares of the diameters of the circles:* (See the figures to the preceding Theorem).

For the triangles APG, *apg*, being similar, we have (101), $AP^2 : ap^2 :: \text{triang. } APG : \text{triang. } apg :: AG^2 : ag^2 :: \text{polyg. } ABDNG : \text{polyg. } abdng$ (102).

Corol. Hence, if we suppose (as in the last Theorem) the circumference of a circle to be the perimeter of a regular polygon, consisting of an infinite or rather an indefinite number of indefinitely short sides, it follows that the surfaces or areas of circles will be as the squares of their diameters. And because the circumferences are directly proportional to the diameters (104, *corol.*) the areas will be as the squares of the circumferences also.

106. *The area or surface of a polygon* (ABDNG) *is equal to a rectangle under half the perimeter and* (CO) *the distance of its centre from the sides.*

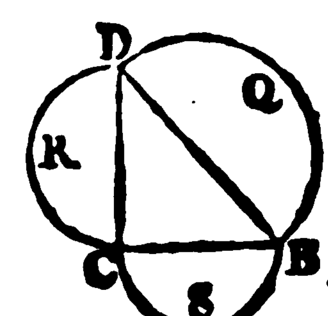
The centre of a regular polygon is a point equally distant from all its sides; and is the same as the centre of the inscribed, or circumscribing circle.



Suppose lines are drawn from the centre to the angular points; then the polygon will be divided into as many equal triangles as it has sides. And because those triangles are isosceles, CO will bisect AG and be perpendicular to it (46): therefore the area of the triangle ACG is half the rectangle CO × AG (89, *cor.* 2), or CO × ½AG; and the area of another of the triangles (GCN) is CO × ½GN, and so on: but the halves of all the sides together make half the perimeter; therefore the rectangle CO × *half the perimeter*, is the area of all the triangles or surface of the polygon.

Corol. Hence it appears, that the area or surface of a circle is equal to a rectangle under the radius and a right line equal to half the circumference. For, if we conceive the circle to be a regular polygon of an indefinite number of indefinitely short sides, the distance (CO) of the centre (C) from the sides, will in that case, be the radius of the circle, and half the perimeter becomes half the circumference.

107. *If semicircles* (Q, R, S,) *are described upon the sides of a right angled triangle* (BCD), *that which is upon the longest side* (DB) *will be equal to both the other two taken together.*



For circles being similar, and in the same ratio as the squares of their diameters (105, *corol.*) their halves must also be similar, and in like proportion, therefore

$S : R :: CB^2 : CD^2$, and by composition

$S + R : R :: CB^2 + CD^2 (= BD^2, 83) : CD^2 :: Q : R$,

or $S + R ; R :: Q : R$; therefore $S+R$ is equal to Q (31).

Hence, if similar figures are described on the sides of a right-angled triangle, that on the longest side will be equal to the other two taken together.

Of Planes and Solids.

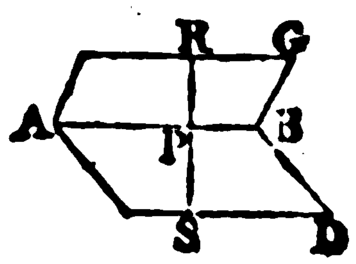
Definitions.

108. A right line is perpendicular to a plane when it is at right angles to all the straight lines that can be drawn in that plane, from the point on which it insists.

109. The distance of a point from a Plane is a right line conceived to be drawn from that point perpendicular to the plane.

Corol. From the two preceding Definitions, and *Art.* 48, it follows, that a perpendicular is the shortest line which can be drawn from any point to the Plane.

110. The inclination of one plane to another is measured by the inclination of two right lines in those planes, drawn from any point in their common intersection, and at right angles to the same: Thus if AB is the line of intersection of the two parallelograms AG, AD; and PR, PS are perpendicular to AB, the inclination of the planes or parallelograms is the angle included by the lines PS, PR.

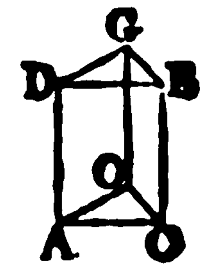


111. Parallel planes are those which are not inclined to each other, or are every where at an equal perpendicular distance.

112. A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

113. Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes.

114. A Prism is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms.



Thus AB is a triangular prism, its ends being the parallel and equal triangles AOC, DGB.

115. An upright prism is that which has the planes of the sides perpendicular to he ends or base.

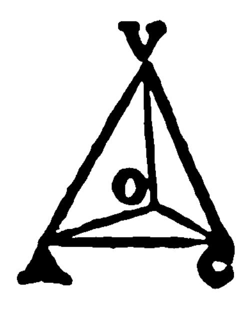
Thus AB is an upright prism; the sides, or parallelograms CG, GA, CD, being perpendicular to the ends or triangles AOC, DGB.

116. A Parallelopiped, or Parallelopipedon, is a prism bounded by six parallelograms, whereof the opposite ones are parallel, equal, and like to each other.

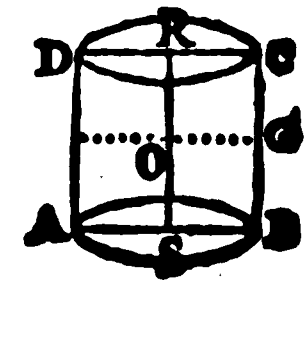
117. A rectangular parallelopipedon, or prism, is that whose bounding planes are all rectangles, and which stand at right angles one to another.

118. When all the bounding planes are squares, the prism or rectangular parallelopipedon, is called a Cube.

119. A Pyramid is a solid whose base is any right lined plane figure, and whose sides are triangles having all their vertices united in a point above the base, called the vertex of the pyramid.

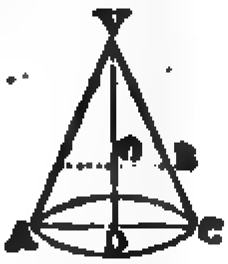


Thus AOCV is a triangular pyramid, its base being the triangle AOC, and its vertex V.



120. A Cylinder ABCD (sometimes called a round prism) is a solid conceived to be generated by the rotation of a rectangle SBCR about one of its sides SR, supposed at rest: which side SR is called the axis of the cylinder.

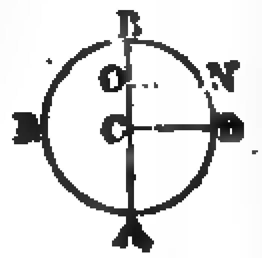
OG is parallel to SB, those lines will describe therefore every section of a cylinder parallel to its cle equal to the base.



Cone or round pyramid AVC is a ed by the rotation of a right angled about its perpendicular DV, called he cone.

f OB is parallel to DC, it will describe a circle; section of a cone parallel to the base is a circle.

ilar Cones, and Cylinders, are such as have their l the diameters of their bases proportional.



Sphere ARBD, is a solid supposed ted by the revolution of a semi-) about the diameter (AB) which l, and is called the axis.

f ON is at right angles to the axis AB, it will rcle; therefore any section of a sphere, made by a ircle.

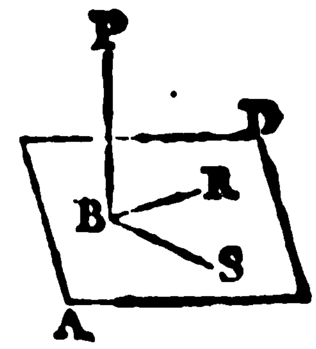
e altitude of a pyramid, or prism, is the perpendi- e of the vertex, or upper plane thereof, from the base.

Theorems.

e common section (GH) *of two planes* (AB, CD) *ne.*

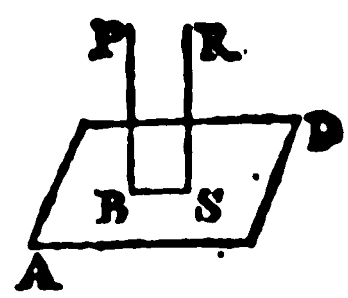
e extreme points G, H, of the ction be joined by the line GH, e being in the plane AB, and plane CD (7.) it therefore must non section of both.

126. *If a right line* PB *be perpendicular to two right lines* RB, SB, *at their point of concourse* B, *it will be perpendicular to* AD *the plane of those lines.*



For suppose PB is perpendicular to a plane passing through the point B; then all right lines in that plane which meet in B will be at right angles to BP (108), therefore conversely, all right lines (RB, SB) which form right angles with BP at the point B, must fall in that plane.

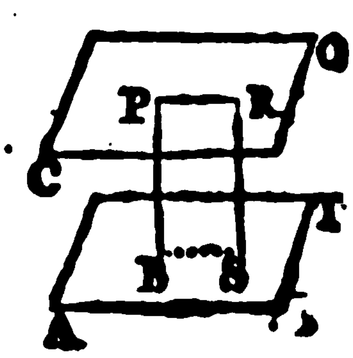
127. *If two right lines* (PB, RS) *are perpendicular to a plane* (AD,) *they will be parallel to each other.*



Join the points B, S. Then because BP is perpendicular to the plane AD, it must lie in (or is the common intersection of) every plane that passes through the point B which is perpendicular to the plane AD, it is therefore in the perpendicular plane that intersects AD in the line BS. In like manner SR must also lie in that same plane, or the perpendicular plane intersecting AD in the line SB; therefore as the angles PBS, RSB are right angles in the same plane, PB, RS, will be parallel to each other (40, *corol.* 2.).

Corol. Hence if several right lines are perpendicular to the same plane, they will be parallel to each other.

128. *If two planes* (AI, CO) *are parallel to each other, then a right line* (PB) *which is perpendicular to one* (AI) *will also be perpendicular to the other* (CO).

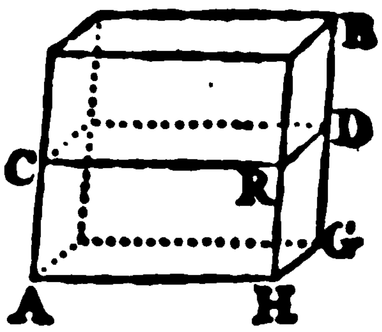


From any point S in the plane AI erect another perpendicular to that plane meeting the other plane in R, and draw PR, BS; then the planes being parallel, the two perpendiculars will be equal (111), and parallel (127); and as the angles at B and S are right angles,

in whatever direction BS may be drawn upon the plane AI, the quadrilateral BPRS will always be a rectangle; consequently BP is perpendicular to PR or to the plane CO.

Corol. Since BS, PR are parallel, therefore the sections (BS, PR) made by a plane (BPRS) intersecting two parallel planes, are also parallel. And it is also manifest, that a plane will cut any number of parallel planes in like angles.

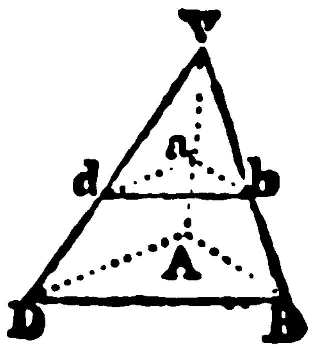
129. *If a Parallelopipedon or Prism* (AB) *be cut by a plane* (CD) *parallel to its base* (AG); *the section will be like and equal to the base.*



For by supposition the plane CD is parallel to the plane AG, therefore (128, *corol.*), the sections of those planes with the four sides of the prism are also parallel, namely, CR parallel to AH, RD parallel to HG, &c. and because the sides of the prism are parallelograms, the sides of the section CD will be equal to the corresponding sides of the base AG; therefore the section CD is a parallelogram like and equal to the base AG.

Corol. And the like is evident when the base is a polygon of any kind whatever: for the method of demonstration will be exactly the same if the sides of the prism are parallelograms.

130. *If a Pyramid* (DVAB) *be cut by a plane* (dba) *parallel to the base* (DBA), *the section* (dba) *will be similar to the base.*



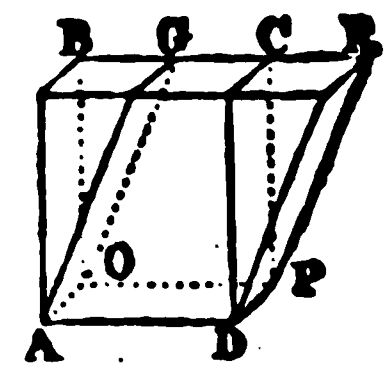
For (128, *corol.*) the sections *db*, *da*, *ab* are respectively parallel to DB, DA, AB, therefore the triangle *dVb* is similar to the triangle DVB, the triangle *dVa* to DVA, and *aVb* to AVB (94).

Hence DB : *db* :: BV : *b*V :: BA : *ba* :: AV : *a*V

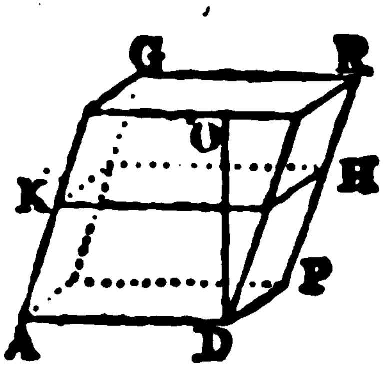
:: AD : *ad*; therefore *db*, *ba*, *ad* are as the corresponding sides of the base; and consequently the triangles *dba*, DBA are similar.

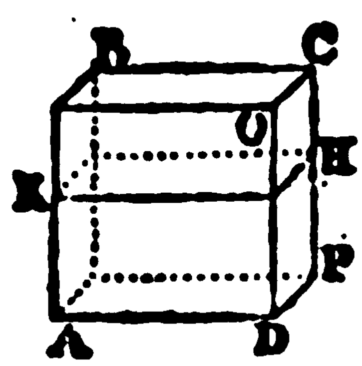
Corol. In like manner it is proved that all sections of a pyramid parallel to its base are similar, and similar to the base, whatever be the number of sides.

131. *Parallelopipeds or Prisms* (ABCD, AGRD) *on the same base* (AOPD), *and having equal altitudes, are equal to each other.*



By substituting surfaces for lines, and solids for surfaces, the demonstration will be similar to that in *Art.* 82, for parallelograms when BR is one right line. Thus, because the plane AB is parallel and equal to the plane DC, and the planes AG, DR also parallel and equal to each other, therefore BC is equal to GR; and taking GC, which is common to both those lines, from each, there remains BG equal to CR; consequently the solids ABGO, DCRP are bounded by like and equal planes, alike situated, and therefore are indentical: now if the solid ABGO is taken from the whole solid AR, the remainder is the prism AGRD; and the same whole AR lessened by the solid DCRP leaves the prism ABCD: therefore the two remainders or prisms AC, AR are equal (33).



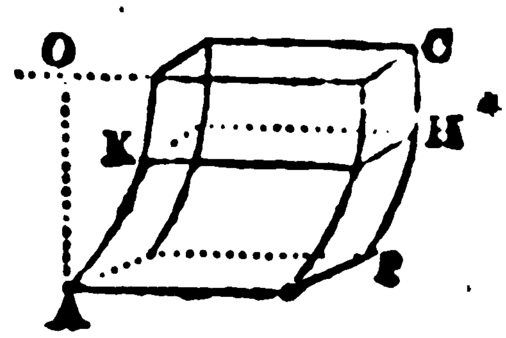


But the same conclusion is manifest from the *Method of Indivisibles*, which supposes that solids are composed of an indefinite number of indefinitely thin elementary parallel planes or sections: Thus, let AC, AR be the prisms having like bases AP, AP, and equal altitudes DO, DO; and conceive KH, KH to be two of those

indefinitely thin planes, parallel to the bases AP, AP: Then, as all the sections (KH, KH) are alike, and equal in both prisms, (129.) it is evident each prism is made up of exactly the same number of those equal elementary parts or sections lying one upon the other, those in AC vertically, and the others in AR, obliquely: which positions give their wholes or the two equal solids a different appearance.

The whole number of those indefinitely thin *laminæ* in each prism, is denoted by the perpendicular height DO; for if DO be divided into an indefinite number of parts, those parts, or the number of sections taken together, must again make up the whole line; hence it follows, that the base AP, or any section parallel to it, multiplied by the height DO, gives the sum of all the elements or the content of the prism.

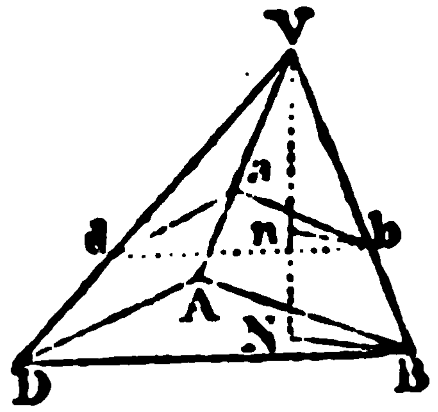
Corol. 1. Hence any solid AC having the base AP and height AO equal to those of the prism, will have the same magnitude as the prism, if all sections (KH, &c.) parallel to the base, are also equal to the base.



Corol. 2. And therefore it follows that prisms and cylinders of equal bases, and altitudes are also equal.

Corol. 3. Also because the base of a prism drawn into its height is the measure of its magnitude, therefore prisms are in the same proportion as their bases multiplied by the heights. Consequently if the bases are equal, the prisms will be as their heights; but in the ratio of their bases when the heights are equal.

132. *Pyramids* DVB, RVH, *standing upon the same, or upon equal bases* DAB, RGH, *and having equal altitudes* NV, RV, *are equal to each other.*



Let *dab* be a section parallel to the base DAB; and V*n*N the perpendicular from the vertex V upon the base DAB; and draw BN, *bn*; (the point *n* being in the plane *dab*).

Then the triangles DVB, *d*V*b* are simil... VN B, are right angles (126), the triangl... also be similar.

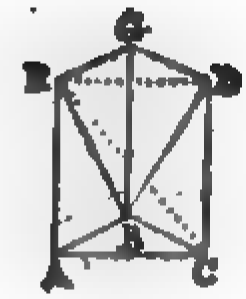
And since the triangles DAB, *dab*, at... surfaces are as the squares of their homolog...

Hence, triang. DAB : triang. *dab* :: DI... :: NV² : *n*V²: Therefore the sections D... squares of their distances from the vertex V... manner it is proved that the sections RGH... rV². Now the bases DAB, RGH, an... being equal, the sections *dab*, *rgh*, at equ... from the vertex, will also be equal. Ther... composed of a like series of indefinitely th... (or *laminæ*); the greatest term of the series... of RGH, and the least *o* at the vertex V; ... pyramids are equal. And when the base... kind whatever, the demonstration will ev... the foregoing.

Corol. Hence, if we suppose a circle t... of an indefinite number of indefinitely sho... it follows that cones having equal bases a... equal. And that cones and pyramids of eq... are likewise equal to each other.

33. *A triangular pyramid is one-th... the same base and altitude.*

Let ABCDGR be a prism upon the triangular base ABC. Then if it be cut through the diagonal RC by the plane RBC; and through the two diagonals BR, BD, by the plane RBD, it will be divided into three equal pyramids ABCR, RGDB, and RDCB.



For if ABC is the base of the pyramid whose vertex is R, and RGD the base of the pyramid whose vertex is B, those pyramids and the prism will have equal bases and altitudes; therefore the two pyramids will be equal (132).

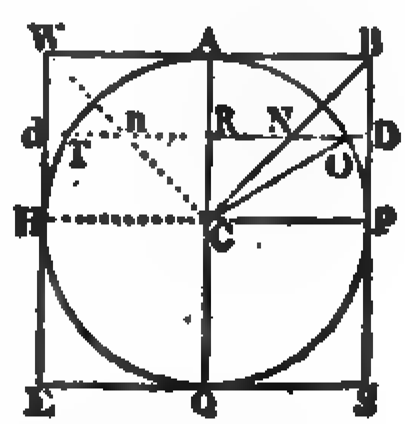
But the pyramids RDCB, ABCR, having the equal bases RAC, RDC, and the common vertex B, must also be equal, because in that case, their altitudes will be the same; therefore the three pyramids are equal to each other. And since the prism and (ABCR) one of the pyramids have the same base and altitude, the truth of the theorem is manifest *.

Corol. 1. Therefore prisms on polygonal bases are triple the pyramids on the same or equal bases, because the prisms may be divided into other prisms having triangular bases.

Corol. 2. And because prisms and cylinders, and pyramids and cones, having equal bases and altitudes, are respectively equal; therefore a cone is the third part of a cylinder of the same base and altitude.

131. *A sphere is two-thirds of its circumscribing cylinder.*

Let C be the centre of the circle circumscribed by the square WBSE; and draw CB.



Then if the rectangle QABS revolve about AQ as a fixed axis, the square CABP will describe the cylinder PHWB; the quadrant APC will

* A Learner will not readily comprehend this Theorem without models of the three pyramids.

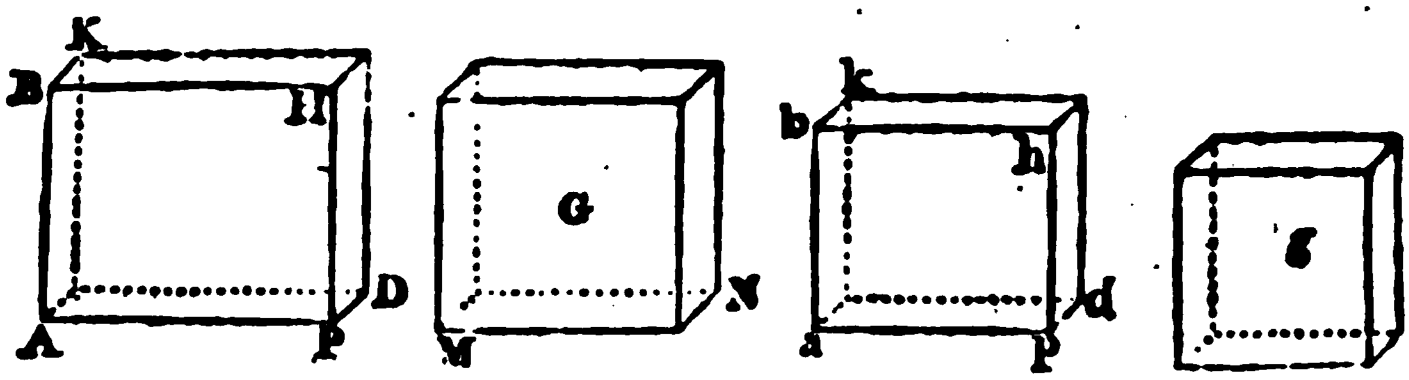
describe the hemisphere CPAH; and the triangle CBA will describe the cone CBW.

Let Dd be parallel to PH, and join CO. Then the radius CO = CP = RD: and because AB = AC, RN will be = RC: but RO^2 (RN^2) + RO^2 = CO^2 (83) = RD^2; or RN^2 + RO^2 = RD^2. But semicircles described on RC (RN), and RO, are together equal to a semicircle described on CO (107.) or RD; therefore circles described on their doubles will also be equal, or the circle on Dd equal to both the circles on Nn, and OT: consequently Nn the section of the cone, and OT the section of the sphere will together (in every section parallel to PH) be equal to Dd the corresponding section of the cylinder. Now supposing the cylinder to be composed of indefinitely thin parallel sections (Dd, &c.) then the cone on the same base (WB) being equal to one third of those sections, or $\frac{1}{3}$ of the cylinder HB (133. *corol.* 2), therefore the hemisphere must be equal to the remaining $\frac{2}{3}$ of that cylinder, or the whole sphere = $\frac{2}{3}$ of the whole cylinder EB.

Corol. 1. A cone, hemisphere, and cylinder, of the same base and altitude, are in the proportion of $\frac{1}{3}$, $\frac{2}{3}$, and 1; or 1, 2, and 3.

Corol. 2. It also appears, that the spherical frustum HTOP, is equal to the difference between the cylinder HdDP and the cone CnN. And that the spherical segment TAO, is equal to the difference between the cylinder dWBD and the conic frustum nWBN.

135. *Similar upright prisms* BD, bd, *are in the same proportion as the cubes of their altitudes.*



Suppose G and g are cubes having heights respectively equal to AB and ab the heights of the prisms. Then prisms of equal altitudes being as their bases (131, *corol.* 2) we have

prism G : prism BD :: base MN : base AD,

or AB^3 : prism BD :: AB^2 : base AD, because the prism G is the cube of AB, and the base MN its square.

And in like manner, ab^3 (or prism g) : prism bd :: ab^2 : base ad.

But the parallelograms ABHP, $abhp$ are similar; and the bases AD, ad, are also similar; therefore (102),

$AB^2 : ab^2$ :: ABHP : $abhp$ :: $AP^2 : ap^2$ :: base AD : base ad;

or $AB^2 : ab^2$ :: base AD : base ad,

or AB^2 : base AD :: ab^2 : base ad;

Whence by equality, AB^3 : prism BD :: ab^3 : prism bd, because the ratio AB^3 : prism BD, is equal to the ratio AB^2 : base AD, by the second of the above proportions, and the ratio ab^3 : prism bd equal to the ratio ab^2 : base ad, by the third.

If AK, ak, are made the bases, and AP, ap the perpendicular heights; then the prisms will be as the cubes of AP and ap: Hence,

Corol. 1. When four right lines AB, AP, ab, ap, are proportional, their squares, and also their cubes, will be proportional.

Corol. 2. And because similar plane figures are as the squares of their heights, or breadths, or other homologous lines in those figures, therefore similar prisms of any kind, and also cylinders, will be as the cubes of their like linear dimensions.

Corol. 3. Hence also, similar pyramids and cones, wh are like parts of similar prisms and cylinders, will be in the sa proportion as the cubes of their heights, or the diameters of th bases. And the like is to be understood of spheres, these bei $\frac{2}{3}$ of similar cylinders.

Scholium. This relation of similar solids is called Triplicate Ratio; an is sometimes demonstrated in parallelopipeds, by considering the ratio o the solids to be compounded of the ratios of the homologous linear dimen sions. To give an exemplification in numbers: Suppose the bases AD, *ad*, are rectangular; and AB, AP, PD, are in the same proportion as 12, 15, 6; and *ab*, *ap*, *pd*, as 8, 10, 4; then the solid *bd*, will be to the solid BD as $8 \times 10 \times 4$ to $12 \times 15 \times 6$; therefore $\frac{8 \times 10 \times 4}{12 \times 15 \times 6}$ will denote the ratio of those products (92, *Arith.*): but this ratio is compounded of the ratios of the homologous sides, namely, of 8 to 12 or $\frac{8}{12}$, 10 to 15 or $\frac{10}{15}$, and 4 to 6 or $\frac{4}{6}$, and the compounded ratio is $\frac{8}{12} \times \frac{10}{15} \times \frac{4}{6}$ (141, *Arith.*) which, in its lowest terms is $\frac{8}{27}$, the ratio of the solids; but $\frac{8}{27}$ is the cube of $\frac{2}{3}$, the ratio of either two homologous sides.

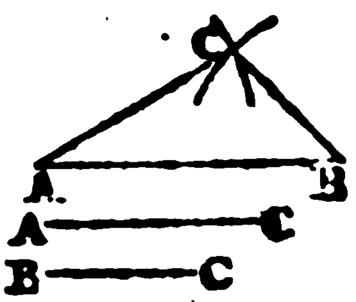
PROBLEMS,

WITH THE

METHOD OF TRACING THE FIGURES ON THE GROUND.

136. ***To make a triangle with three given right lines* AB, AC, BC.**

With the distances AC, BC as radii, about the centres A, and B, the extremities of the longest line, describe two arcs of circles intersecting each other in C; draw CA, CB. Then ABC is the triangle.



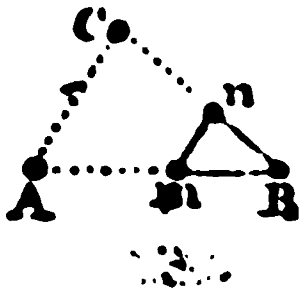
For the radii or two shortest sides of the triangle are, by construction, equal to the given lines AC, BC.

If both the shortest of the given lines together are less than the longest line, it is evident the arcs will not intersect each other, in which case the problem becomes impossible.

By means of this Problem, any right-lined figure may be copied: or a right-lined figure made exactly like another right-lined figure, first dividing the given figure into triangles.

A triangle may be marked on the ground by means of *cords*, or rather *measuring tapes or lines*: thus, suppose it is required to lay down the triangle ABC, whose sides shall be 60, 50, and 40 *feet*.

Having measured out AB = 60 *feet*, fasten the ends of two measuring lines at A and B; then draw them straight on the ground, and bring 50 *feet* on one line to 40 on the other, and where they intersect will give the point C.



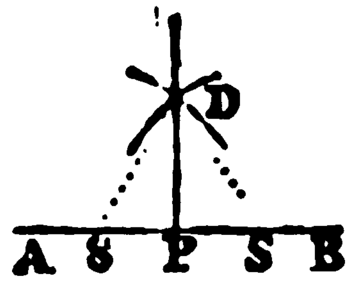
When the sides of the triangle are too long for the common measuring tapes or lines, lay down a triangle similar to that proposed, and then prolong the sides to the length required.

Thus suppose AB = 450, BC = 400, and AC = 300 *feet*. Take the same aliquot part of each side, $\frac{1}{10}$ for example (in the present case), or 45, 40, and 30 *feet*, and with those distances make the triangle B*mn*; then measure out BA = 450, and BC = 400; and if the triangle B*mn* is correctly laid down, AC will measure 300 *feet*. For, by similar triangles, 45 (B*m*) : 30 (*mn*) :: 450 (BA) : 300 (AC).

It is evident that any error in the length of *mn* will produce 10 times that error in AC; and therefore it may sometimes be necessary to repeat the operation more carefully.

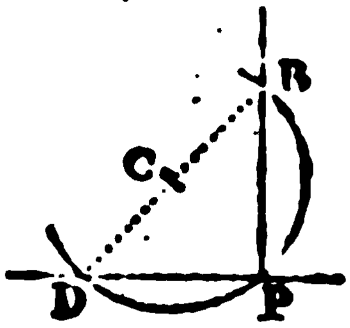
137. *At a given point* P *in a right line* AB, *to raise a perpendicular* PD, *to that line.*

On each side of P take equal distances PS, PS, and about S, S, as centres, with same radius, describe arcs intersecting each other in D; then draw PD for the perpendicular required.



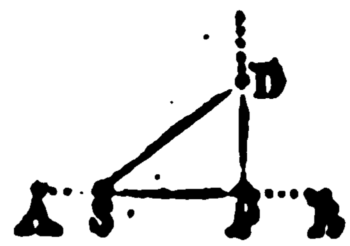
For if DS, DS, are joined, the triangle SDS will be isosceles; therefore (46, *corol.* 1), PD is perpendicular to SS or AB.

When the given point P *is near the end of the line.* About any convenient point C as a centre, describe a circle through P, cutting the given line in D, draw DCB, then join BP, which will be the perpendicular required.



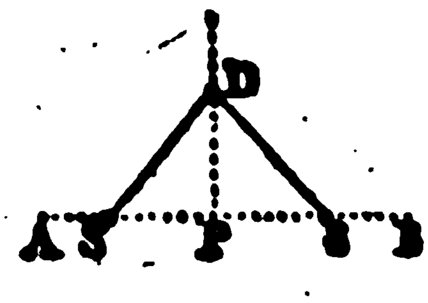
For DPB being a semicircle, the angle at P is a right one (72); therefore BP is perpendicular to DP.

This is readily performed on the ground by means of three *rods* or *lines*, whose lengths are in the proportion of 3, 4, and 5. Thus if the triangle SPD is laid down (by the preceding article) with SP = 16, PD = 12, and SD = 20 *feet*; then PD will be perpendicular to SP or AB (83)



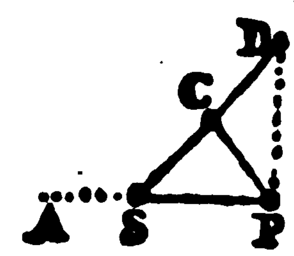
Otherwise thus:

Measure equal distances PS, PS on each side of P; then two rods or lines SD, SD, of an equal length, will make the triangle SDS isosceles; and consequently the direction of the perpendicular from P, is marked by the ends which meet at D.



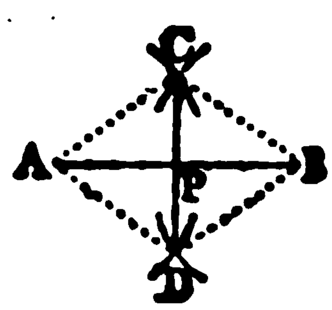
Or thus:

When the point P is near the end of the line. From any convenient point C make CS = CP, and CD = CS; S, C, and D being in a right line; then PD will be perpendicular to PA. For the angle DPS is a right one (72).



138. *To bisect or divide into two equal parts, a given right line* AB.

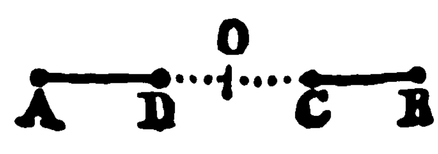
With any radius greater than half the given line, about the extremities A and B as centres, describe arcs intersecting each other in C and D: then draw CD, and it will bisect AB in the point P.



Draw the radii AC, AD, BD, BC: then those radii being equal, and the side CD common to both the triangles CAD, CBD, those triangles are therefore identical; and consequently the angle ACD is equal to the angle BCD. And since the triangle ACB is isosceles, AB is bisected by CP (46, *corol.* 1).

In this manner a line may be divided into 4, 8, 16, &c. equal parts. Thus AP, BP bisected give 4 equal parts; and those again bisected would make 8; and so on.

The most expeditious method of finding the middle of a line on the ground, is to measure equal distances from its extremities. Thus, suppose A and B are the ends of the line, and that AD, BC (found by measuring from A and B) are each 157 *feet*; and the remaining part DC is 19 *feet*; then O the middle of the line will evidently be 9½ *feet* from D or C.



In measuring lines or distances on the ground, it sometimes may be necessary to take *off-sets* when obstacles fall in the way.

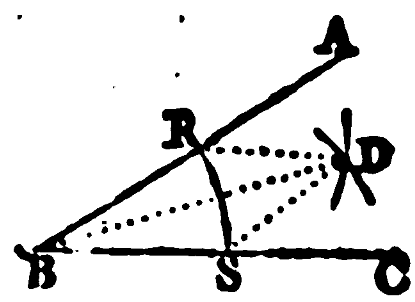
Suppose A and B are the extremities of a line to be measured: and that K and S are pools of water or swamps.

Having set up marks at R, G, D, C, in the line AB, measure equal *off-sets* CN, DO; and GW, RQ, at right angles to AB: then

the quadrilaterals RW, DN being rectangular, QW will be equal to RG, and ON to DC; and the whole line AB equal to AR + QW + GD + ON + CB.

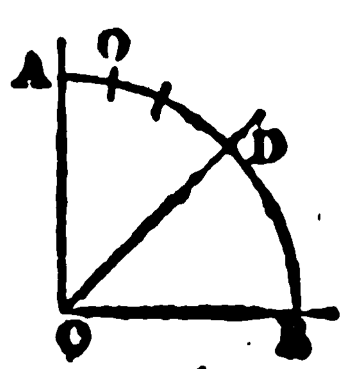
139. *To bisect a given right lined angle* ABC.

With any convenient radius BS, about the angular point B as a centre, describe an arc SR, and from the centres S, R, with any radius longer than half the distance between those points, describe two other arcs intersecting one another in D; then the line joining B and D will bisect the angle ABC, and the arc SR.



For if the radii SD, RD, are drawn, the sides of the triangles BRD, BSD will be respectively equal, each to each, therefore they are also equi-angular (46.), and consequently the angles RBD, SBD are equal.

By such bisections, an angle or its corresponding arc may be divided into 2, 4, 8, &c. equal parts. Thus if ACB be a quadrant, or an angle of 90 degrees (64.); the first bisection divides it into two equal angles, or the arc AB into two parts (DA, DB) of 45 *degrees* each: another bisection divides the arc AD into two equal parts of $22\frac{1}{2}$ *degrees*: the next gives an arc AO of $11\frac{1}{4}$ *degrees*: and if the bisection be continued 7 times, we get an arc of $42\frac{3}{16}$ *minutes*. Such a division is rea-



B) is 5 or 6 *inches*; and will
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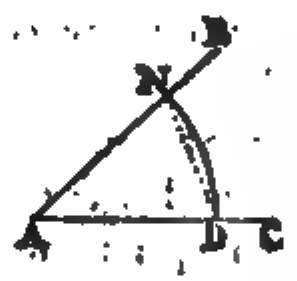

e radius, de-
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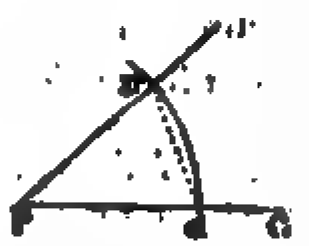

ke an angle which shall contain a
protractor, line of chords, or a sector,

ircular instrument for measuring and
h is divided into 180 equal parts or
d over the intersection of two lines,
shewn by the intercepted arc on the
rotractor for the same purpose is fre-
es, the centre being on one edge, and

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142. A li
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143. *To m*
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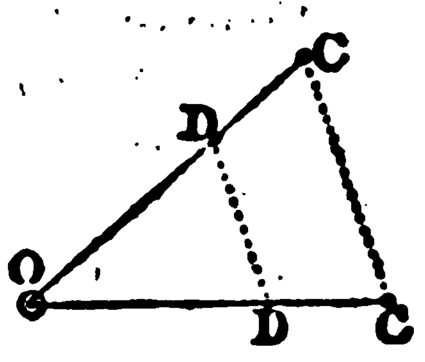
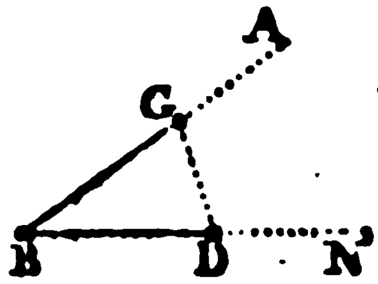
of 60 degree

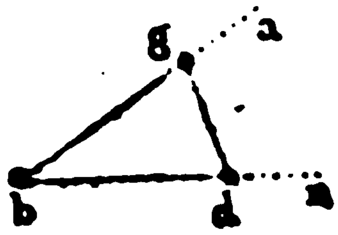
145. *To measure the angle* ACB *with the sector.* See the Fig. to *Art.* 143. About C with any radius CD, describe an intercepted arc DG. Open the sector till the distance between the brass points marked C, C, (the extremities of the chord-lines) is equal to the radius CD. Then if the distance DG be laid cross-ways on those chords, so that its extremities are equally distant from C, C, or from the centre of the instrument, the points of the compass will fall on the number of degrees in the angle. Thus if CO, CO, be the chord-lines of 60 degrees each on the sector, (moveable about the centre O) and DO the chord of any other arc, 40 degrees for example; then by similar triangles CO (the *radius*): DO (the *chord* of 40 degrees) :: CC : DD; therefore if CC be made the radius of any arc, or circle, DD will be the chord of 40 degrees in that arc, or circle.



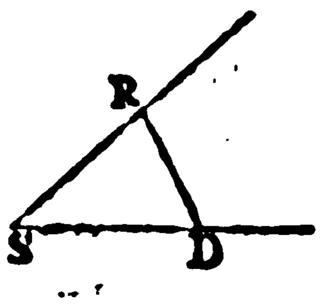
Hence it is, that the sector has frequently the advantage of the protractor, or common line of chords, because it may be set to different radii; the limits being the distance between the brass points C, C, when the instrument is shut, and their distance when it is quite open.

146. When it is proposed to trace an angle on the ground equal to another angle, the operation is similar to that in *Art.* 136. Thus, to lay down the angle *abn* equal to the angle ABN, the direction of *bn* being given. Measure equal distances BD, BG, and also the cross distance GD; then with those three distances lay down the triangle *bdg* (136), and the point *g* gives the direction of *ba*.



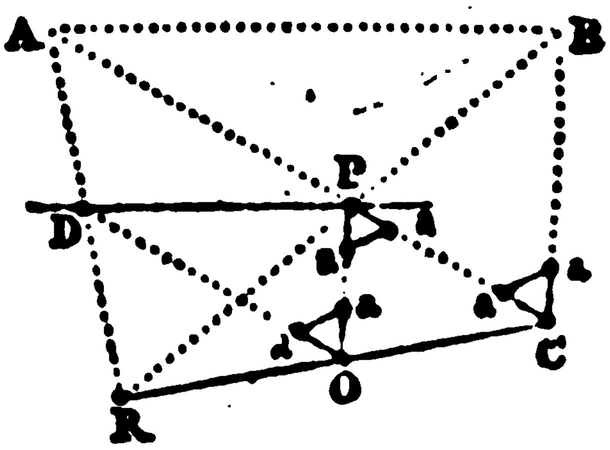


147. If the angle *abn* (when laid down) is to contain a given number of degrees; first, make an angle DSR on paper equal to those degrees; then having measured the equal sides SD, SR, and the opposite side RD on some convenient scale of equal parts, let the triangle *gbd* be traced on the ground with three corresponding distances in *feet* or *yards*, &c. (136). Thus, suppose the angle RSD is 41 degrees, then if SR, SD are each 40 on a scale of equal parts, RD will be 28 on the same scale, nearly: consequently if the triangle *gbd* is traced on the ground, with 40, 40, and 28 *feet*, the angle *abn* will be 41 degrees.



found by similar triangles. For CR : CB :: CK : CP. And a perpendicular from B on CR will give the distance GP at another proportion.

151. When it is proposed to trace a line through a given point P parallel to an inaccessible line AB, set up marks at any convenient points C, R, in the directions AP, BP; next, by means of three equal isosceles triangles Caa, Paa, Oaa, trace PO parallel to CB, and OD parallel to PC; then the direction DP is parallel to AB.



For by construction OD is parallel to CA, and OP to CB; therefore the triangles ORD, CRA; and OPR, CBR, are respectively similar;

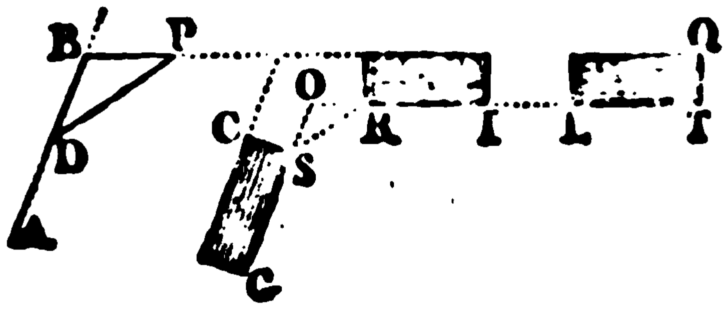
Hence RO : RC :: OD : CA,

And RO : RC :: OP : CB; therefore by equality of ratios, OD : OP :: CA : CB.

Now the sides about the equal angles DOP, ACB of the triangles DOP, ACB being proportional, those triangles are therefore similar (94, *corol.* 1); and since the homologous sides are respectively parallel and like situated, the third sides DP, AB must also be parallel.

Corol. Because the quadrilaterals RDPO, RABC are similar, if we measure the sides RO, DP, RC, the inaccessible distance AB may be found at one proportion; for RO : DP :: RC : AB.

151*. In *castramentation* it is sometimes necessary to change the direction instead of continuing the fronts of all the battalions or divisions in the same line. Let QR be two divisions of the encampment, the fronts being in the same line OT, and IL the distance between them; and let it be required to place the other divisions GC, &c. that the fronts SG, &c. may be in a given direction or parallel to a given line BA, the distance between the divisions remaining as before or RS = IL, and (as is usual) the two prolongations RO, SO of the fronts equal to each other:



In BQ and BA take two equal distances BP, BD, and measure the sides of the isosceles triangle DBP; then the point O is found thus, DP : PB ::

RS (or IL) : RO. Suppose BP =
20 feet ; then 50 : 30 :: 20 : 12 fe
12 feet, a string or tape OS = 12,
from O and R will giv the point
triangles DBP, SOR being simila
SOR, DBP are equal, and conse

132. *From a given poi*
PG *upon a given line* AB.

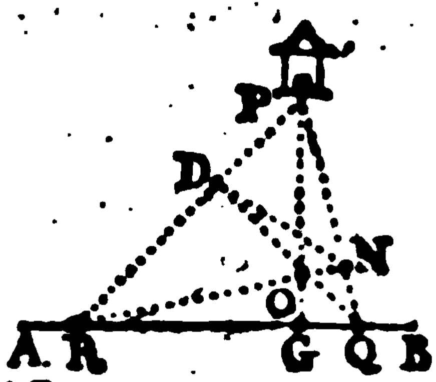
About P as a centre with
greater then the distance of
scribe an arc DC; and from
a radius greater than half D
intersecting each other in R
PG is the perpendicular requ

Draw the radii RC, RD
DP, respectively, and the sid
RCP, RDP, those triangles
the angles CPG, DPG are eq
therefore PG is perpendicular

When the point is nearly
end of the line. From an
AB, describe an arc PDR;
to DP; then join RP: an
perpendicular to AB.

For by construction CD b
PG is perpendicular to CD (

When a perpendicular is to b
trace the line CPD paralle
perpendicular to CD at the
perpendicular to AB.



153. *If the object P is inaccessible:* Set up marks at any two convenient points R, Q, in AB; then on RP, QP, trace the perpendiculars QD, RN; and the point of intersection O gives the direction of the perpendicular POG.

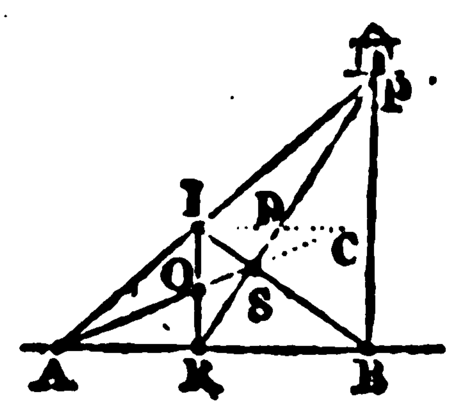
We have to prove that POG is perpendicular to AB.

Conceive DN to be joined: Then because the opposite angles ODP, ONP, of the quadrilateral ODPN are right angles, a circle will pass through the points O D, P, N. (72), therefore the angles ODN, OPN, standing on the same chord ON (of the circle) will be equal to each other (70).

And since RDO, QNO are right angles, and the angle ROD equal to the angle QON, therefore the triangles RDO, QNO are equi-angular; hence DO : ON :: RO : OQ; therefore the triangles ODN, ORQ, are also equi-angular (94, *corol* 4), consequently the angle ORQ = ODN = OPN. But the angles ORQ, GQN, together are equal to a right angle (41, *corol.* 2); therefore OPN and GQN make a right angle, and consequently PGQ is a right angle.

Corol. Hence the three perpendiculars let fall from the angles of a triangle upon the opposite sides, will intersect one another in the same point.

Or thus:



Let AB be the line, and P the inaccessible object as before. At any convenient point R in AB, trace a perpendicular RO to AB, which continue till OI = RO. Make PIA a right line, then mark the point S where the lines AOS, and RP meet, also the point B or concourse of the lines AB and ISB. And PB will be perpendicular to AB.

For let IC parallel to AB meet AOC in C; then AR : RB :: CD : DI (95), and by composition (94, *schol.*) AB : RB :: CI : DI. But because the triangles OIC, ORC are similar, and OI = OR, therefore CI = AR, hence the last proportion becomes

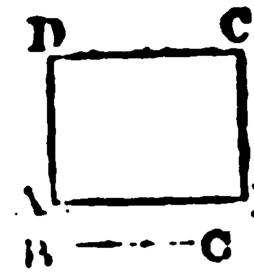
AB : RB :: AR : DI.

And AP : IP :: AR : DI, by the sim. triang. ARP, IDP;

Therefore AB : RB :: AP : IP (by equality); therefore RI, BP are parallel (94).

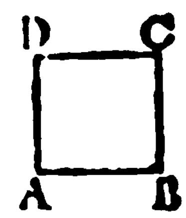
Corol. By this problem we may find the distance of an inaccessible object P from an accessible line AB. For if we measure AR and RB, it will be AR : RB :: AB : BP.

134. *On a given base* AB *to make a Rectangle whose height shall be equal to a given line* BC.



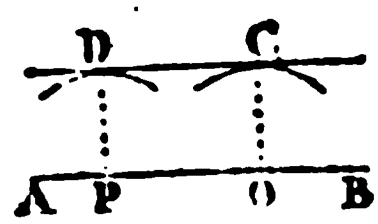
At the extremities of the base AB erect the perpendiculars AD, BC, each equal to BC; then join DC; and ADCB is the rectangle required.

For AB and DC being at the same perpendicular distance, they must therefore be parallel; and since the angles at A and B are right angles, the parallels AC, BC, will meet DC in right angles (40, *corol.* 2); therefore DB is a rectangle (22).



Corol. 1. In like manner a square is constructed on a given line AB by making the perpendiculars AD, BC, each equal to AB.

Corol. 2. Hence also, a line (DC) is drawn parallel to a given line (AB) at a given distance (BC).



155. The following is also a practical method of drawing a line DC parallel to another line AB, at a given distance PD.

With the given distance PD in the compasses, about any two points P, O, in AB, as centres, describe arcs D and C; then lay the edge of a ruler to touch those arcs, and draw the line required. For if PD, OC are drawn to the points of contact, PDCO will be a rectangle (67).

To trace a Rectangle on the Ground. Having measured out one side (the direction being given) to the required length, erect perpendiculars at its ends; then if those perpendiculars are prolonged to the distance proposed, their extremities will evidently mark the angular points of the Rectangle.

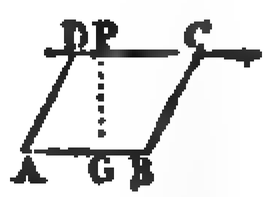
…r a line is traced parallel to another line inaccessible…
…a proposed distance from that
…it be required to trace the line
…the face of the bastion CD, at the
…ards.

…the point G in the direction CD,
…ndicular to DG, and equal to 300
…BA is traced perpendicular to GB, it will be parallel

…is constructed at A against the bastion, the shot (at
…B will strike its face CD in a perpendicular direction,
…st force possible.

…*given base* AB *to make a parallelogram* DB
…*ight* GP, *so that the sides* AD, AB *shall form*

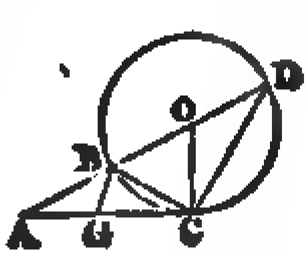
…oint G in AB erect the perpen-
…ual to the given height (137);
…w DC parallel to AB (140);
…ngle DAB equal to that pro-
…h n draw BC parallel to AD; and ADCB is
…am. For the opposite angles being equal, the
…hose angles will also be respectively equal (60).



…ce from a given point (A) to draw a line AD) to
…ne (DC) in a given angle (BCP). At any point
…line, make an angle DCB equal to the angle
…from the given point A draw AD parallel to CB;
…done.

…*ide a given line* AC *according to mean and*
…*tion; or so, that the rectangle under the whole*
…*art, shall be equal to the square on the other*
…CG :: CG : GA.

…pendicular to, and $= \frac{1}{2}$AC;
…re with OC describe a circle;
…join DC, and parallel to it
…en CA : CG :: CG : GA.



Join CB. Then the triangles ABC, ACD being (99) we have, AD : AC :: AC : AB, or AD : BD : AB (because BD = AC); therefore AD is divided in ding to mean and extreme proportion: And because B rallel to DC, it divides CA in the same proportion in G is divided in B (94, *corol.* 2).

Corol. Hence AB = GC. For because of the parallels

$$AD : BD :: AC : GC \ (94).$$

$$\text{And } AD : BD :: BD : AB;$$

whence (by equality) AC : GC :: BD : AB; now the dents being equal, the consequents GC, AB, are neces

135. *In a given circle to inscribe a regular Pent*

Having divided the radius CA (by the foregoing Problem) according to mean and extreme proportion in G, make GB = GC; take AD = AB; then draw BD, which will be the side of the pentagon, or the chord of $\frac{1}{5}$ of the circumference of the circle.

Draw GR parallel to AB: then CR = CG, and GA (94).

By construction, CA : CG :: CG : GA, or because G

$$CB : GB :: GB : RB.$$

But the angle GBR is = the angle GCB, therefore CB, GB; GB, RB about those equal angles, are prop hence the triangles BGC, BRG are equi-angular (94, c therefore the former being isosceles, the latter BRG be isosceles, consequently RG = RB. But the outwa GRC of the triangle GRB is equal to both the inward angles, and therefore equal to twice the angle GBR quently the angle ABC, which is equal to GRC, is angle GBR; therefore BG bisects the angle ABC.

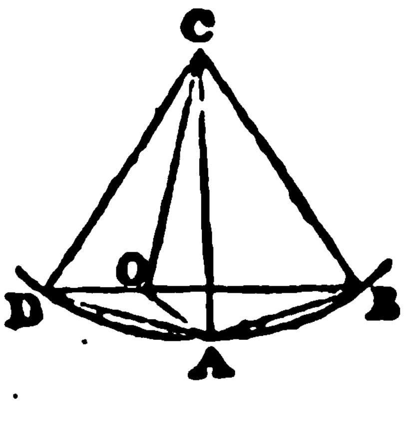
the isosceles triangle ACB, each of the angles at A and B is double the other angle ACB.

Now all the angles of the triangle ACB being $\frac{5}{5}$ of two right angles, the angle ACB is $\frac{1}{5}$ of two right angles, and its double, or the angle DCB $= \frac{1}{5}$ of 4 right angles: therefore DB is the chord of $\frac{1}{5}$ of the circumference: and 5 of those chords form the pentagon.

Corol. 1. Because the angles ABG, ACB are equal, and the angle CAB common to the triangles CAB, GAB, and the former isosceles, the latter GAB is also isosceles, and consequently AB $=$ BG ($=$ GC); therefore if the radius of a circle is divided according to mean and extreme proportion, the greater segment (GC $=$ GB $=$ AB) will be the side (AB) of a regular decagon in that circle.

Corol. 2. Hence also, BD bisects GA, and the angle GBA.

159. THEOREM. The square on the side DB of a regular pentagon inscribed in a circle, is equal to the square on the radius CB, and the square on DA the side of the decagon taken together (*Euclid, B.* 13. *Pr.* 10.):



Let CO bisect the angle DCA; and join OA.

The angle DCB is equal to $\frac{4}{10}$ $\Big\}$ of 2 right angles:
and DCO ...... to $\frac{1}{10}$
Therefore OCB is equal to $\frac{3}{10}$ of 2 right angles.

And each of the angles CDB, CBD is also equal to $\frac{3}{10}$ of 2 right angles:

Therefore the triangle COB is isosceles, and OC $=$ OB:
Consequently the triangles COB, DCB are equi-angular;

Hence, OB : BC :: BC
C is equal to the rectangle un

And because the triangles D
gle ODA common, those tr

Therefore AD : DO :: D
D is equal to the rectangle un

And therefore the m of
equal to the sum of the
DB. But OB × DB +
square on BC + the squa

160. *On a given line A*
tagon.

Make CO perpendicular to
½AC ; through O draw AD to
OD = OC: join CD, and th
be the radius of the circle in
AC is the side of the Pentago

Take OB = OC. Then
that in *Art.* 157; AD will
mean and extreme proportion
made the radius of a circle, B
the same circle (158, *corol.*).

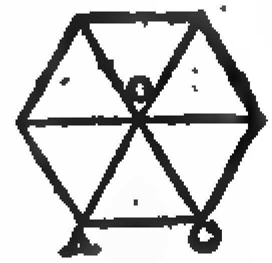
But the triangles ADC, A
Hence AD : DC :: A
or AD : DB :: D
are in the ratio of the radius
scribed decagon. Hence, be
$CB^2 = BD^2$ (83); theref
Pentagon, CB will be that of
of the circumscribing circle (

Therefore make AG, CG,
centre of the circumscrib

S

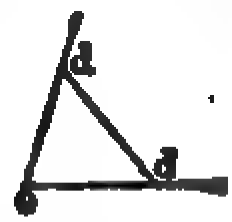
a given line AC *to construct a regular Hex*.

), CO each equal to AC (136);
ıgle AOC will be equilateral and
'; then 6 of those triangles,
an angular point at O, will evi-
the required hexagon.



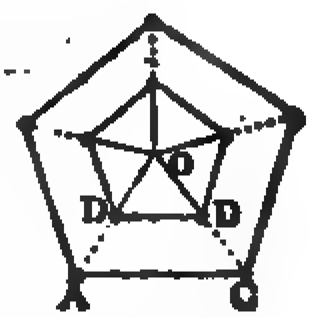
, or Hexagon, when the extent of the sides are too great
on measuring tapes or lines, may be traced on the ground
proportional distances (136). Thus, suppose it is required
Pentagon whose side AC shall be 100 *yards*.

le the angle *dOd* = 72 degrees on paper
e equal distances *Od*, *Od* on a scale of
uppose 80 each, then the distance *dd* will
n the same scale.



5 triangles DOD, &c. with the equal
), &c. each equal 80, and DD, &c.
et (136).

nilar triangles, 94 (DD) : 80 (OD)
: 255 *feet* nearly = OC: therefore
&c. are measured out to 255 *feet*
tremities will mark the angular points of the pentagon.



exagon may be traced on the ground in the same manner by
uilateral triangles.

ng large and regular Works where exactnes is required, the
centres should be laid down with a Theodolite, and the
the angular points of the Polygons computed trigono-

the common Geometrical Problems, the foregoing are
ost simple and necessary in Field-practice. It is easy to
ver, that great accuracy cannot be expected, particularly
und is not level.

divide a given line AB *into a proposed number*
rts: suppose 3.

From the extremities draw AC, I
parallel to each other; in those li
take 5 equal parts of any conveni
length (BO = OG &c. = AS = !
&c.) join the opposite points of d
sion; and AB will be divided int
equal parts.

For BQ being parallel and equ
also be parallel and equal (80); tl
opposite points divide AB in the
AR, BQ are divided (94).

Or *thus*:—Having drawn AC
take the proposed number of equ
fore; then join RB, and parallc
points of division in AR, and the
quired number of equal parts (93).

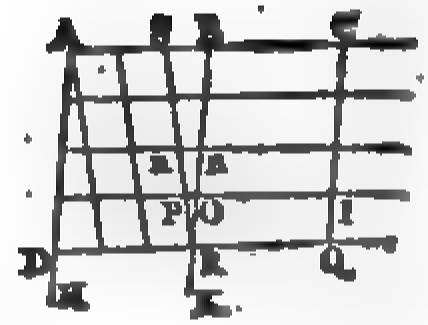
When the given line is too sl
sions, the following method is sor
same purpose.

Suppose AB is a given line to
In a line CQ of any convenien
length, take 7 equal parts, suppos
from C to A; with CB = CA
and AB = the given line AB
make the isosceles triangle CBA
(136); take GC = DC, CR = C
points of division.

Then by similar triangles, CA
cause CD is $\frac{6}{7}$ of CA, DG will
OR = $\frac{4}{7}$, &c. and the shortest lin

103. Hence is derived the method
scale be constructed to 12ths, of the l

Having divided AB into 3 equal parts, draw two parallel lines AH, BK making any convenient angles with AB; in those lines take 4 equal distances, suppose from A to D, and from B to R; and through the points of division draw 4 lines parallel to AB; next, divide DR into 3 equal parts; then if the points of division in AB and DR are joined diagonally, the scale is constructed.



For by similar triangles, RB : BS :: RO : OP; therefore RO being $\frac{1}{4}$ of RB; OP will be $\frac{1}{4}$ of BS, or $\frac{1}{4}$ of $\frac{1}{3}$ (or $\frac{1}{12}$) of BA; and the next division *nn* is $\frac{2}{12}$, &c.

If QR = CB = BA is the scale for a *foot*, OP is an *inch*, *nn* = 2 *inches*, IP = 13 *inches*, &c.

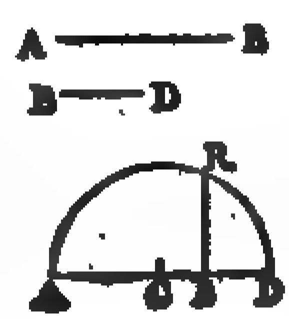
But if we divide AB into 4 equal parts, only 3 must be taken in AH and BK to make 12*ths*. of AB (because 4 × 3 = 12).

Generally:—Resolve the number to which the divisions are to be extended, into two factors, then divide the given line (AB) into as many equal parts as there are units in one factor, and take as many equal parts in the other lines (AH, BK) as there are units in the other. Thus if AB is divided into 3 equal parts, and 5 are taken in AH, BK; or if AB is divided into 5, and 3 are taken in AH, BK, in either case the scale gives 15*ths*. of AB. On the common Plain Scales, the equal parts in each line are 10, which give the divisions in 100*ths*.

A line divided into equal parts, and one of the parts subdivided, as in *Art.* 162, or else diagonally, is called a *Line or Scale of Equal Parts*. A variety are to be found on the common Plain Scale belonging to a Case of Instruments.

164. *To find a mean Proportional between two given lines* AB *and* BD.

Take AB and BD in one line AD, which bisect in C; and about C as a centre, with CA or CD describe a semicircle; then if BR be drawn perpendicular to AD, it will be the mean proportional required (97, *corol.* 1).



65. *To find a* ... CD.

Draw two lines AG ... venient angle at A; ... and AO each = C... to which draw OP ... proportional require...

For OP being parallel to ... ilar; therefore AB : AL...

66. *To find a 4th P...* , BC, CD.

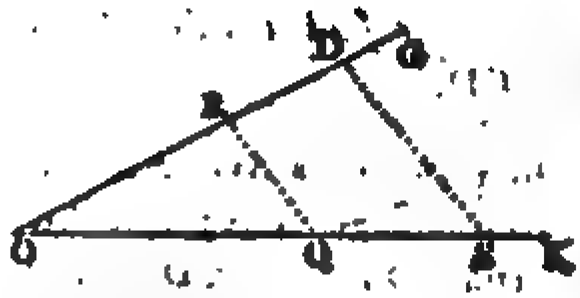
Having taken two lines AC ... the foregoing Problem, ma... , AC = BC, and join BC; ... = CD, and draw DP paralle... similar triangles AB : AC ... the 4th. proportional require...

67. This Problem is of very ... *Scales, Plans, and Maps.* We...

If ABCD be the Plan ... country, and suppose ... distance between the ob... O, P, is 1700 paces of ... at 5¼ feet each; it is ... required to make a Scale of ... to the Plan.

1700 ... = 1580 yards.

m two indefinite
forming any an-
OS equal to the
ad from any scale
, set off OD =
R = 1000; join
el to it draw RQ,

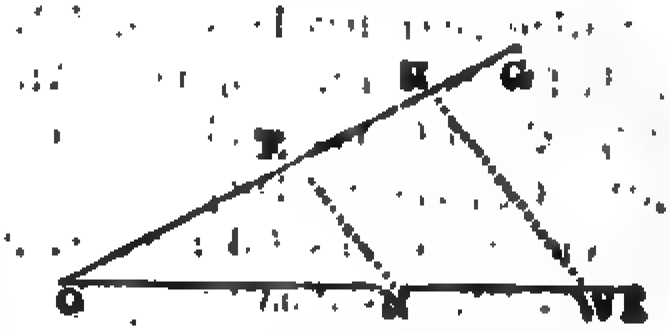


cale of 1000 *yards*. This divided, and subdivided is the
hich each of the least divisions is 100 *yards*.

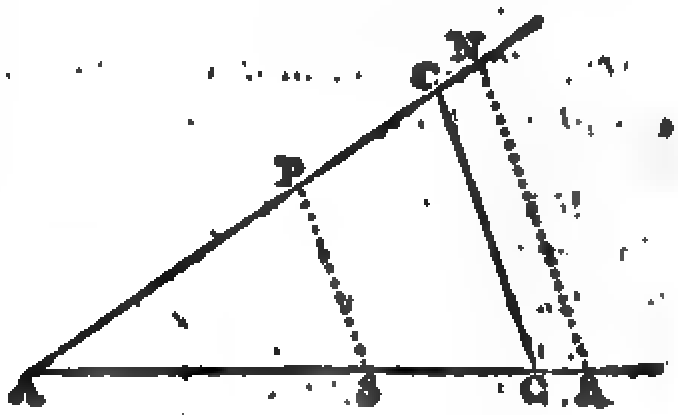
e construction thus) The distance OP measured on a
hes.

8 : 1·53 :: 1000 : 0·98 *of an inch*, the length of *on* the
yrds.

lan in the last Example be reduced to a Scale of 1 *inch to*



lefinite lines
in the last
f OE = 1000,
) (the yards
any conve-
equal parts;
= the scale
nd parallel to
then OW is
mile to the



C = OW, and CC = 1,
ales) make the isosceles tri-
n because any two corre-
es on the Plan must be in
ion as the two scales, if AR
to the length of the Plan
= AB its breadth, RN and

parallel to CC) will be the length and breadth of the
d.

Or, without the construction:

As 1338 : 1·53 *inch.* (OP) :: 1760
scale to the Plan ABCD.

The length and breadth of the Pl
respectively;

Hence, 1·73 : 1 :: 1·89 : 1·09 *inc*
1·73 : 1 :: 1·15 : 0·67 ....

By means of the triangle ACC we
one Plan to the other exactly in th
breadth *ad*, *ab*, were found. For any
AC, the proportional distances on the
ing parallels to CC. But the *Pro*
adapted for expedition in operations
of the instrument till, at the same ope
end is equal to one of the Scales (AC
other end equal to the other Scale (C
of the points at one end, will give
distance at the other. Or any two line
may be used instead of the Scales ther

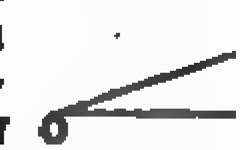
And *vice versâ*, any two correspond
Maps, and the length of one Scale, wi

3. Suppose a Map is laid down t
let it be required to adapt a Scale (P
to the same Map.

The Toise is = 2·1315 yards
(*Arith.*).

Therefore $\frac{2·1315 \times 4000}{1760}$
= 4·84 *miles* nearly the Scale AB.

On two indefinite
OS, OG, making
angle at O, set
OS = 4·84, and
= 4 from any
convenient scale of
parts, make
= the scale AB; join SG and d
is a scale of 4 miles.

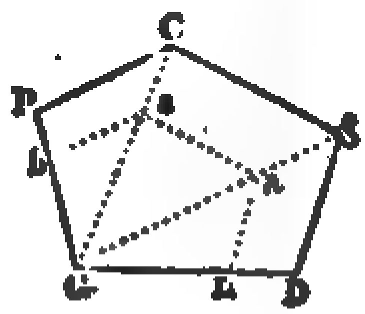


e length of the scale AB is 1·73 *inches?*

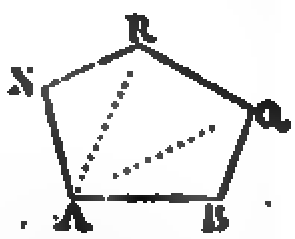
4·84m. : 1·73in. :: 4m. : 1·43in. the length of the 4 mile

, or any part of it, may be enlarged, or diminished to a after the manner of *Examp.* 2. For we can suppose ABCD rt of a large Plan.

en line AB to make a figure NB similar to a right lined

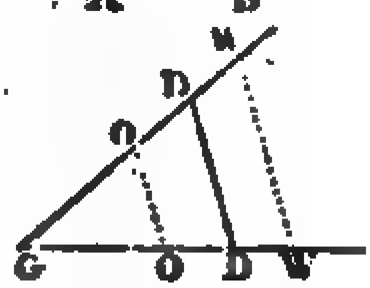


l AB make the isosceles triangle lraw the diagonals GS, GC: foregoing Examp. any lines of being laid on GD, the corre- of the required figure will be DD. Thus if GW = the di- l GO = DS; WW, and OO nal AQ, and side BQ.



e may be constructed on the

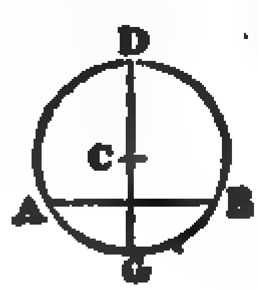
B, then draw *ua*, *ax*, and *ab* C, and CP.



angle is preferable to any other ns, because the parallels (WW, ortionals are found with greater

going constructions which respect the reduction of figures, nfined to a small scale; but the method may be extended of any size whatever.

nd the centre of a given Circle.



d AB be bisected at right angles therefore, will be a diameter to n C the centre of GD, will also f the circle (65, *corol.*).

gh three given points, not lying in a right line, circle.

Let A, B, D, be the three points. Draw BA, BD, and bisect those lines with the perpendiculars RC, PC: then the intersection C is the centre of the circle (65, corol. 4.), which described with the radius CA, CB, or CD, and the thing is done.

And in the same manner a circle is described

70. *Through a given point P to draw* *circle.*

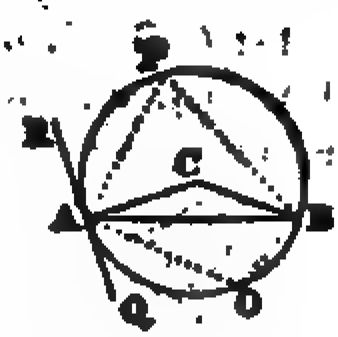
If P is in the circumference of the circ draw the radius CP, then a line through P right angles to PC is the tangent required (6 corol. 1).

When the given point P is without the ci to the centre C; and on PC describe a semi-circle; then PA drawn to the intersection of the circles will be at right angles to the radius CA (72), and therefore a tangent to the circle.

71. *To draw a Tangent to a circle par* line AB.

Draw the radius CR parallel to AB, and make the radius CP perpendicular to CR; then a line through P, parallel to CR (and AB) will touch the circle in that point because it forms a right angle with the radius PC.

72. *On a given line* AB *to describe a S* *cle that shall contain a given angle.*

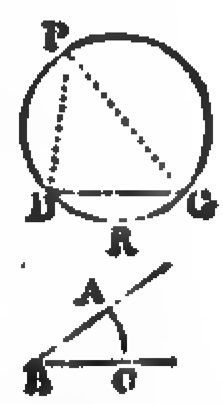


At the extremities A, B, of the given e, make each of the angles CAB, CBA ıal to the difference of the proposed angle d a right one; and with CA or CB de- ibe a circle: Then the segment APB on : same side of AB as the centre C, will ıtain the given angle when it is *less* than a right one; d the opposite segment AOB will contain it when it is *eater.*

For if RQ be a tangent at the point A, it will be perpendi- ıar to the radius AC (67, *corol.* 1); then the angle CAB the difference of the right angle CAQ and the angle ıQ; but BAQ is equal to the angle (APB) in the segment ᴾB (73).

And the angle CAB is equal to the difference of the right gle RAC and the angle RAB, but this latter angle is equal to · angle (AOB) contained in the segment AOB (73).

173. *To cut off a segment from a given circle that shall ıtain an angle equal to a given angle* ABC. *Or to draw hord in a given circle that shall subtend a given angle at circumference.*



About B with the radius of the circle, describe : arc AC; make the arc DRG = double the : AC, and draw the chord DG: Then the an- (DPG) in the segment DPG is equal to the gle ABC (69).

Or thus:—At any point (P) in the circumference, make an gle (DPG) equal to the given angle; then the line (DG) ıing the extremities of the sides including the angle, is the ord required.

174. *To inscribe an equilateral triangle in a given cle.*

Bisect the radius CD at right an
he chord AB; join BP, AP; and A
iangle.

Draw AD, BD, AC, BC. Then
C = OD, and the side AO common
ngles ACO, ADO will be identical
al to the radius AC or CD; conseq
CD, BCD, are equilateral; but the
A = PDA (70); and the remai
D or DCB (71); therefore the tria
equilateral,

Square is inscribed in a circle b
wo diameters which intersect each o

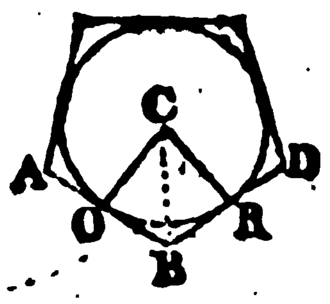
75. *To inscribe a circle in a giv*

isect two of the angles, ABD, D
n the intersection C of the bisecti
fall perpendiculars CS, CG, CF
; then if a circle be described
either of those perpendiculars, it w
sides of the triangle in S, G, and R

or the two angles at A being equa
ght ones, and the side AC commo
C, those triangles are therefore ide
sides CG, CS are equal. A
ner it is proved that CR and CG
the sides of the triangle will be ta
and S (67).

orol. Hence three lines bisecting
intersect one another in the same

76. *To inscribe a circle in a re*

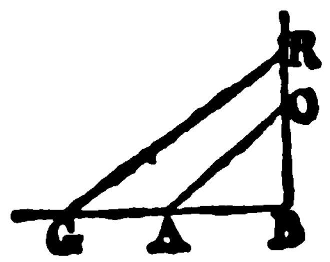


Bisect any two adjacent sides (BA, BD) with perpendiculars CO, CR; then their intersection C is the centre of the inscribed circle.

Draw BC. Then the hypotenuse BC being common to both the right angled triangles BOC, BRC, and BO = BR, the squares on OC, RC, will therefore be equal to each other (83, *corol.*), and consequently OC = RC. In like manner it is proved that the perpendiculars bisecting the other sides are all equal and meet in the same point C. Therefore a circle described with CO or CR will touch all the sides of the polygon (67).

And it is also evident that CB is the radius of the *circumscribing circle*; but this line bisects the angle ABD: Therefore to circumscribe a regular Polygon with a circle; bisect an two of its angles (except opposite ones) and the intersection of the bisecting lines is the centre of the circle.

177. *To make a square equal to two given squares.*

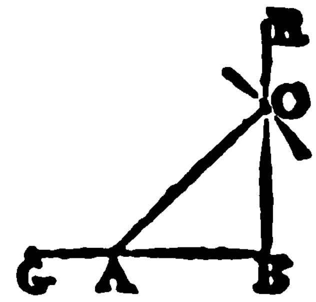


Let BA, BC, the sides of the given squares be drawn to form a right angle ABC; join AC, which will be the side of the square required (83).

And in the same manner a square may be made equal to three, or more squares. For example, suppose the sides of three given squares are AB, BC, and BG; then because the square on AC is equal to the squares on AB, BC, if BR be made equal to AC, it follows that a square on GR will be equal to the three proposed squares.

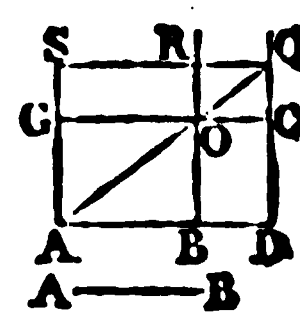
178. *To make a square equal to the difference of two given squares.*

With two indefinite right lines BG, BR, make a right angle B; take BA equal to the side of the less square; and about A as a centre with AC the side of the greater, describe an arc to intersect BR in C; then BC is the side of the required square (83, *corol.*).



179. *On a given line* AB *to make a rectangle equal to a given rectangle* AGCD.

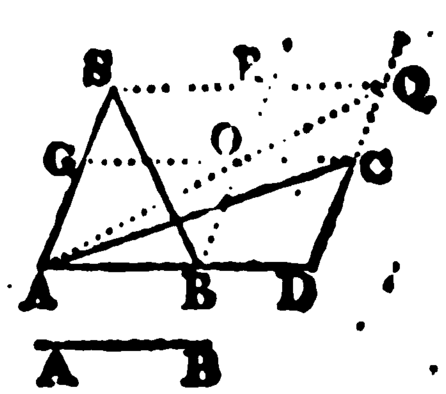
On AD (produced if necessary) take AB = the line AB: draw BR parallel to DC; and through O let AQ be drawn to meet DC produced; then if QS is made parallel to DA, BRSA will be the rectangle required.



For the triangles ASQ, ADQ; and ORQ, OCQ being respectively equal (80), the quadrilaterals ASRO, AOCD must therefore be equal (33); but the former, together with the triangle AOB, and the latter with the triangle AOG, make the two rectangles BRSA, AGCD, those rectangles must therefore be equal to each other, because the triangles AOB, AOG are equal.

180. *On a given line* AB *to make a triangle* ASB *equal to a given triangle* ADC.

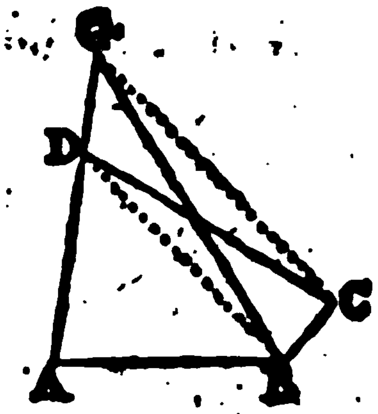
Draw AG and CG parallel to DC, and DA, respectively; then the parallelogram GD will be double the given triangle ADC (82, *corol.* 1): take AB equal to the given line AB; and by the construction in the preceding Problem, make the parallelogram AR equal to the parallelogram GD; draw the diagonal SB; and the triangle ASB will be equal to the given triangle ADC.



For the parallelograms AGCD, BRSA being equal, their halves must also be equal.

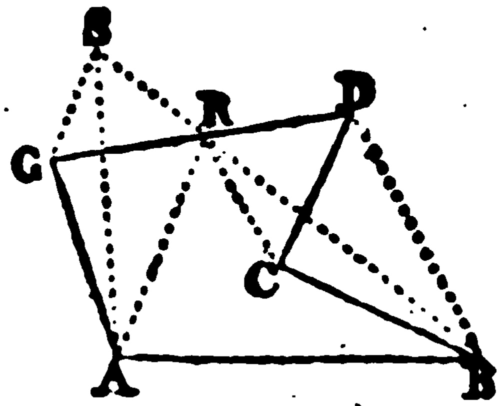
181. *To make a Triangle equal to a given Quadrilateral* ABCD.

Parallel to the diagonal BD draw CG to meet AD produced; join BG: then the triangle ABG is equal to the given quadrilateral.



For the triangles BCD, BGD on the same base BD, and between the same parallels BD, CG, are equal (82^a), therefore the triangle ABD, together with the triangle DBG is equal to the same triangle ABD together with the triangle BCD (32), or the triangle ABG equal to the quadrilateral ABCD.

182. *To make a Triangle equal to the irregular pentangular figure* ABCDG *on the side* AB.

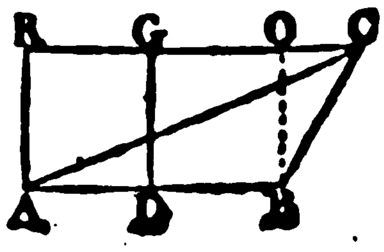


Let CR be drawn parallel to BD, and join BR. Then the triangles CBR, CDR, on the side CR and between the parallels CR, BD are equal (82^a); therefore the figure ABCRG with the triangle CBR, is equal to the same figure together with the triangle CDR, and consequently the given figure ABCDG is reduced to the quadrilateral ABRG.

Now the quadrilateral ABRG is reduced to a triangle by the preceding Problem, thus:—Parallel to the diagonal AR draw GS to meet BR produced; then join AS; and the triangle ASB will be equal to the quadrilateral ABRG, and therefore equal to the given figure ABCDG.

And in like manner any multi-lateral right lined figure may be reduced to a triangle.

183. *To make a rectangle equal to a given triangle* ABC.

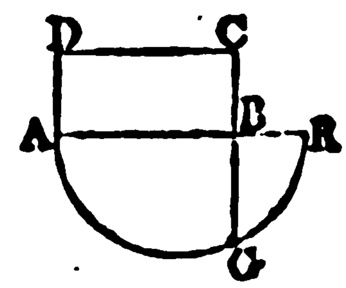


Let the base AB be bisected in D; and draw CR parallel to AB; then if AR, DG are made perpendicular to AB, the rectangle RD will be equal to the triangle ABC.

Draw BO parallel to DG. Then the triangle ABC is equal to half the rectangle RB (82, *corol.* 1): but RD is half the rectangle, therefore it is equal to the triangle ABC.

And therefore a rectangle whose height is half AR, and base AB will also be equal to the triangle.

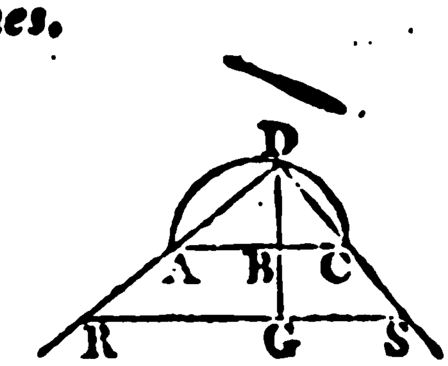
184. *To make a square equal to a given rectangle* ABCD.



Extend AB till BR = BC, and on AR describe a semicircle; then produce CB to G; and the square on BG will be equal to the rectangle under AB, BR or BC (164).

Schol. Hence by this, and the preceding Problem, a square may be made equal to a given triangle: and consequently equal to any given right-lined figure (182).

185. *To make a rectangle of a given magnitude having its sides in the ratio of two given right lines.*



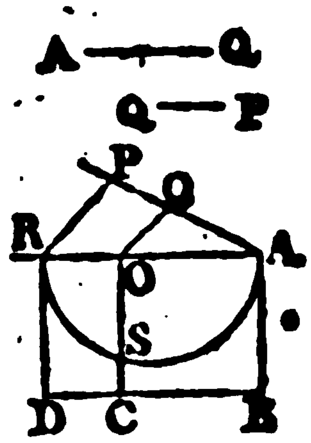
Let AB and BC be the given lines. Upon their sum AC describe a semicircle, and make BD perpendicular to AC; produce DB (if necessary) till DG is the side of a square equal to the given magnitude; join DA, DC, and through G draw RS parallel to AC meeting DA, DC produced (when necessary): Then RG, GS are the sides of the required rectangle.

For the angle RDS being a right one (72), and DG perpendicular to RS, therefore the rectangle RS × GS is equal to the square on DG (97, *corol.* 2), or equal to the given magnitude (by the construction); and because RS is parallel to AC, the sides RG, GS, are in the given ratio of AB to BC (95).

186. *If* AC *is a square on the line* AO, *and* AQ, QP *two given right lines; to find another square that*

shall be to the square AC, *as* AQ *is to* QP. *Or, to find two squares having the ratio of two given right lines.*

In any convenient direction from A, take the given lines AQ, QP; join QO, and parallel thereto draw PR to meet AO produced (if necessary): then if a semicircle be described on AR, OS will be the side of the required square.



Complete the rectangle RC. Then because QO, PR are parallel, the triangles AQO, APR will be similar,

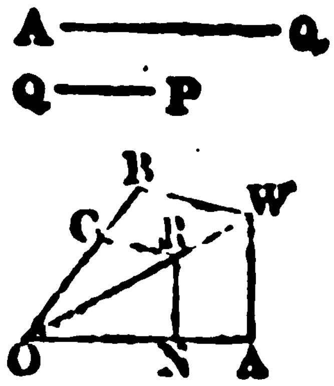
Hence, AQ : QP :: AO : OR (94, *corol.* 2). But the parallelograms or rectangles AC, OD having the same height (AB or RD) will be in the ratio of their bases AO, OR (87):

Therefore AQ : PQ :: AO^2 (*rectang.* AC) : *rectang.* OD (= OR × OC or OA):

But the rectangle OR × OA is equal to OS^2 (97, *corol.* 2): and consequently AQ : QP :: AO^2 : OS^2, the required square.

187. *To describe a figure* (CRNO) *similar to a given right-lined figure* BWAO, *so that the latter may be to the former, as the line* AQ *is to the line* QP.

Find, by the last Problem, a square (OS^2) so that AQ : QP :: OA^2 : OS^2; and make ON = OS; draw NR, RC parallel to AW, WB, respectively; and CN is the figure required.

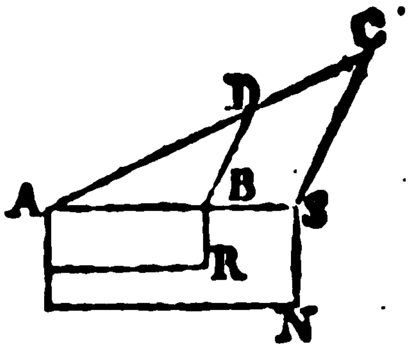


For the figures CRNO, BWAO being similar (165, *Ex.* 4); and because similar plane figures are as the squares of their homologous sides (102), we have BA : CN :: AO^2 : ON^2 (OS^2) :: AQ : QP (by the construction.)

By this Problem, plane figures are augmented, or reduced in Area according to any given proportion.

188. *To make a triangle* (ACS) *of a given magnitude, which shall also be similar to a given triangle* ADB.

On AB make the rectangle AR = to the given triangle ADB (183); then on AB (produced if necessary) let the rectangle AN be constructed equal to the magnitude of the required triangle, having its sides AS, SN in the ratio of AB to BR (18), draw CS parallel to BD, meeting AD produced: and ACS is the triangle.



For the triangles ADB, ACS being similar, and also the rectangles AR, AN, we have (102),

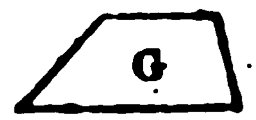
rectang. AR : *rectang.* AN :: $AB^2 : AS^2$ :: *triang.* ADB : *triang.* ACS :

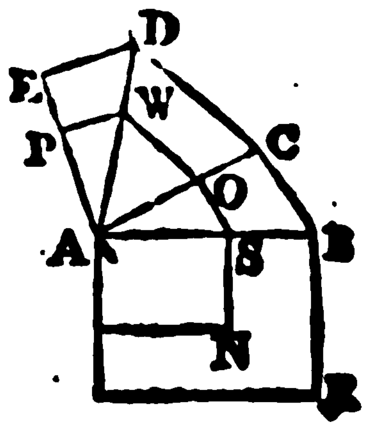
Or, *rectang.* AR : *rectang.* AN :: ADB : ACS; and the antecedents being equal (by the construction) the consequents AN, ACS must also be equal, or the triangle ACS = the given magnitude (by construction.)

Schol. Therefore a triangle may be made similar to one triangle and equal to another.

189. *To describe a figure* (ASOWP) *similar to a given figure* ABCDE, *and equal to a given right-lined figure* G.

Let the two figures EB, and G be reduced to squares (181, 184.). Then the construction will evidently be exactly the same as that of the preceding problem. For if the rectangle AR be made equal to the figure EB, and a similar rectangle AN equal to G (185), the side AS of that rectangle will be the base of the required figure: then the sides SO, OW, WP being drawn parallel to the corres-

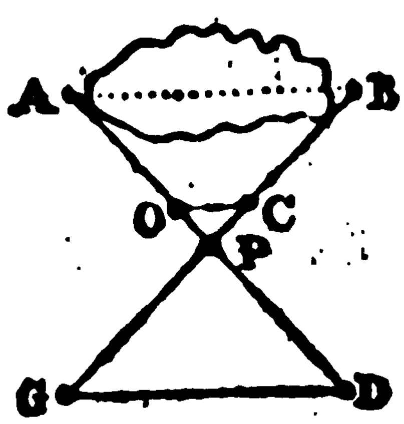




ponding sides of EB, the figure PS will be similar to EB, and equal to AN or G.

Methods of determining distances by means of similar Triangles traced on the Ground.

190. *To find the length of the line* AB *accessible only at both ends.*



Having fixed on some convenient point P, measure BP and AP; and prolong those lines till PG = PB, and PD = PA; then the distance between the points D and G will be equal to AB.

For the sides of the triangles GPD, BPA about the equal angles at P are respectively equal, therefore the third sides GD, BA will also be equal (38).

Or thus,

Having measured PB, PA (as before), take PC some convenient aliquot part of PB, and PO the same aliquot part of PA; then measure the cross distance OC, which will be the like aliquot part of the required distance AB.

For the sides PO, PA; PC, PB being proportional, the triangles OPC, APB will be similar;

Hence PO : PA :: OC : AB; therefore whatever part PO is of PA, the like OC will be of AB (94).

Suppose PA = 392, and PB = 414 *feet*; and let PO, PC be $\frac{1}{5}$ of PA PB, or equal to 78*f.* 5*in.* and 82*f.* 9$\frac{3}{5}$*in.* And suppose OC measures 93$\frac{1}{2}$ *feet*; then AB = 93$\frac{1}{2}$ × 5 = 467$\frac{1}{2}$ *feet.*

191. *When the line* (AB) *is accessible at one end* (B) *only.*

We suppose some object at ... accessible end A: and let a m... set up at B; then in the directio... take BG (the longer the better)... through a convenient point P, as i... preceding problem, let the dista... BD, GR be measured, so that PI... PB, and PR = l G; then if a m... be set up at K the intersection of ... DR, will be equal to AB.

For the triangles PBG, PDR ... all respects, the triangles PBA, P... equal (§ 98).

Or BA may be found without ... this take PC, PO, like aliquot pa... will be the same aliquot part of BA (

For PO : PB :: OS : B.

Suppose PB = 442, PG = 464 *feet*; and ... as ... of PB and PG), also, suppose OS me... feet: for 110½ : 442 :: 113 : 452.

109. *Let O be an object on the op... to find the distance DO.*

Lay down an isosceles triangle DBA, the side DB being in any convenient direction; then having measured a base DR, set up a mark at R; and in the same direction take another base and make the triangle *dba* similar and equal to DBA (*da* being parallel and equal to DA); then find the concourse (C) of the lines ORC, *da*C, and mea...

By similar triangles, R*d* : *d*C :: RI...

$\ldots = 300$, $Rd = 60$, and $dC = 270\frac{1}{2}$ *feet*,

$\ldots : 270\frac{1}{2} :: 300 : 1014$ *feet nearly* $=$ DO.

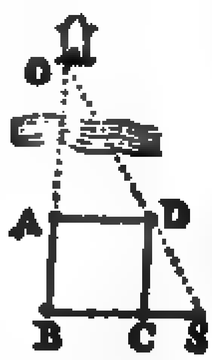
t the most expeditious method of finding the dis-
naccessible object, is by means of a Rhombus, as

O *the object, and* OB *the required*
With a line or measuring tape whose
ial to the side of the intended rhom-
vn one side BA in the direction BO,
another side be in any convenient di-
en two ends of two of those lines at
en the other ends (at D) being kept
l the lines stretched on the ground, those lines AD,
n the other two sides of the rhombus. Set up a
here CO, AD, intersect; and measure RD:

ides of the triangles RDC, CBO being respectively
triangles will be similar; hence, RD : DC :: CB

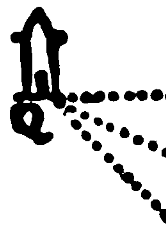
side of the rhombus is 100 *feet*, and RD $=$ 11*f*. 7*in*. then
100 : 863 *feet nearly* $=$ BO.

-Having laid down the rhombus,
course of the lines ODS, BCS, and
: Then CS : CD :: AD : AO.



is nearly level, a rhombus whose side is 100 *feet* will deter-
to the extent of 300 yards within a very few feet of the

find the length of an inaccessible line (QR) *by
hombus.*



At some convenient point B, lay down the rhombus (BADC), so that two of its sides BA, BC are directed to the extremities of the line. Mark the intersections O and P (as in the first case of the preceding problem): then the triangle ODP will be similar to and OP parallel to QR.

For each of the rectangles DO × BQ, DP to the square on the side of the rhombus (prob.) they must therefore be equal to each = DP × BR; therefore DO : DP :: BR the angles at D and B are equal, the triangle be similar (94, *corol.* 1). Therefore OD :

Suppose OD = 9*f.* 5*in.* PD = 11*f.* 10*in.* OP = of the rhombus = 100 *feet.*

Then $11\frac{10}{12} : 100 :: 100 : \frac{10000}{11\frac{10}{12}}$ = RB.

Therefore $9\frac{5}{12}$ (OD) : $13\frac{4}{12}$ (OP) :: $\frac{10000}{11\frac{10}{12}}$ (R 121·9 *feet* = RQ.

Therefore the inaccessible distance RQ is ing the square of the side of the rhombus by that product by the product of OD and PD.

The length of an inaccessible line may tracing a quadrilateral, as in *Art.* 151. B ever, are necessarily confined to moderate quire much care in the execution in order factory results.

PLANE TRIGONOMETRY.

Definitions.

A Triangle has three sides and three angles: And of those being given (the three angles excepted) the found by means of similar triangles: This is the of Trigonometry.

Hence it follows that Plane Trigonometry will admit erent Cases: For the *data* may be

: side and two angles.

erefore the 3*d*. angle is also given, *Art.* 41).

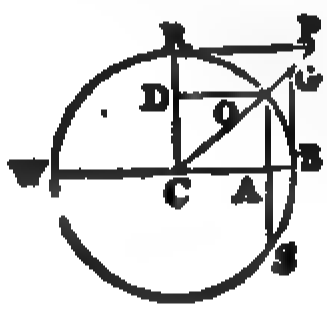
o sides and an angle opposite to one of them.

o sides and their included angle.

: three sides.

he sides of the similar triangles (or lines proportional ides) which enter into the computations, are called *gents, secants,* &c.

Let C be the centre of a circle, as perpendicular to the diameter PCB an angle, its measure be- OB (64, 144).



e chord OS and BG perpendicular to the radius CB; RP perpendicular to the radius CR.

Then,

AO is the Sine AC or OD the Cosine BG the Tangent RP the Cotangent CG the Secant CP the Cosecant	of or

199. The *Cosine*, *Cotangent*, &c. &c. of the *complement* of the angle right angle (*co* being a contraction of

Thus, OD or AC is the *Sine*,
RP the *Tangent*,
CP the *Secant* of the angle plement of the angle PCB to a right an PCR together make the right angle B

200. The *Sine*, *Tangent*, and *Se* also the *Sine*, *Tangent*, and *Secant* of the difference of PCB and 180 *degree*

201. AB is the *versed sine* of th and AW the *versed sine* of the angle

202. When the arc is a quadrant the radius, and *cosine* 0: But the *t* infinite, because they become parallel an

Thus, CR is the *sine* of 90 *degrees*

203. The degrees, minutes, &c. co are usually marked thus, °, ′, ″, &c. 90 *degrees*, 57 *minutes*, 42 *seconds*.

204. *Corollaries.*

1. Hence it appears that (AO) th half the chord (OS) of twice that arc (

2. Because the lines in and about the quadrant RCB form equiangular triangles, we have,

CA : AO :: CB : BG,
or, *cosine* : *sine* :: *radius* : *tangent*.

And, CA : CO :: CB : CG. Therefore the *radius* is mean proportional between the *cosine* and *secant* of an angle.

And BG : BC :: CR : RP. Hence the *radius* is also mean proportional between the *tangent* and *cotangent*.

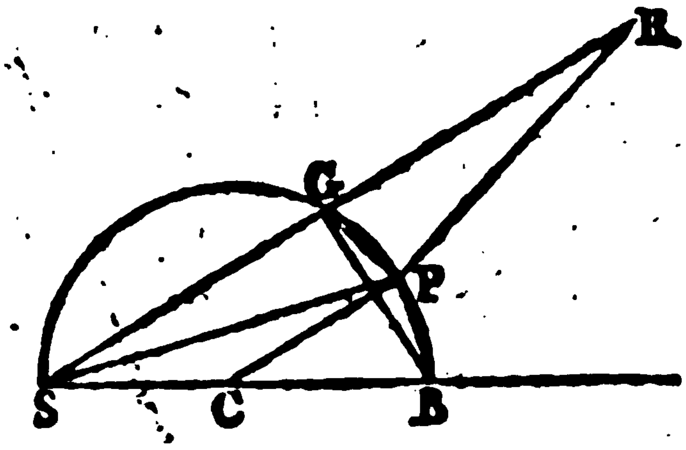
3. When the angle is 45° or half a right angle, the *sine* and *cosine* are equal; and the *tangent* and *cotangent* each equal to the *radius*.

And other properties are easily derived from the same figure

Of computing the Sines, Cosines, &c.

205. SINCE 4 right angles contain 360° × 60′ or 21600 *minutes*, it follows that ½ the side of a regular polygon of 10800 sides inscribed in a circle, is the *sine* of an arc or angle of 1′. Half the side of a polygon of 5400 sides, is the *sine* of 2′. And half the side of a polygon of 2700 sides, the sine of 4′, &c. But these figures cannot be inscribed geometrically for which reason the formation of the *Trigonometrical Canon* or Tables of *Sines*, *Tangents*, &c. has been attended with much labour. Before fluxions were invented, the method of approximation was by continual bisections, which brought out chords corresponding to arcs in a descending geometrical progression; in this manner, the chord of a small arc being obtained, the chords of other small arcs were inferred from analogy on a supposition that the chords and arcs are nearly proportional when the angles are small: To explain this,

Produce SG till GR is equal to the diameter SB, and join PR; then SR is equal to SG and SB.



Now the quadrilateral SGPB being in a circle, the external angle PGR is equal to the angle PBS (75). And because BS = GR, and PB = PG, therefore in the triangles PGR, PBS, the sides about the equal angles PGR, PBS are equal, therefore the triangles are identical, and consequently the third sides PR, PS are also equal. And because the angles PSC, PSG are equal (70, *corol.*), the isosceles triangles SPR, SCP will therefore be equiangular.

Hence CP : SP :: SP : SR, or $SP^2 = CP \times SR$.

Now if the radius CP be 1, SP^2 will be equal to SR, and SP equal to the square root of SR, or equal to the square root of the sum 2 + SG, (because GR = SB = 2).

Hence, if the supplemental chord (SG) of any arc (BG) be increased by the diameter (2), the square root of the sum will be the supplemental chord (SP) of half the arc (BG).

208. Let the chord BG be equal to the radius, then BPG is an arc of 60°. And because the angle SGB is a right one (72), SG is equal to the square root of the difference of the squares of SB and BG (83, *corol.*).

The square of SB is 4, and the square of BG is 1, therefore SG the supplemental chord of the arc BG or $\frac{1}{6}$ of the circumference is 1·7320508075688 &c. the square root of 3.

Consequently SR is = 2 + 1·7320508075688 &c. and its square root is 1·9318516525781 &c. ∴ SP the supplemental chord of the arc BP or $\frac{1}{12}$ of the circumference.

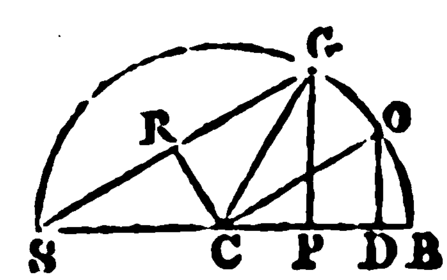
And the square root of 2 + 1·9318516525781 &c. is = 1·9828897227476 &c. the supplemental chord of $\frac{1}{6}$ the arc BP, or $\frac{1}{12}$ of the circumference.

In this manner, after eleven bisections, we get 2 + 1·99999973854478 the square of the supplemental chord of $\frac{1}{12288}$ of the circumference or 1′ 45″ $\frac{15}{32}$: Which taken from 4, the square of the diameter, leaves 0·00000026145522 the square of the chord of 1′ 45″ $\frac{15}{32}$: And the square root is 0·00051132692 the chord of 1′ 45″ $\frac{15}{32}$, or the side of the inscribed polygon of 12288 sides *.

Now the chords of small arcs being nearly in the same proportion as the arcs themselves, we have, 1′ 45″ $\frac{15}{32}$: 0·00051132692 :: 2′ : 0·0005817764 the chord of the arc of 2′; and its half or 0·0002908882 is the sine of 1′.

And the *cosine* is = 0·9999999577 the square root of the difference of the squares of the radius 1, and the *sine*.

209. The *sine* and *cosine* of 1′ being given, the *sine* of 2′ *will be equal to twice the product of that sine and cosine.*



For let B be the centre of a circle, and OD, DC the *sine* and *cosine* of the arc OB or angle OCB, and GP the sine of GB or twice the arc BO. Then if CR be perpendicular to SG it will also bisect it (65). And because the angles OCB, GSP are equal (71), and CO equal to SC, the triangles SRC, CDO will be equal, therefore SG is equal to twice the *cosine* CD, and the triangles SPG, CDO are similar:

Whence, CO : OD :: SG (2CD) : GP;

Therefore when the radius CO is = 1, GP is = 2CD × OD (69).

Again CO : CD :: SG (2CD) : SP:

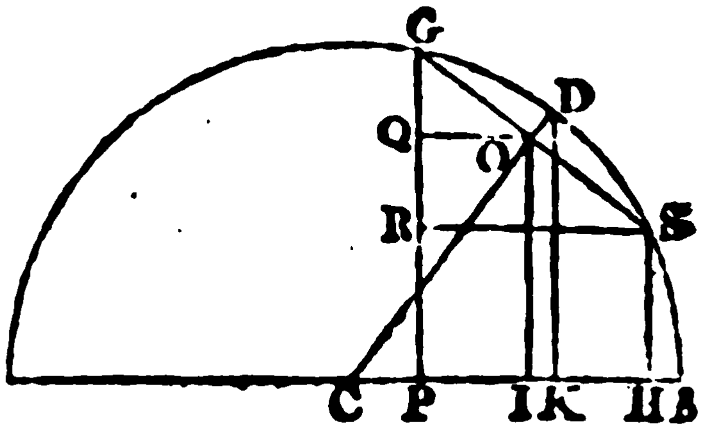
* See Ludolph Van Ceulen *de Circulo et Adscriptis*, where the bisections are continued 30 times, and the supplemental chords brought out to 40 places of figures.

And if CO is 1, SP is = 2CD × CD, or $2CD^2$; and *cosine* CP equal to the difference of SP and SC, or the differe of $2CD^2$ and 1.

Therefore if the angle OCB is 1′, GP the *sine* of 2′ will = 2 × 0·9999999577 × 0·0002908882 = 0·0005817764.

And the difference of twice the square of 0·9999999577 a 1, is 0·9999998308, the *cosine* of 2.

210. But to find the *sine* of 3′, or the *sine* of triple an a when its *sine* and *cosine* are given,



Let SH be the *sine* of the arc BS; OS or OG the *sine* of half the arc SDG; and GP the *sine* of the sum of both arcs BS and SG. Then DS is the common difference of the arcs BS and BD, and BD and BG; and therefore the arc BD is an arithmetical mean between the arcs BS, BG.

On CB let fall the perpendiculars DK, OI, and draw SR, OQ parallel to BC: Then because OS, OG are equal; QG and QR will also be equal (93): and RG being the difference of the *sines* HS, PG, therefore QG or QR is half their difference, and OI half their sum.

Now the triangles COI, CDK being similar, we have CD : DK :: CO : OI; therefore when the radius CD is 1, OI will be = CO × DK:

But DK is the *sine* of the mean arc BD: therefore the product of DK the *sine* of the mean arc, and CO the *cosine* of the common difference DS, is equal to OI: consequently 2CO × DK is equal to twice OI or the sum of HS and PG: and therefore PG is equal to the difference of 2CO × DK and HS.

Now if the arcs BS, SD, DG are each 1′, the
sine of 1′; CO is the *cosine* of 1′; and DK is th
arc BD of 2′:

Therefore to find PG the *sine* of 3′, multiply tw
of 1′ by the *sine* of 2′, and subtract the *sine* o
product.

And if the arc BS is 2′, and SD, DG each 1′
the *sine* of 3′, and PG that of 4′:

And PG is equal to twice the *cosine* of 1′ × *si*
the *sine* of 2′.

In like manner, if BS = 3′; SD and DG each
sine of 5′ will be = twice the *cosine* of 1′ × *sin*
the *sine* of 3′: and so on for the *sine* of any m
arc 1′.

Corol. If the mean arc BD is 60°, then CK
will be equal to ½ CD (203); and because the
the complement of IOC to a right angle, the tr
similar to the triangle OCI or DCK, therefore R
½ CGS (the arc BD being 60°) or the *sine* of the a
consequently PG (or PR + RG) will be equal t

Therefore if two arcs be taken, one greater tha
other as much less, the *sine* of the greater arc w
the *sine* of the less arc, together with the sine o
is the common difference from 60°.

Thus if the two arcs are 15° and 45°; then
of 15° added to 0·7071 &c. the *sine* of 45°,
the *sine* of 75°.

221. The *sines* and *cosines* being found, th
gents, &c. are obtained from similar triangl
198):

: AO :: CB : BG,
e : sine :: radius : tangent. (904).

) : AC :: CR : RP,
e : cosine :: radius : cotangent.

) : CO :: BC : CG,
e : radius :: radius : secant.

) : CO :: CR : CP,
e : radius :: radius : cosecant.

n the *sines, cosines*, &c. are computed to every .5°, and arranged in columns, they form a Table *l sines, cosines*, &c. to every minute of the ese are called *natural sines*, &c. because they gths in parts of the radius : ... the *Logarithms* ers or natural sines, &c. compose the *artificial c Canon.*

the Table of Sines and Tangents.

Table contains the Logarithms of the Sines and ery minute of the quadrant. Two degrees are in d the minutes in the left, and right hand columns, for both.

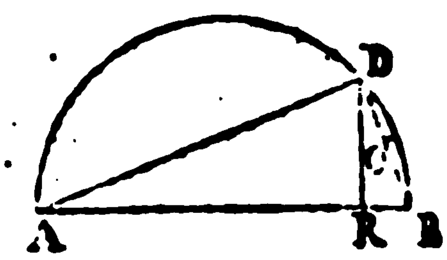
up to 45 are at top, the minutes being in the nn; but the degrees from 45 to 90 necessarily ary order at bottom, and the minutes are num- on the right.

c or angle be 15° 17' (page 39):

sine	9·420933 .... the *cosine*		of 74° 43'.
cosine	9·984363 .... *sine*		
tang.	9·436570 .... *cotang.*		
cotang.	10·563430 .... *tang.*		

And the index of the arithmetical complement of the *log. sine* increased by 10 gives the *log. cosecant.*

216. To find the *log. versed sine* of an arc, add the log. of the number 2 to twice the *log. sine* of $\frac{1}{2}$ the arc, and the sum, rejecting 10 in the index, is the log. required.



For let ADB be a semi-circle, DR and RB the *sine* and *versed sine* of the arc DB: Then ADB, DRB being right angles, the triangles ADB, DRB will be similar, and we have

$$AB : DB :: DB : RB,$$

consequently DB is a mean proportional between AB and RB: and when the *radius* is 1, the diameter AB will be 2, therefore, if for AB and DB we substitute their measures, $\frac{DB^2}{2}$ is the value of RB.

Bisect DB in O; then DO or BO is the *sine* of $\frac{1}{2}$ the arc DB: and because the square on any line is equal to 4 times the square on $\frac{1}{2}$ that line, $4BO^2$ will be equal to DB^2; therefore $\frac{4BO^2}{2}$ or $2BO^2$ is the *versed sine* RB.

Suppose the arc DB = 30°, then BO is the *sine* of 15°:

15° *log. sine*	9·412996	
	2	
	18·825992	log. BO^2
2 log.	0·301030	
Versed sine of 30° log.	9·127022	

Sine RB : RD :: RD : RA; therefore RA, the versed sine of the supplement, is a third proportional to the *versed sine* and *sine* of an arc.

arc, and the

d.

Let the arc DB = 30°. Then 30° *log. sine*

log. versed s

Suppl. versed sine

217. If at any time it should be tho

of a *log. sine* or *tangent* to parts of

and tolerably near by taking the p

erence of the *log. sines* or *tangents*

(174, *Arith.*).

Thus, suppose the *log. tangent* of 17′ 20″ is

17′ *log. tang.*	7·694179	
18′ *log. tang.*	7·719003	
	24824	difference

Then, as 60″ : 24824 :: 20″ : 8275 which

es 7·702454 the *log. tang.* of 17′ 20″ nea

e in this part of the Quadrant, the dif

ession, decrease; for example, the d

and 19′ is less than that between the *log.*

d the foregoing operation reversed

nding to a given *log. sine* or *tang*

, to find the arc or angle answering to

given log.	8·643714	
next less	8·642563	*log. sine* 2° 31
diff.	1151	

d the difference of the *log. sines* of 2° 31

en, as 2865 : 60″ :: 1151 : 24″; then

18. To find the *natural sine*,

logarithmic sine, &c. when th

ber answering to the given logarit

ithms of the natural numbers; th

will be as many places to the righ

10; or as far to the left of un

14).

O o 2

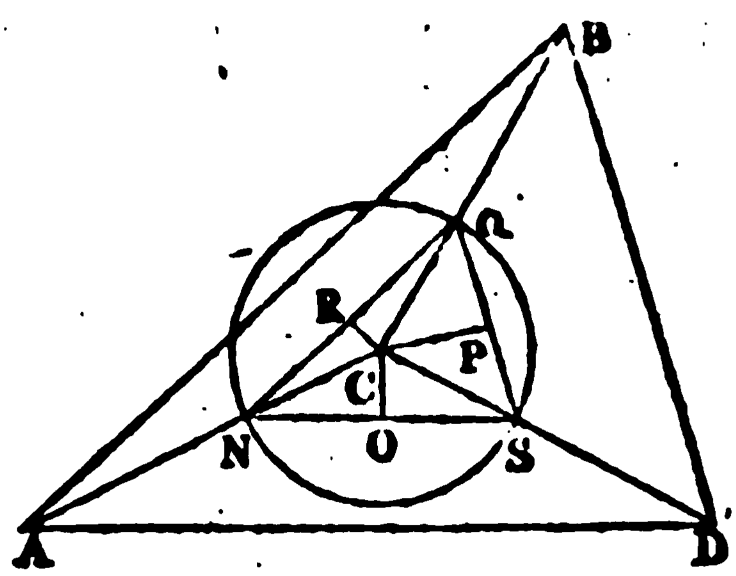
Thus, 7·241877 is the *log. sine* of 6′; and the number answering to the logarithm ·241877 is 17453, therefore 0·0017453 is the *natural sine* of 6′ to the *radius* 1.

Again, suppose we would find the *natural tangent* of 59° 24′.

The *log. tang.* is 10·228120, and the number to the log. ·228120 is 169091; now the index being 10, the first figure on the left will be an integer; therefore 1·69091 is the *natural tangent* of 59° 24′. In like manner, the *natural tangent* corresponding to the *log. tang.* 12·104901 is 127·321 &c.

219. The use of the sines in the resolution of Plane Triangles will appear from the following

THEOREM. *The sides of every plane Triangle are in the same proportion as the sines of their opposite angles.*



Let ABD be a triangle; C the centre of its circumscribing circle: then the radii CA, CB, CD being equal, the triangles ACB, ACD, BCD, are isosceles.

About C with any radius CN, describe a circle, and draw the chords NS, NQ, QS, which bisect with the perpendiculars CO, CR, CP; and the angles NCS, NCQ, QCS, will also be bisected (46, *corol.* 1).

And since the sides or radii CN, CQ, CS, are equal, the triangles NCS, NCQ, QCS will be isosceles and similar to ACD, ACB, BCD, respectively;

whence NS : AD :: NQ : AB :: QS : BD;
and NO : AD :: NR : AB :: QP : BD, because the halves of any lines must have the same proportion as the wholes.

But NO is the *sine* of the angle NCO; NR the *sine*

of NCR; and QP the *sine* of QCP to the same radius (204, *corol.* 1): And (71) the angle NCO is equal to NQS (or ABD); NCR equal to NSQ (or ADB); and QCP equal to QNS (or BAD). Therefore the sides AD, AB, BD, have the same proportion as NO, NR, QP, the *sines* of their opposite angles.

Thus if the angle A = 42°, B = 64°, D = 74°. Then the radius CQ, CS or CN being = 1,

NO = ·8988 &c. *sine* of 64° the angle B,
NR = ·9613 &c. *sine* of 74° angle D,
QP = ·6691 &c. *sine* of 42° angle A; and their doubles, or NS = 1·7976, NQ = 1·9226, QS = 1·3382 are the sides of the triangle NQS which is similar to the triangle ABD. Hence if one side of the triangle ABD be given, the other sides are found by proportion. Let DB (for example) = 100 *yards*:

Then QS : NS :: BD : AD,
viz. 1·3382 : 1·7976 :: 100 :
or ·6691 : ·8988 :: 100 : 134·3 *yards* nearly, by using the *sines* or halves of QS and NS, which have the same ratio as the wholes.

And QS : QN :: BD : BA,
or ·6691 : ·9613 :: 100 : 143·7 *yards* nearly, by taking the halves of QS and QN. But it is much more expeditious to work with the logarithms of the sines.

220. But independent of computation by the table of Sines, Tangents, &c. the several cases of Trigonometry are also resolved *geometrically*; and *instrumentally*. A scale of equal parts, with a Line of Chords or a Protractor for laying down or measuring angles, are sufficient for the *geometrical construction*, which is the most simple but least accurate method of solution.

The Sector is an instrument particularly adapted for trigonometrical operations. On each of its legs are laid down the natural sines, tangents, &c. together with the corresponding radius divided (on the 6-inch Sectors) into 100 equal parts: by those lines, the common proportions in trigonometry may be

wrought tolerably correct: But the Logarithmic or Gunter's Scale is the most commodious for that purpose. This Scale on the sector usually consists of three contiguous lines, namely, the line of numbers, that of sines, and the other of tangents, marked N, S, T; part lies on one leg, and part on the other, and therefore the sector must be quite open when it is used.

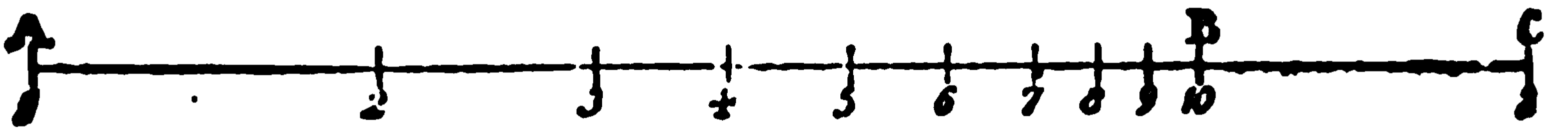
The Line of numbers is nothing more than the logarithms of the natural numbers from 1 to 10 taken from a scale of equal parts, and each extended from the beginning of the line on the left hand, towards the right: Thus,

From a scale of equal parts take ·301 the log. of 2, which set from 1 to 2.

And from the same scale set off ·477 the log. of 3, from 1 to 3:

And ·602 the log. of 4, from 1 to 4: and so on to 10.

Then the line AB will be the log. scale of numbers from 1 to 10, or from 10 to 100, or from 100 to 1000, &c.



And because the logarithms of 10 and 2 added together make the log. of 20, if the distance between 1 and 2 be set from B to C, then AC is the log. scale of 20 when AB is that of 10, or of 200 when AB is 100, &c.

But in taking the logarithms from a scale of equal parts, it is not necessary to consider them as decimals; for instead of ·301, ·477, ·602, &c. we may use any convenient numbers in the same proportion, as 3·01, 4·77, 6·02, &c. or, 30·1, 47·7, 60·2, &c. And when the scale is of sufficient length, these primary divisions may be divided and subdivided by laying off the logarithms of 1·1, 1·2, 1·3, &c. &c. as we find them on the 2 feet ruler called the Gunter's Scale.

In adapting the Sines and Tangents to the Scale of Numbers, the line AB is considered as the logarithm of the radius; for which reason the sine of 90° and the tangent of 45° are coincident with 10 (or B) on the scale. And when the sines and tangents correspond to a radius of 10, their logarithms are laid down from the left towards the right by means of the same scale of equal parts used for the logarithms of the natural numbers: Thus, the radius being 10, the sine of 30° is 5 (206), and therefore 30° on the line of sines answers to 5 on the line of numbers.

But because the radius is a mean proportional between the tangent and cotangent of an arc (204), it follows that the log. tang. and cotang. together always make double the log. of the radius or tang. of 45°, whence it is that the degrees above 45 on the line of tangents are numbered in a contrary order: thus 20° is also marked 70°; for the log. tang. of 70° is equal to the log. tang. of 45° together with the difference of the log. tangents of 20° and 45°. This inverted order of the tangents above 45° may be said to reduce the scale to half its length with the same extent of divisions.

Having premised what may be thought necessary respecting the Trigonometrical Canon, and the Logarithmic Scale; we shall now proceed to resolve the several Cases of Plane Triangles.

CASE I.

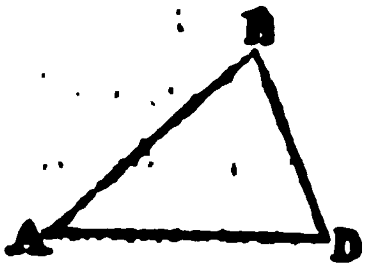
221. When one side and the angles are given.

Examp. 1. Given AD = 360.

$$\text{Angles}\begin{cases} A = 43^\circ\ 15' \\ D = 72\quad 51 \\ B = 63\quad 54 \end{cases}$$

Required the sides AB and DB?

Geometrically.



From any convenient scale of equal parts, make AD = 360; then at the extremities A and D lay down the angle A = 43° 15′, and the angle D = 72° 51′ (144); produce AB and DB till they meet, and ABD is the triangle.

AB, and DB measured on the scale upon which AD was taken, will be found 380, and 275 nearly.

Arithmetically, or by computation.

By the Theorem *Art.* 219, As the sine of any angle,
Is to its opposite side,
So is the sine of any other angle,
To its opposite side,

The natural sines of $\left\{ \begin{matrix} 43° & 15' \\ 72 & 51 \\ 63 & 54 \end{matrix} \right\}$ are $\left\{ \begin{matrix} 0{\cdot}6852 \\ 0{\cdot}9555 \\ 0{\cdot}8980 \end{matrix} \right\}$ nearly, (218).
to the radius 1

Therefore ·8980 (*sin.* ang. B) : 360 (AD) : : ·9555 (*sin.* ang. D) : 383·1 nearly, = AB.

And ·8980 : 360 : : ·6852 : (*sin.* ang. A) : 274·7 nearly, = BD.

But the usual method by the logarithmic sines is much shorter: thus,

As the *sine* of the angle B, 63° 54′	log.	9·953290
		0·046710 *arith. comp.* (186, *Arith.*)
To the opposite side AD = 360	log.	2·556303
So is the *sine* of the ang. D, 72° 51′	log.	9·980247
To its opposite side AB, 383·1	log.	2·583260

And

As *sine* 63° 54′	log.	9·953290
		0·046710
To AD	log.	2·556303
So is the *sine* of the angle A, 43° 15′	log.	9·835807
To the opposite side BD, 274·7	log.	2·438820

222. When the two first terms of the proportion are repeated, as in the present example, the operation may be somewhat abridged by taking the sum of the arithmetical complement and the log. of the 2*d*. term, instead of setting them down separately a second time;

Thus, 2·603013 is the sum of $\begin{cases} 0·046710 \\ 2·556303 \end{cases}$

9·835807 *log. sine* 43° 15'

2·438820 log. of 274·7 as before.

Instrumentally, by the Logarithmic or Gunter's Scale.

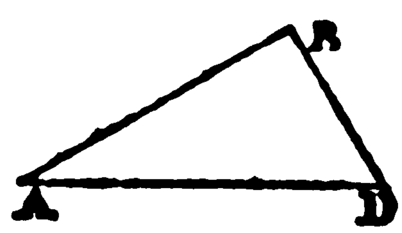
Set one foot of a pair of Compasses at 63° 54' on the line of Sines and extend the other to 72° 51', then that extent will reach the same way from 360 to 383 on the line of Numbers.

And the extent from 63° 54' to 43° 15' will reach from 360 to 275.

The reason of this operation is evident from the nature of logarithms: for when 4 numbers are directly proportional, the second divided by the first, is equal to the fourth divided by the third, and *vice versâ* (22, *Arith.*); therefore the difference of the logarithms of the first and second terms is equal to the difference between the logarithms of the third and fourth (183, *Arith.*): Thus the difference of the log. sines of 63° 54' and 43° 15' is equal to the difference of the logarithms of 360 and 274·7.

Examp. 2. Given AD = 33·15.

Angles $\begin{cases} A = 29° \ 0' \\ D = 56 \ 11 \\ B = 94 \ 49 \end{cases}$

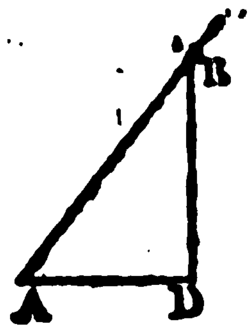


Required the sides AB and DB?

Here the first term of the proportions is the sine of 94° 49' or 85° 11' (200); and the sides will be, AB = 27·64, and BD = 16·13.

Examp. 2. Given AD = 1863.

Angles $\begin{cases} A = 49° \ 17' \\ D = 90 \ \ \ 0 \end{cases}$



Required the other two sides?

Construction. Take the base AD = 1863 from a scale of equal parts; and make the angle A = 49° 17′; then if DB be erected perpendicular to AD, the triangle is constructed.

Computation. Since the triangle is right-angled at D, the angle B is the complement of the angle A:

Therefore,

As *cosine* of angle A	log.	9·814160
		0·185540 *arith. comp.*
To AD, 1863	log.	3·270213
So *sine* of angle A, 49° 17′	log.	9·879637
To DB ....2164·7	log.	3·335390

And,

As *cosine* of the angle A		0·185540 *arith. comp.*
To AD	log.	3·270213
So is *sine* of angle D, 90°	log.	10·000000
To AB 2856	log.	3·455753

By the Logarithmic Scale.

The extent from 40° 43′ to 49° 17′ on the line of sines, will reach on the line of numbers from 1863 to 2165 nearly, for DB.

And from 40° 43′ to 90° will reach from 1863 to 2855, the hypothenuse AB.

223. But the angle at D being a right one, the operation for finding the perpendicular DB is rather more simple by means of the *tangent* of its opposite angle A;

Thus,

As the *radius*	log.	10·000000
To the *tang.* of the angle A, 49° 17′	log.	10·065178
So is AD, 1863	log.	3·270213
To DB, 2164·7	log.	3·335391

And the *secant* of 49° 17′ taken for the second term of the proportion, instead of the *tangent*, will bring out the side AB.

By the Log. Scale.

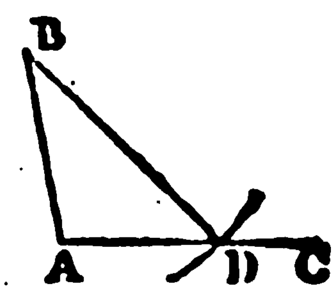
The extent from 45° to 49° 17′ (10° 43′) on the line of tangents (220) will reach (the contrary way) from 1863 to 2165 nearly, on the line of numbers.

CASE II.

224. WHEN two sides and an angle opposite to one of them are given.

Examp. 1. Given $\begin{cases} AB = 246{\cdot}5 \\ BD = 370{\cdot}5 \\ \text{Ang. } A = 101^\circ\, 21' \end{cases}$

Required AD, and the other two angles?



Construction. At A the extremity of an indefinite right line AC, make the angle CAB = 101° 21′, and set off AB = 246·5 from any convenient scale of equal parts; about B with BD = 370·5 taken from the same scale, describe an arc intersecting AC in D; draw BD; and ABD is the triangle.

The measure of the angle ADB is 41°, and that of B, 38°, nearly: and AD is 230 on the scale of equal parts.

Computation. The proportion in this case for finding an angle will be

As the side opposite the given angle,
Is to the *sine* of that angle,
So is the other given side,
To the *sine* of its opposite angle: Being the reverse of that in the former Case for finding a side.

As BD, 370·5	log.	2·568788
		7·431212 *arith. comp.*
To *sine* of angle A, 101° 21′	log.	9·991422
So is AB, 246·5	log.	2·391817
To the *sine* of the angle D, 40° 43′	log.	9·814451

Now the two angles A and D together make 142° 4′, therefore the third angle B is 37° 56′ (41).

Then by *Case* 1:

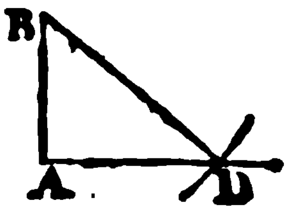
As the *sine* of the angle ADB, 40° 43′	log.	9·814451
		0·185549
To the opposite side AB................	log	2·391817
So *sine* of angle B, 37° 56′	log.	9·788694
To AD, 232·3	log.	2·366060

By the Log. Scale. The extent from 370½ to 245½ on the line of numbers, will reach from 78° 39′ (the supplement of 101° 21′) to 41° on the line of sines, for the angle ADB.

Examp. 2. Given $\begin{cases} AB = 49{\cdot}6 \\ BD = 81 \\ \text{Ang. } A = 90^\circ \end{cases}$

Required AD, and the other two angles?

This is constructed in the same manner as the preceding example.



Computation.

As BD, 81	log.	1·908485
		8·091515
To *sine* of the opposite angle A, 90°	log	10·000000
So is AB, 49·6	log.	1·695482
To *sine* of the angle ADB, 37° 46′	log.	9·786997

And 52° 14′ the complement of 37° 46′ is the angle B.

As the *sine* of the angle A, 90°	log.	10·000000
To BD	log.	1·908485
So is the *sine* of B, 52° 14′	log.	9·897908
To AD, 64·03	log.	1·806393

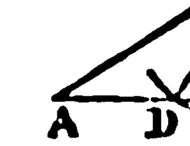
225. But AD may be found independent of the angles, thus (83, *corol.*):

Square of BD = 6561
of AB = 2460.16
diff. 4100·84, and its square root is 64·03 nearly, th

Examp. 3. Given $\begin{cases} AB = 4516 \\ BD = 2721 \\ \text{Ang. } A = 29^\circ\ 20' \end{cases}$

Required the other angles, and side?

Construction. Having made the angle A = 29° 20′, and AB = 4516, about B with 2721 describe an arc D*d* to intersect AC; draw BD, B*d* to the points of intersection; then either of the triangles ADB, A*d*B i quired.



For it is manifest that when the arc cuts the base two points, either AD, or A*d* will be the unknown this ambiguity must always take place when the opposite the given angle (A) is *less* than the other (AB), except the arc, instead of intersecting A touch it; in which case the angle opposite AB right one (67, *corol.* 1). The single answers ar limited to examples where an angle opposite a give right one, and such as have the side opposite the g greater than the other given side.

Computation.

As BD or B*d*, 2721 .... log. 3·434
6·5…
Is to the *sine* of the opposite angle A, 29°20′ log. 9·690
So is AB, 4516 ... . log. 3·651
To the *sine* of 34°24′ or its supplement 125°36′ log. 9·910

Therefore the angle ADB = 125° 36′ And A*d*B (BD
ABD = 25 4 AB*d*

Consequently,

As the *sine* of the angle A .... log. 9·690098
0·309902
Is to BD .... log 3·434729
So is the *sine* of the angle ABD, 25° 4′ log. 9·627030
To AD, 2353·2 .... log. 3·371661

0·308908 arith. comp.
A .... log. 3·434729
.... log. 9·997397
le ABd, 96° 18' log. 9·997397
.... log. 3·741028

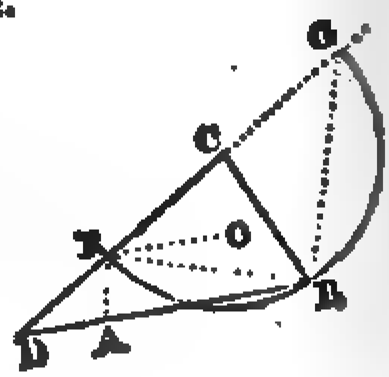
...biguous Case in Trigonometry.

CASE III.

...sides and their included angle are given.

...ng angles will be found from the following

...f the given sides,
...ference,
...gent of half the sum of the two unknown angles,
...ent of half their difference.



...n. Let DCB be the ...e; CD, CB the given ... the given angle DCB. ... and about C with the ... scribe a semi circle: join ...d draw RO parallel to

... sum of the angles CRB + CBR is equal to the ...e unknown angles CDB + CBD, each sum being ...ment of the angle DCB to two right angles; Therefore ... triangle RCB is isosceles, each of the equal angles ...R, is equal to *half the sum* of the unknown angles ...BD.

...because RO, BD are parallel, the angles RBD, BRO, ...equal, and the angle CRO equal to the angle CDB: ...e angle RBD added to CBR (half the sum of the ...wn angles) is the greater angle CBD; and the angle BRO ...from CRB (the like half sum) is the less angle CRO

(CDB); therefore BRO or RBD is *half the difference* of the unknown angles CBD, CDB *.

Let RA be parallel to BG. Then the angle RBG being a right one (72), BG and RA will be perpendicular to BR. Now if an arc was described about R with the radius RB, and another arc about B with the same radius, BG would be the tangent of the angle GRB or half the sum of the unknown angles; and RA the tangent of the angle ABR or half their difference.

But CG, CB, CR are equal, therefore DG is the sum, and DR the difference of the given sides CD and CB.

And because GB and RA are parallel, the triangles DRA, DGB will be similar; whence we have,

$$DG : DR :: GB : RA;$$

That is, as the sum of the sides, is to their difference, so is the *tangent* of half the sum of the unknown or opposite angles, to the *tangent* of half the difference of those angles.

Examp. 1. Let CD = 4100
CB = 2265
Angle DCB = 87° 52′.

Required the other two angles, and the side DB?

Construction. Make the angle DCB = 87° 52′; and from a scale of equal parts set off CD = 4100, and CB = 2265; join DB; and the triangle is constructed.

---

* Half the difference of any two numbers or lines added to, and subtracted from half their sum, give the greater, and less, respectively. Let BD, BR, be each equal to half the difference of two lines, and BS, BM, each equal to half their sum: then RS is the greater, and RM the less. For SM is the sum, and RD the difference of those lines.

DB measured on the same scale of equal parts is 4610 nearly.

And the measures of the angles D and B are 29° and 63° nearly.

Calculation.

CD = 4100			180°
CB = 2265	Included angle DCB .... =		87 52
sum 6365	Sum of the unknown angles =		92 8
diff. 1835	Angle CBR or CRB .... =		46 4 half.

As 6365	log.	3·803798
		6·196202
To 1835	log.	3·263636
So is the *tangent* of 46° 4′ (BRG)	log.	10·016174
To the *tangent* of 16·40 the angle RBA	log.	9·476012

Greater ang. CBD = 62·44 sum

Less CDB = 29·24 diff.

The side DB is found by *Case* I. thus,

As the *sin.* of CBD, 62° 44′,

Is to CD, 4100,

So is the *sine* of the angle DCB, 87° 52′,

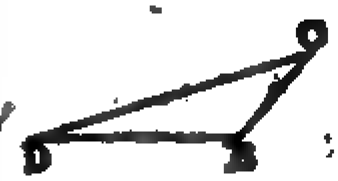
To the opposite side DB, 4609·3.

By the Logarithmic Scale.

Having taken the extent from 6365, the sum of the sides, to 1835 the difference, on the line of numbers, set one foot of the compasses at 45° on the line of tangents, and let the other rest on that line while the foot which was on 45° is moved back to 43° 56′ (or 46° 4′); take off the compasses and set one foot on 45° again; then the other will extend to 16° 30′ nearly, the 4*th*. term of the proportion.

To explain this operation, it may be necessary to observe, that if the tangents above 45° were laid down on the scale in their natural order to the right of 45°, the extent from 6365 to 1835 would reach from 46° 4′ to 16° 30′ on the left; therefore the distance of 16° 30′ from 45° must be *less* than that extent by the distance from 45° to 43° 56′ (220); now the difference was found by moving one foot of the compasses from 45° while the other rested, and consequently that difference or extent when laid from 45° will give the 4*th*. term of the proportion, as in the last step of the process.

Examp. 2. Given $\begin{cases} CD = 94 \\ CB = 26 \\ \text{included angle } 22^\circ\, 20' \end{cases}$



Required the other angles, and the third side?

Answer. Angle D = 8° 8′
B = 149 30
DB = 70·7.

By the Logarithmic Scale.

CD =	94		180° ′
CB =	26	Included angle	22 20
Sum	120	2)	157 40
diff	68		78 50 half sum of unknown ang.

The extent from 120 to 68 on the line of numbers will reach from 78° 50′ (or 11° 10′) to 70° 30′ (or 19° 30) nearly, on the line of tangents. Here the extent from the first term of the proportion to the second is from right to left on the line of numbers, but the contrary way from the third to the fourth on the line of tangents, because (as it has been observed) the tangents above 45° are counted to the left.

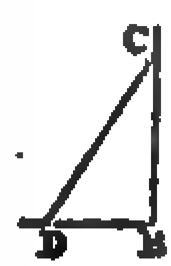
Half sum of the unknown angles	78° 50′	
Half difference	70 30	
Angle B	149 20	sum
Angle D	8 20	diff.

Now the extent from 149° 20′ (the angle B) to 22° 20′ (angle C) on the line of sines, will reach the same way on the line of numbers from 94 (DC) to 70, DB.

Examp. 3. Given $\begin{cases} BD = 92{\cdot}64 \\ BC = 36{\cdot}4 \\ \text{Angle } B = 90^\circ \end{cases}$

Required the angles at D and C, and the side DC?

Construction. Erect BC perpendicular to BD; then from a scale of equal parts (which should have a diagonal scale decimally divided) set off BD = 92·64, and BC = 36·4; join DC; and DBC is the triangle.



DC on the same scale measures 43.

And the angles D and C with the chords, will be found 58° and 32°.

Computation.

BC =	36·4
BD =	22·64
sum	59·04
diff.	13·76

As 59·04	log.	1·771146
		8·228854
Is to 13·76	log.	1·138618
So is the *tang.* of 45°, half the sum of angles D and C	log.	10·000000
To the *tang.* of 13 7′, half their difference	log.	9·367472

sum 58 7 angle D.
diff. 31 53 angle C.

And DC found by *Case* I. is 42·86 &c.

By the Logarithmic Scale.

The extent from 59·04 to 13·76 on the line of numbers, will reach from 45° to 13° 10′ on the tangents, for half the difference of the angles D and C.

Half sum	45° 0′
Half diff	13 10
Ang. D	58 10
C	31 50

Then the extent from 31° 50′ to 90° on the line of sines, will reach from 22·64 to 43 nearly, for DC on the line of numbers.

227. But when the included angle is a right one, as in the present example, if either of the given sides be made *radius*, the other will be the *tangent* of its opposite angle (198). Therefore to find an unknown angle, suppose D,

As BD, 22·64	log.	1·354876
		8·645124
Is to BC, 36·4	log.	1·561101
So is the *radius*	log.	10·000000
To the *tang.* of 58° 7′, the angle D, as before ;	log.	10·206225

By the Logarithmic Scale.

The extent from 22·64 to 36·4 on the line of numbers, will reach, on the line of tangents, from 45° to 58° 10′ (31° 50′) the angle D. For the 2d. term being greater than the first, the 4*th*. must be greater than 45°.

But the unknown side DC may be found without the angles, thus (83, *corol.*):

Square of BD = 515·5696
of BC = 1321·96

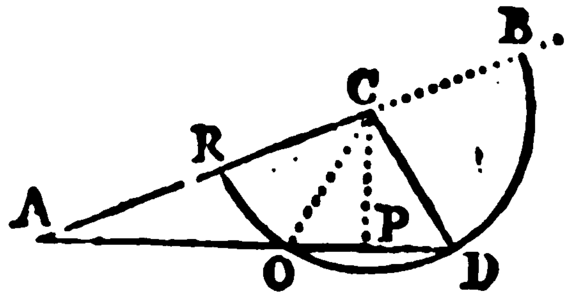
Sum 1837·5296; and the square root of this sum is 42·86 &c. the hypotenuse DC, as before.

CASE IV.

228. WHEN the three sides are given.

We shall lay down two methods of finding the Angles.

1. Suppose ACD the proposed triangle; and let the perpendicular CP divide it into two right-angled triangles APC, DPC:



Then,

As the side AD,
Is to the sum of the other two sides AC, DC,
So is the difference of those sides AC, DC,
To the difference of PA and PD, the segments of the base AD.

Demonstration. Produce AC; and about C with the side CD describe a semi-circle, and draw CO. Then the radii CR, CD, CB being equal, AB is the sum of the sides CA and CD, and AR is their difference.

And because PO and PD are equal (65), AO will be the difference of the segments PA and PD: therefore (98),

AD : AB :: AR : AO which being taken from AD, and the remainder OD divided by 2, gives PD (or PO) one of the segments; and the sum of PO and AO is the other. Then the angles of the triangles APC, DPC are found by *Case* II.

Examp. 1. Let AD = 462, CA = 384, CD = 169, required the angles?

The Construction from a Scale of equal parts is according to *Art.* 136.

Calculation.

CA = 384
CD = 169
Sum 553 = AR.
Diff. 215 = AR.

As 462 : 553 :: 215 : 257·35 = AO, nearly.
462 = AD.
Diff. 204·65 = OD.
Half 102·33 = PD or PO.
257·35 = AO.
359·68 = AP.

Now in the triangles APC, DPC,

are given AC = 384		and DC = 169
AP = 359·68		DP = 102·33.

And the angles found by *Case* II. will

be PCA = 69° 30′	and PCD = 37° 16′
PAC = 20 30	PDC = 52 44

Therefore the angles are, C = 106° 46′
D = 52 44
A = 20 30

By the Logarithmic Scale.

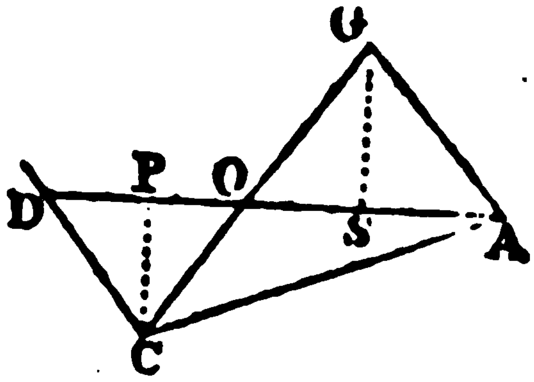
The extent from 462 to 553 on the line of numbers, will reach, the same way, from 215 to 257 nearly, on the same line.

The perpendicular however, may be drawn from either an-

gle; but when it falls without the triangle, it must meet the opposite side produced; in which case the calculation is no ways different from the preceding: Thus, suppose ACO to be the triangle; and let the perpendicular CP meet AO produced;

Then AO : AB (CA+CO) :: AR : AD; and half the difference of AD and AO is the segment PO as before: now the angles PCO, PCA being found (by *Case* II.), their *difference*, instead of the *sum*, will be the angle (ACO) opposite the base.

229. *Method* 2. This is principally derived from the preceding Demonstration. Thus, suppose CGA is the proposed triangle; and let it be required to find the angle CGA opposite the base CA:



Make GO = GA; and OC will be the difference of the sides GC and GA: then

As the rectangle of the sides GC and GA,
Is to the rectangle of half the sum and half the difference of CA and CO,
So is the square of the *radius*,
To the square of the *sine* of half the angle CGA.

Demonstration. Let AO produced meet CD drawn parallel to GA, and make GS and CP perpendicular to AD:

Then the triangles OCD, OGA will be isosceles and similar; and the angles OCD, OGA, and also the opposite sides are bisected by the perpendiculars CP, GS.

Now if AD is the base of the triangle ACD, and AC, DC, the other two sides, AO will be the difference of the segments PA, PD, exactly as in the preceding demonstration:

Therefore,

As the side AD,
Is to CA + CD, the sum of the other two sides,
So is CA — CD, the difference of those sides,
To OA.—Or because CO = CD, it will be
As AD : CA + CO :: CA — CO : OA:

And their halves will also be proportional,

or, As ½ AD,
To half the sum of CA + CO,
So is ½ the diff. of CA — CO,
To ½ OA.

Therefore the rectangle ½AD × ½AO is equal to the rectangle under the ½ sum and ½ diff. of CA and CO (89).

But OP = ½ OD, and OS = ½ OA, therefore OP + OS or PS = ½ AD; and consequently the rectangle PS × OS (= ½ AD × ½ AO) is equal to the rectangle of the aforesaid ½ sum and ½ difference.

Now the triangles OPC, OGS being similar, we have
OC : OG :: OP : OS; and by composition (94, *schol.*)
OC + OG (GC) : OG :: OP + OS (PS) : OS,
or GC : OG :: PS : OS:

And GC × OG : OG × OG :: PS × OS : OS × OS, by taking equimultiples of the two first terms of the proportion, and also of the two last (Arith. 95):
or GC × OG : OG^2 :: PS × OS : OS^2:
whence GC × OG : PS × OS :: OG^2 : OS^2.

But PS × OS is = the rectangle under the ½ sum and ½ difference of CA and CO, hence the last proportion becomes,
As GC × OG, or GC × GA,
Is to the rectangle of ½ the sum and ½ the diff. of CA and CO,
So is OG^2, to OS^2:

Therefore, if OG or GA be made the *radius*, OS will be the *sine* of the ang'e OGS, or of half the angle CGA.

Example 2. Le' 'he sides of the Triangle CGA be as in the preceding example, namely, CA = 462, GC = 384, GA = 169.

Then,

GC = 384			
GA = 169			
215 diff.		7·415669	*arith. comp.* log. 384 = GC.
CA = 462		7·772113	*arith. comp.* log. 169 = GA.
sum 677	half = 338·5 log.	2·529559	(186, *Arith.*)
diff. 247	half = 123·5 log.	2·091667	
	radius square, log.	20·000000	
	2)	19·809008	
Angle OGS 53° 23′	log. *sine*	9·904504	
2			
Angle CGA = 106 46 as before.			

The other two angles are found by *Case* II.

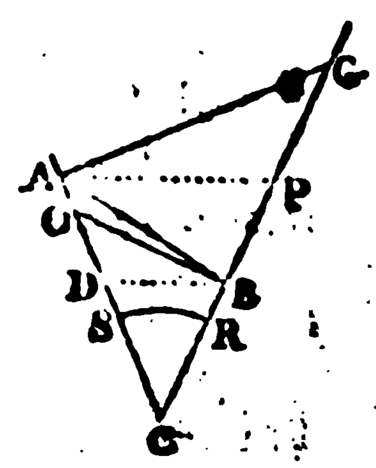
The method of working the last proportion by the Logarithmic Scale is omitted, it being rather complex, and therefore may produce considerable uncertainty in the results, particularly on the six-inch Sectors. We may also remark in general respecting these operations, that when the sides of the triangles exceed 1000, the calculations should be made with the pen, because there is too much *guess-work* on the Scales when the integers are more than three.

Application of Trigonometry to measuring Heights and Distances.

230. THE Instrument proper for measuring horizontal and vertical angles in common Trigonometrical operations is a Theodolite furnished with one or two Telescopes, and a Vertical arc: And if the horizontal circle is not less than about 6½ inches in diameter, the observed angles may be read off to half a minute. The student, however, would benefit little from a

description of the Instrument, because the method of examining and correcting the Spirit-levels, &c. and adjusting the whole for observation, must be acquired under the eye of a Master.

But after all the care that may have been bestowed in correcting the line of collimation, telescope-level, &c. it seldom happens that the elevations or depressions shown by the Instrument are correct. It is therefore always adviseable to determine the *error*, or how much the elevations or depressions are too great, or too little. This may be done in the following manner:



Let C be the centre of the earth, SR an arc on its surface, A the place of the telescope when the Theodolite stands in the vertical line CA, B the place of the telescope when it stands in the vertical CB, AG (perpendicular to AC) the horizontal line at A drawn to meet CG, and BO (at right angles to BC) the horizontal line at B.

Then, if the telescope at B be directed to a mark or object at A, the elevation of that object above the horizontal line BO is the angle OBA; and when the telescope is at A, and directed to an object at B, its depression below the horizon AG will be the angle GAB.

Let SD = RB, and RP = SA. Then because the triangles APC, DBC are isosceles, and the angles CAG, CBO right ones, the angle CAP + angle PAG = a right angle; but the angle CAP + half the angle ACP also make a right angle, therefore the angle PAG or its equal DBO, is equal to half the angle C.

Now the depression or angle GAB = GAP + PAB (or ABD); or GAB = PAG + DBO + OBA; but PAG + DBO = angle C:

Therefore the depression GAB = ang. C + *elev.* OBA; or depr. GAB + *elev.* OBA = ang. C + twice the *elev.* OBA; Therefore the elevation and depression together, lessened by the angle C, is equal to twice the elevation: consequently *half the difference between the sum of the elevation and depression, and the angle C, is the elevation.*

Now, whatever be the error in elevation or depression, their sum will be constant; for one is always diminished by the same quantity that the other is augmented; hence the preceding rule gives the *true elevation*, except the angle C be greater than the elevation and depression together, in which case, the said *half difference* is the *true depression* of the highest of the two points or objects A, B.

And when the observations are both elevations, or both depressions, their *difference* is constant, and *half the difference between the angle* C *and that constant difference will be the true elevation of the highest of the two points* A, B, *if the angle* C *be the less, but equal to the true depression of that highest point or object, when it is the greater.*

Should both the reciprocal observations be depressions (or both elevations), and equal to each other, the vertical heights SA, and RB are equal; and the true depressions will be half the angle C.

Examp. The following observations were made with a Theodolite for determining the error in the vertical angles taken with that instrument.

Two marks, A and B, were set up exactly at the same height above the ground as the height of the telescope; and at A, the depression of B, or the angle GAB was 24′; and at B, the elevation of A, or the angle OBA =12′ The distance of the stations or arc SR was 2600 *yards*, which, allowing 69½ miles to a degree, gives 1′·28 of a degree nearly, the angle C.

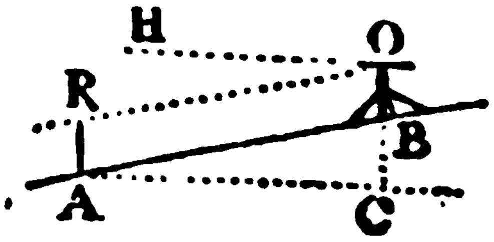
Then, $\frac{24' + 12' - 1'{\cdot}28}{2} = 17'{\cdot}36$ or about 17′½ the true elevation or angle OBA; consequently 17′½ − 12′ = 5′½ is the error, or what the altitudes shown by the instrument were too little, or the depressions too great.

A distance of 600 or 700 *yards* however, is sufficient for trying a common Theodolite. In which case the angle C may be neglected, and the verticals SA and RB considered as parallels: the expressions then become more simple. Thus if one observation be an elevation = 17′, and the other a depression = 13′, then *half their sum* = 15′ is the true elevation or depression; and 17′ − 15′ = 2′ is what the instrument gives elevations too great.—If both are elevations, or both depressions *half the difference* is the true elevation of one station, and the true depression of the other.

Here the observations themselves are supposed to be correctly made; for the result will evidently partake of any error that may arise in consequence of a mistake.

231. Short Bases for temporary use only, are sometimes measured with Rods, or the Gunter's Chain of 66 feet. But the common 50, or 100 feet Tapes are much better adapted for expedition: with these lines, when the ground is tolerable level, and the direction or *alignement* of the base pretty correct, the error in distance will probably be about 3 inches in 50 feet, or $\frac{1}{200}$ of the whole measurement as long as the Tapes are kept dry: after frequent use however, they should be tried on a level pavement, or long floor, for which purpose a distance of 50 feet may be laid down by means of one or more Rods properly adjusted in respect of length.

232. When a Base is measured on sloping ground, it must be reduced to the corresponding horizontal line, if horizontal angles at its extremities are taken with a Theodolite. Suppose AB is a base of 300 *yards*; OB a Theodolite; and let the height of the staff AR be equal to OB the height of the instrument; also suppose HOR, the angle



of depression of the top R below the horizontal line HO is 5°; then if OC is perpendicular to HO, the line AC, parallel to HO, will be the horizontal base corresponding to the measured base AB.

Now the angles HOR, BAC being
Case I.)

As *radius* ....
To AB, 300 ....
So is *cosine* of 5° (the angle BAC)
To AC, 298·9 ....

The difference of AB and AC is only
uction of this kind seems unnecessar
e is inclined to the horizon in a small
on is intended to produce a very accur

33. EXAMPLES.

. To find the distances AO, BO from
he inaccessible object O, I measured A
ground being nearly level; and havin
B, the angles at those stations, taken

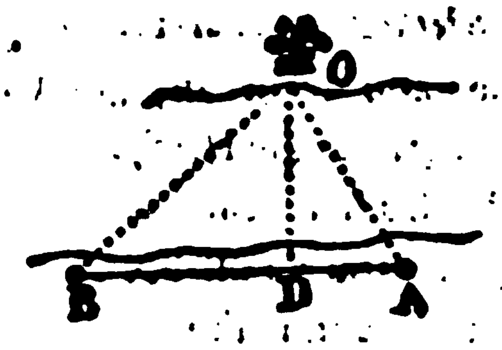
ere $\begin{cases} A = 57° \ 12', \\ B = 24 \ 45. \end{cases}$ Whence the d
required?

he angle at O, or supplement of the angle
nd B is 98° 3'. And the *Construction* and
ulation will be exactly the same as in the two
examples, *Case* I. (221).

And $\begin{cases} \text{AO will be found} = 308·6 \textit{ feet}, \\ \text{BO} \qquad\qquad\quad = 619·7 \end{cases}$

. Wanting to know the breadth (D
d a base AB of 400 *yards* along the
nities A and B took angles to an obj

mely $\begin{cases} \text{angle OBA} = 37° \ 40', \\ \text{angle OAB} = 59 \ 15. \end{cases}$ Hen
required?



Construction. Make BA = 400 from any convenient scale of equal parts; and at the extremities B and A, lay down the respective angles 37° 40′ and 59° 15; then the perpendicular OD upon the base BA (152), will be the breadth required. And its measure is 212 nearly.

Calculation. By *Case* I. (221).

As the *sine* of the angle BOA, 83° 5′ (the supplement of the angles B and A)	log.	9·996828
		0·003172
Is to BA, 400	log.	2·602060
So is *sine* of angle B, 37° 40′	log.	9·786089
To AO	log.	2·391321

Then,

As *sine* of the angle ODA. 90°	log.	10·000000
Is to AO	log.	2·391321
So is the *sine* of the angle A, 59° 15′	log	9·934199
To OD, 211·6 *yards*	log.	2·325 20

By the Logarithmic Scale. The extent from 83° 5′ to 37° 40′ on the sines, will reach from 400 to 245 (AO) on the line of numbers.

Then, the extent from 90° to 59° 15′ on the sines, will reach from 245 to 210, for OD, on the line of numbers.

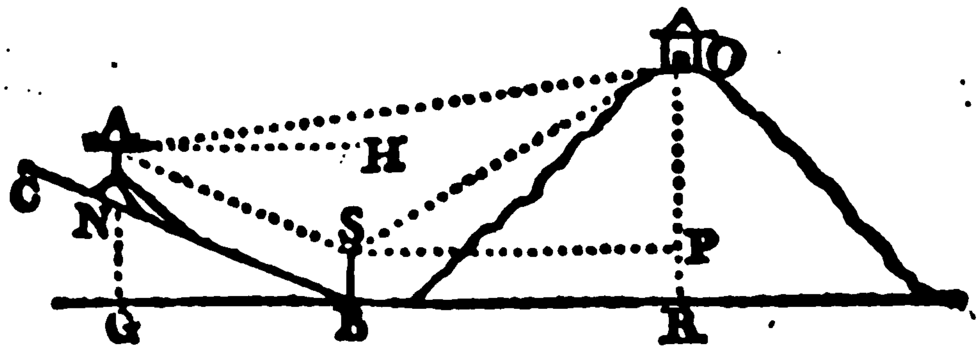
3. To find the height, and the distance of the object O on the top of a hill from the station B, we measured a base BN of 642 *yards* up the sloping ground BC, directly from the object O, the points O, B, N, being in the same vertical plane, then having set up a staff BS whose length was equal to the height of the Theodolite, we found the angles of elevation and depression to be as follows;

At the station N, $\begin{cases} \text{object O elev. } 3° 59' = \text{ang. OAH,} \\ \text{top of staff S depr. } 39' = \text{ang. HAS.} \end{cases}$

At the other station B, the elev. of O = 5° 52′ = ang. PSO.

Hence the horizontal distance BR, the height RO, and also GN the height of the station N above B, are required?

Method of Construction. Draw RG indefinitely to represent an horizontal line, and from any point B draw the slope BC making the angle CBG = 39′ (the angle HAS): then from a scale of equal parts set off BN = 642, and make BS perpendicular to BG and equal to the height of the Theodolite NA; let SA be parallel to BC and equal to BN, and AG parallel to SB; also draw the horizontal lines, AH, SP: then if the angles OSP, OAH are made equal to 5° 52′, and 3° 59′, the angles of elevation respectively, and OR is perpendicular to GR, the figure will be constructed.



Calculation.

Angle OAH = 3° 59′			
HAS = 39	.... its supplement	179° 21′	angle ASP
Angle OAS = 4 38	subtract	5 52	angle OSP
		173 29	angle OSA

Therefore the angles of the triangle OAS are OSA = 173° 29′
OAS = 4 38
AOS = 1 53

By *Case* I. (221).

As *sine* AOS, 1° 53′	log.	8·516726
		1·483274
To AS, 642	log.	2·807535
So is *sine* of OAS, 4° 38′	log.	8·907297
To SO	log.	3·198106

Then, as *sine* SPO, 90°	log	10·000000
To SO	log.	3·198106
So is *sine* OSP 5° 52′	log.	9·009515
To the height OP, 161·3	log.	2·207621

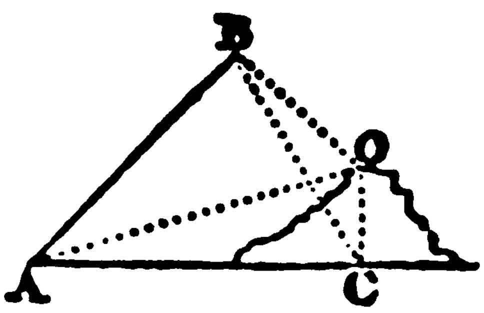
SO ..... log. 3·198106 (222)
5° 52′ *cosine* ..... 9·997719
Distance SP = BR = 1569·7 log. 3·195825

As *sine* ang. NGB 90°	log.	10·000000
To NB 642	log.	2·807535
So *sine* NBG 39′	log.	8·054781
To NG 7·3 *yards*, nearly	log.	0·862316

And if SB (PR) the height of the Theodolite when standing on the ground, be added to OP, we shall have the height of O above the horizontal line GR.

N. B. If a correct result is required from an operation of this kind, the error (if any) in angles of elevation should be determined (230); and care must be taken to adjust the height of the instrument when at B, so that the telescope may be exactly at the height BS from the ground.

4. Wanting to know the distance (AC), of a hill from the station A, and also the height (OC); we measured a base AB of 298 *yards* on ground nearly horizontal, and at the extremities A and B observed the horizontal angles, BAO (or BAC) = 42° 17′, ABO (or ABC) = 79° 29′; and at A the angle of elevation OAC was 4° 51′. Required the distance AC, and height CO?



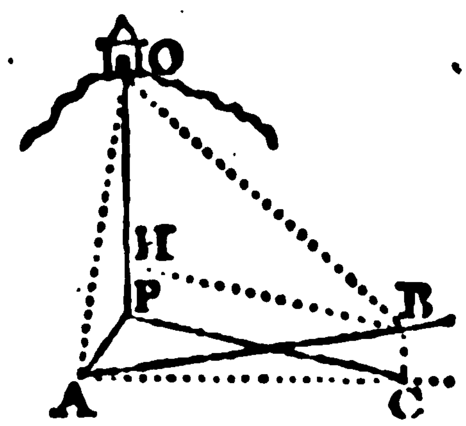
Method of Construction. The three points A, B, C being supposed in a plane parallel to the horizon, and the plane of the instrument at A and B in that plane, the angles taken to the point O in the perpendicular CO will be the same as they would be if the telescope was directed to the point C, because the horizontal circle of the Theodolite is not moved by elevating or depressing the telescope. Therefore, having made AB = 298, and the angle BAC = 42° 17′, ABC = 79° 29′, and OAC = 4° 51′, raise the perpendicular CO; then AC is the distance, and CO the height sought.

Calculation. The angle ACB is 58° 14′ the supplement of the horizontal angles at A and B.

As *sine* of 58° 14′	log.	9·929521
		0·070479
To AB, 298	log.	2·474216
So is *sine* of ABC, 79° 29′	log.	9·992643
To AC, 344·6	log.	2·537338
Ang. OAC = 4° 51′	*tang.*	8·928658 (223)
Height CO = 29·2	log.	1·465996

And the height of the instrument being added to 29·2 *yards* will give the whole height of the top O.

5. To find the distance of the object O on the top of a hill from the station A, and also its height, we measured a base AB of 210 *yards* up sloping ground, its inclination with the horizontal line AC being 9° 30′ the angle BAC; and the horizontal angles at A and B (found by directing the telescope to O) were PCA = 64° 10′, and PAC = 76° 17′; also the angle of elevation OBH (HB being an horizontal line) was 5° 34′. From hence the height, and distance of the object O are required?



Method of Calculation. Let the horizontal lines BH, AP meet OP the line from O perpendicular to the horizon; and suppose AC is the horizontal base (232), and BC perpendicular to AC.

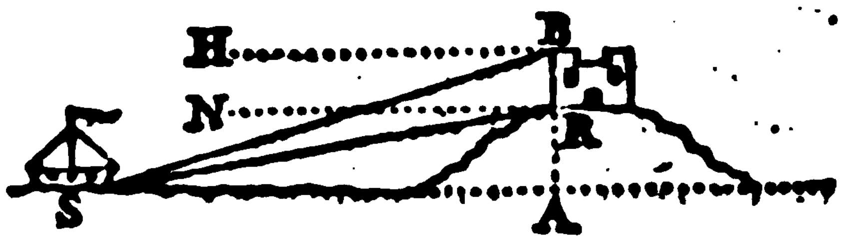
In the right angled triangle ACB, the hypotenuse AB and all the angles are given, whence CB the height of the station B above A will be found = 31·7 *yards*: and AC the reduced base = 207·1 *yards*.

Then AC, and the angles of the triangle ACP, $\left\{\begin{array}{l} PCA = 64° \ 10' \\ PAC = 76 \ \ 17 \\ APC = 39 \ \ 33 \end{array}\right.$ will give AP the horizontal distance from A = 292·8 *yards*; and CP the horizontal distance from C = 316 *yards* = BH.

Now in the right angled triangle OHB the side BH and all the angles are given, whence HO = 30·8 *yards* the height of O above B; to this add BC and we have OP = 65·5 *yards* the height above A. To this also should be added the height of the instrument for the whole height of O above the ground at A.

6. At B, the top of a castle which stood on a hill near the sea shore, the depression of a ship at anchor was 4° 52′ (the angle HBS), and at R, the bottom of the castle, its depression was 4° 2′ (the angle NRS). Required the horizontal distance of the vessel, and also the height of the castle above the level of the sea, supposing RB the castle itself to be 54 feet high?

Method of Construction. From any scale of equal parts make BR = 54, and draw the horizontal lines RN, BH at right angles to BR: let the angles HBS, NRS be made = 4° 52′, and 4° 2′, respectively; then if SA is drawn perpendicular to BR produced, it will be the horizontal distance, and AR the height of the bottom of the castle.



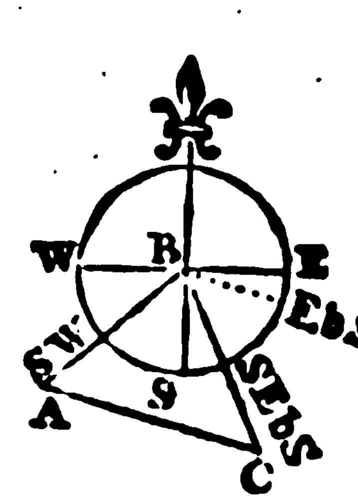
Method of Calculation. The angle BSR is equal to 50′ the difference of the angles of depression, therefore by *Case* I. (221).

As the *sine* of 50′
Is to BR, 54,
So is the *sine* of the angle SBR (the *cosine* of 4° 52′),
To SR.

And as the *sine* of ang. A, 90°
To SR,
So is *sine* of ang. RSA (NRS) 4° 2′,
To AR. 260 *feet.*
And, so is *cosine* of 4° 2′, to 3690 *feet* = AS the horizontal dist.

7. In surveying with the compass, an object C bore SE b S, and when we had gone 240 *yards* in a SW direction, the object bore E b S. Required its distance from the stations B and A?

Construction. Let the circle whose centre is B represent the compass; E, W, S, the east, west, and south points; draw E b S one point or 11¼ deg. from E; S E b S three points or 33¾ deg. from S; and SW four points or 45° from S; and make BA = 240 from a scale of equal parts; then if AC be drawn parallel to the E b S direction, C will be the place of the object.



Method of Calculation.

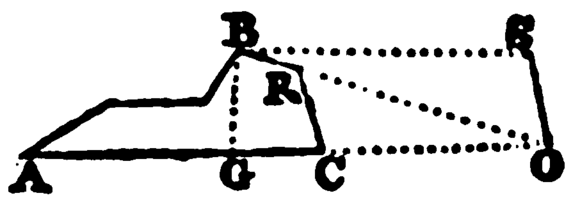
In the triang. ABC $\begin{cases} \text{ang. ABC} = 7 \textit{ points} \text{ or } 78°\ 45' \\ \text{ACB} = 4 \textit{ points} \text{ or } 45 \\ \text{BAC} = 5 \textit{ points} \text{ or } 56\ \ 15, \end{cases}$

And the side BA = 240, whence, by *Case* I. AC = 333, and BC = 282 *yards*.

8. If BG the height of the rampart ABRC be 16 *feet*, and the exterior talus BR of the parapet is inclined to the horizon in an angle of 4°; what is the difference in the distance (BO) of a musket shot made directly in front, and another (BS) inclined to that direction in an angle (OBS) of 40°, both shots being made in the plane of the talus?

Calculation.

As *sine* of ang. GOB, 4°	log.	8·843585
		0·156415
To GB 16	log.	1·204120
So *sine* of ang. G 90°	log.	10·000000
To BO, 229·4	log.	2·360535



Now OS is the intersection of the plane of the horizon and that of the talus; therefore the direct shot, or the line BO is at right angles to OS, and consequently the angle BSO is the complement of OBS;

Whence, as *cosine* of 40°	log.	9·884254
		0·115746
To BO	log.	2·360535
So *sine* of BOS, 90°	log.	10·000000
To BS = 299·4	log.	2·476281
BO = 229·4		
diff. 70 *feet, Ans.*		

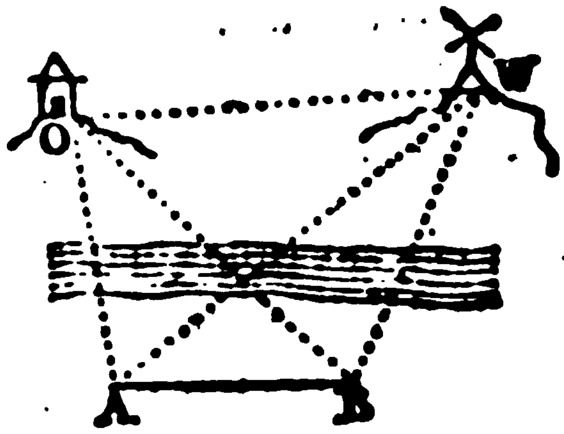
9. Wanting to know the horizontal distance between the inaccessible objects O, W, and also their heights, we measured a base AB of 670 *yards* on ground nearly horizontal; and at the extremities A and B took the following angles:

At A, ang. $\begin{cases} \text{BAW} = 40° \ 16'. \\ \text{WAO} = 37 \ \ 40. \end{cases}$ Elevation of W = 3° 46′. of O = 3 33.

At B, ang. $\begin{cases} \text{ABO} = 42° \ 22' \\ \text{OBW} = 71 \ \ 7. \end{cases}$

Hence the heights, and distance OW are required?

Construction. From any convenient scale of equal parts, make AB = 670, and lay down the respective horizontal angles at A and B; join O, W, the points of concourse of the lines from A and B; and OW measured on the scale from which AB was taken, will be 1170 *yds.* nearly, the distance between the objects.



Calculation.

The angles of the triangle AOB are $\begin{cases} ABO = 42° \ 22' \\ OAB = 97 \ \ 56 \\ AOB = 39 \ \ 42 \end{cases}$

Whence, by *Case* I. (221), AO will be found = 706·8 *yards.*

And the angles of the triangle AWB $\begin{cases} BAW = \ 40° \ 16' \\ ABW = 113 \ \ 29 \\ AWB = \ 26 \ \ 15 \end{cases}$

will give AW = 1389·4 *yards.*

Now in the triangle OAW we have given $\begin{cases} AW = 1389·4 \\ AO = \ \ 706·8 \end{cases}$

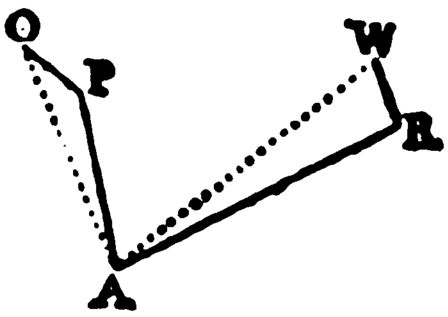
and the included angle OAW = 57° 40′.

To find OW (by *Case* III.)

		AW =	1389·4		
		AO =	706·8		
	180° 0′	sum	2096·2	.... log.	3·321433
	57 40				6·678567
sum of unknown ang.	122 20	diff.	682·6	.... log.	2·834166
half	61 10			.... *tang.*	10·259233
half diff.	30 36			.... *tang.*	9·771966
ang. OWA =	30 34				

Then, as *sine* OWA, 30° 34′	log.	9·706326
		0·293674
to AO	log.	2·849310
so is *sine* ang. OAW = 57° 40′	log.	9·926831
to OW, 1174 *yards*	log.	3·069815

The lines AW, AO in the preceding figure are represented as drawn to the objects W and O on the elevated situations, because the telescope of the Theodolite is pointed to them when the angles are observed; but the distances by *construction* and *calculation* are the *horizontal lines*



standing on the ground at any considerable distance, seldom appear, even through good telescopes, sufficiently defined to permit the angles to be taken to that precision which is evidently necessary when a satisfactory result is required; for a small error in either angle will produce a very considerable one in the distance. Thus, in the foregoing example, suppose an error or variation of 3" in the angle OCB, or let it be 2' 12" instead of 2' 15",

Then, as 12" (the difference of 2' 12" and 2'), is to 2', so is 400 to 4000 *yards* the distance CB, instead of 3200.

Again, Let the base AC = 300 *yards*, and suppose the angles at A and C are 3' 20" and 4', respectively;

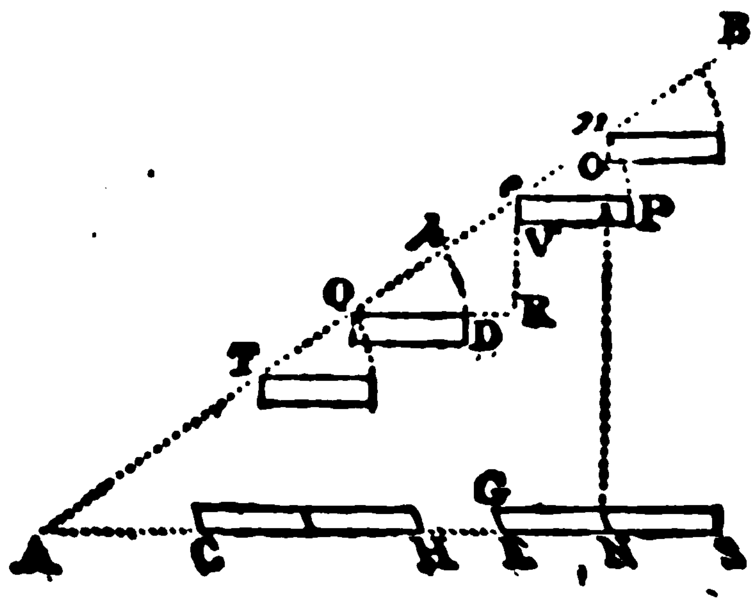
Then, as 40" (their difference), is to 3' 20", so is 300 *yards*, to 1500 *yards* = CB.

Now admit an uncertainty of 3" in each angle, and take that at A = 3' 23", and the other at C = 3' 57", and we have

As 34" (the difference) : 3' 23" :: 300 *yards* : 1794 *yards*; the difference is 294 *yards* in about a mile; an uncertainty perhaps as great as that in an estimate by the eye at the same distance.

11. If CS be a line of cavalry; to determine the wheeling intervals between half squadrons marching *en echellon* from the right, when halted and formed on the line AB which is inclined to CS in a given angle.

Construction. Let CS consist of two squadrons CH and ES; the extent of each = 48 *yards*, depth GE = 7½ *yards*, the interval HE = 16 *yards*, and suppose the angle BAS = 35°.



Drawn Nn perpendicular to CS (*the half squadron NS being supposed to march from N to O in a direction perpendicular to CS*),

and make ne = NE, oh = EH; hQ, QT each = NE; and from n, o, Q, T, draw lines parallel to CS, and on those lines make parallelograms each equal to GN for the half squadrons: then if the half squadrons wheel on the pivots n, e, Q, T, till their fronts are in the line AB, the extent TB will be equal to CS, with the proper interval between the squadrons, or he = HE.

Calculation. We want the perpendicular distances OI, IP, and VR, DR.

en = EN = 24 *yards* = 72 *feet*.
eQ = NH = 40 *yards* = 120 *feet*.

In the right angled triangles eIn, QRe the angles at Q and e are 35°.

As *rad.* : 72 (= en) :: *sin.* 35° : 41·3 *feet* = nI, whence OI = 19 *feet*, nearly.
rad. : 72 :: *cosin.* 35° : 59 *feet* = eI, whence IP = 13 *feet*, nearly.
rad. : 120 (= eQ) :: *sin.* 35° : 68·8 *feet* = eR, whence VR = 46 *feet*, nearly.
rad. : 120 :: *cosin.* 35° : 98 *feet*, nearly, whence DR = 26 *feet*.

But the measurement of those lines from construction, will be sufficiently correct for practical purposes.

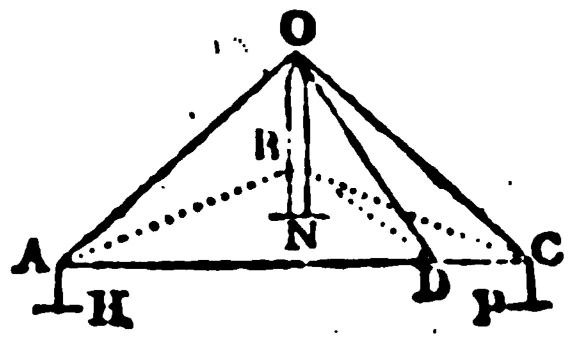
234. In the preceding examples, the angles subtended by distant objects are supposed to be in an horizontal, or in a vertical plane: We shall now give the method of computation when they are measured in planes oblique to the horizon.

Angles oblique to the horizon are usually taken with a sextant or Hadley's quadrant, which is held in a position so that its plane passes through both objects and the eye of the observer. And elevations are found by reflecting the object from an artificial horizon. But whoever intends to observe with a sextant must acquire the method of using it from *practice* under the direction of a person who is master of the several adjustments, &c.; for which reason we shall not attempt a description of the instrument.

EXAMPLES.

1. Suppose ON is an object standing on the horizontal plane HNP; HA and PC two staves or rods equal in height to that

of the eye; and let the plane ABC be parallel to the horizontal plane HNP; also suppose HP or AC is a base of 250 *yards*; and that the angles taken in the plane OCA, are OAC = 56° 46′, and OCA = 62° 54′; the angles of elevation OAB, OCB being 6° 40′, and 7° 6′, respectively. Hence the height, and horizontal distances AB, CB, are required?



When one of the sides (AC) including an angle (OAC) oblique to the plane of the horizon, is horizontal, the angle is reduced to the corresponding horizontal angle by the following proportion,

> As the *cosine* of the angle of elevation (OAB),
> Is to the *cosine* of the given ang. (OAC),
> So is the *radius* or *sine* of 90°,
> To the *cosine* of the reduced angle (BAC).

For let DBO be a vertical plane and the angle ADO a right one; then the triangles ABO, DBO being also right angled at B, we shall have, (*Case* I. 221).

Sine ABO, 90° : AO :: *sine* AOB : AB;
Sine ADO, 90° : AO :: *sine* AOD : AD;

Therefore by equality,

sine AOB : *sine* AOD :: AB : AD :: *sine* ADB, 90° : *sine* ABD;

or, *sine* AOB : *sine* AOD :: *sine* 90° : *sine* ABD:

But AOB is the complement of the elevation; AOD the complement of the observed angle OAC; and ABD that of the reduced angle BAC; therefore, &c.

As *cosine* 6° 40′	log.	9·997053
		0·002947
To *cosine* 56° 46′	log.	9·738820
So *sine* 90°	log.	10·000000
To *cosine* 56° 31′ the reduced angle BAC	log.	9·741767

As *cosine* 7° 6′	log.	9·996657
		0·003343
To *cosine* 62° 54′	log.	9·658531
So *sine* 90°	log.	10·000000
To *cosine* 62° 40′ the reduced angle ACB		9·661874

Therefore the angles of the triangle AOC reduced to the horizontal plane are $\begin{cases} BAC = 56° \ 31' \\ ACB = 62 \ \ 40 \\ ABC = 60 \ \ 49 \end{cases}$

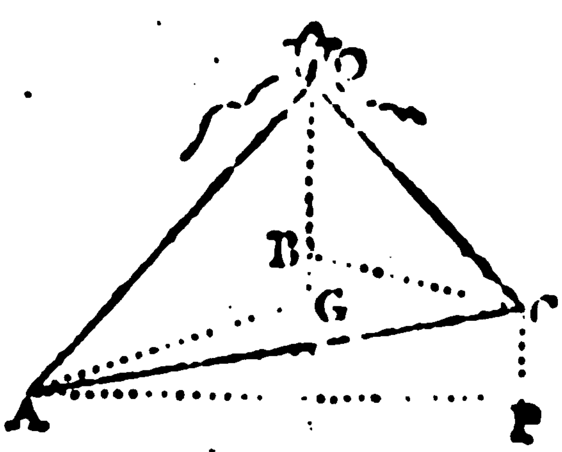
And the side AC being 250 *yards*, we shall have (by *Case* I. 221) AB = 234·4, and CB = 238·8 *yards*; whence BO = 29·7 *yards*: to this add NB the height of the observer's eye above the horizontal plane HNP, and the sum will be the whole height NO.

But the distances AB, CB, and height BO may be calculated without any reduction of angles; for AC and all the angles of the triangle AOC being given, the sides AO, CO are found by *Case* I. and then the right angled triangles ABO, CBO, will give AB, CB, and BO at three proportions.

And should it be necessary, the reduced angles may be found from the sides of the triangle ABC, by *Case* IV. (228).

2. If A and C are two stations on sloping ground; O an object on the top of a hill: and the angles OCA, OAC (measured with a sextant) equal to 79° 29′, and 63° 11′, respectively; also suppose the angle of elevation at A is = 6° 30′, at C = 5° 22′: What are the horizontal distances and height of the object; AC being = 410 *yards*?

Let OG be perpendicular, and AG, CB, parallel to the horizon: then AG, CB are the horizontal distances.



In the triangle AOC the angles are $\begin{cases} OCA = 79° \ 29' \\ OAC = 63 \ \ 11 \\ AOC = 37 \ \ 20 \end{cases}$

And AC = 410 *yards*.

Whence (221) AO = 664·7, CO = 603·4, these hypotenuses, with the angles of elevation OAG, OCB, in the right angled triangles AGO, CBO, give AG = 660·3, OG = 76·4, CB = 600·7, OB = 56·4 *yards*.

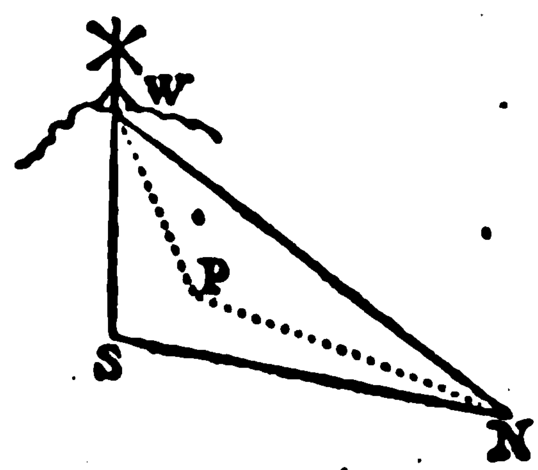
And the difference of OG and OB is 20 *yards* = BG = CP the difference in the heights of the stations, AP being supposed horizontal.

The sides AC, CP, will give AP. And the angles of the triangle AOC when reduced to the horizon, may be found from the horizontal distances AP, AG, CB, taken as the sides of a triangle (228).

3. At a mile-stone N on the ascending road NS we observed the angle SNW between the next mile-stone S and the windmill W on the top of a hill, and found it to be 46° 37′; the elevation of W, or angle WNP was 3° 49′; next, at the mile-stone S, the angle NSW measured 91° 4′. Hence the horizontal distance NP, and height PW are required?

The angles of the triangle SWN are $\begin{cases} SNW = 46° 37' \\ NSW = 91 \;\; 4 \\ SWN = 42 \;\; 19, \text{ and} \end{cases}$
NS = 1760 *yards*:

these give NW = 2614:



Then in the triangle WPN, right angled at P, the hypotenuse NW and all the angles are given, whence NP = 2608; and PW = 174 *yards*.

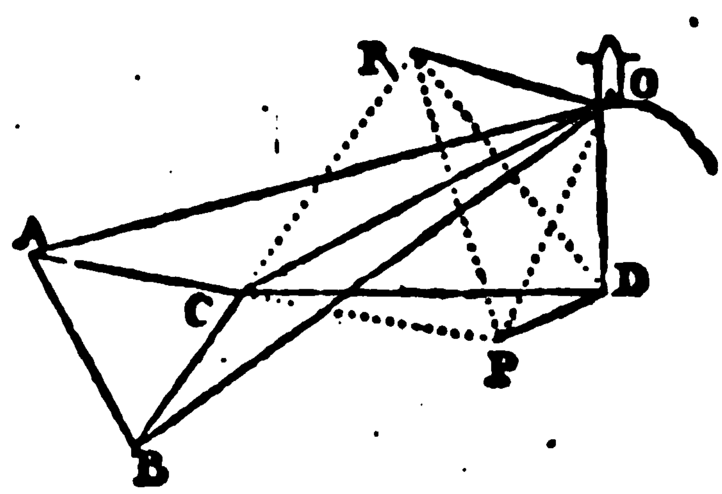
In this example, no reduction is necessary on account of the inclination of the base NS to the horizon.

4. Let BC be a measured base of 370 *yards* on the plane ABC; and suppose marks are set up at the stations A, B, C, and the following angles taken with a sextant to the elevated object O:

At A $\begin{cases} OAC = 20° \; 50' \\ OAB = 80 \;\; 18 \end{cases}$

At B $\begin{cases} OBA = 73 \;\; 44 \\ OBC = 16 \;\;\; 4 \end{cases}$

At C $\begin{cases} OCB = 149 \;\; 10 \\ OCA = 140 \;\;\; 6 \end{cases}$



Required the distance of the object O from the station C, and its height above the plane of the base BC.

The angles of the triangles OAC, OAB, OBC, are

OAC = 20° 50′	OAB = 80° 18′	OBC = 16° 4′
OCA = 140 6	OBA = 73 44	OCB = 149 10
AOC = 19 4.	AOB = 25 58.	BOC = 14 46.

These three triangles form the sides of the pyramid whose vertex is O, and base ACB: we have therefore to find its height OD, and the point D where the perpendicular OD meets the plane of the base.

Calculation.

14° 46′ *sin.*	9·406314		
	0·593659	sum 3·161861	
BC = 370 log.	2·568202		
OBC = 16° 4′ *sin*	9·442096	9·709730 *sin.* 149° 10′ = OCB	
CO log.	2·603957	2·871591 log. OB	
OAC = 20° 50′ *ar.co.sin.*	0.448976	0·006254 *ar.co.sin.* 80° 18′ = OAB	
AOC = 19° 4′ *sin.*	9·514107	9·641324 *sin.* 25° 58′ = AOB	
AC = 369 log.	2·567040	2·519169 log. AB = 330·5	
		0·358676 *ar. co. sin.* 25° 58′, AOB	
		9·982257 *sin.* 73° 44′, OBA	
		2·860102 log. AO.	

The sides of the triangle ABC $\begin{cases} BC = 370 \\ AC = 369 \\ AB = 330{\cdot}5 \end{cases}$ give the angle ACB = 53° 8′ (228).

Let OP, OR, meet AC, BC produced, at right angles in P and R; and suppose OD is the perpendicular on the plane of the base, and join PD, CD, RD. Then OCP = 39° 54′ (the supplement of OCA); and OCR = 30° 50′ (the supplement of OCB);

Then, 39° 54′ *cosine* 9·884889	30° 50′ *cosine* 9·933822
CO log. 2·603957	CO log. 2·603957
CP = 308·2 log. 2·488846	CR = 345 log. 2·537779

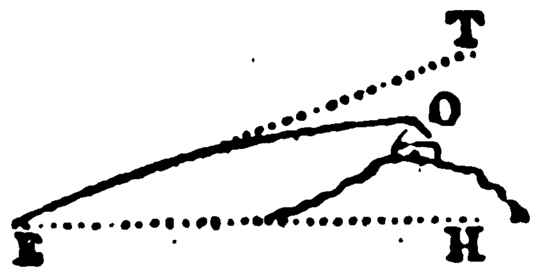
Now in the quadrilateral CRDP (in the plane of the base ABC) we have the sides CP, CR, and their included angle = 53° 8′, whence (226) we get the angle CRP = 57° = CDP (because the angles CPD, CRD being right ones, a circle will circumscribe the quadrilateral), therefore CP and all the angles of the right angled triangle CPD are given; whence the distance CD = 367·5 *yards*; from this side and the hypotenuse CO, the perpendicular OD will be found = 162·3 *yards*.

If the triangle ABC is on level ground, CD is the horizontal distance of the object O from the station C, and OD its height.

OF TERRESTRIAL REFRACTION.

235. As the *apparent* or *observed elevations* of objects are always greater than the *true*, it may not be improper to give a short explanation of Refraction.

Let E be the place of an observer's eye, EH the horizontal line, and O an object, suppose on the summit of a distant hill.



Then if the rays of light proceeded from the object O to the eye at E in a straight line, the object would appear in its true place at O, and OEH would be the elevation (considering EO as a right line); but the rays in passing through the atmosphere are continually attracted or bent downwards from a rectilineal direction, by which means the object is seen in the direction ET, which is supposed to be a tangent to the curve at E, and therefore the apparent or observed elevation is the angle TEH; and the angle TEO, or rather the angle comprehended by TE and a right line from O to E, will be the refraction.

This Refraction which is called the terrestrial, to distinguish it from that which affects the altitudes of the heavenly bodies, is not constant at the same elevation and distance, but is found to vary with the changes in the atmosphere, as heat, a different density, moist vapours, &c. &c. At the distance of 8 or 10 miles it is sometimes no more than about 30 *seconds*, but in particular states of the air we find it amount to upwards of 2 *minutes*.

236. It is a difficult operation to determine the exact quantity of refraction at any particular time. The following

If one of the objects (B) instead of being depressed, is elevated, suppose to the point R; then the sum of the angles $dAB + dBA$ will be greater than the sum $OAB + OBA$ (or angle C) by the angle of elevation RAG; but if from the sum $dAB + dBA$ we take the depression HBA, there will remain $dAB + aBA$ the sum of the two refractions; therefore, *if the depression be subtracted from the sum of the contained arc and elevation, half the remainder is the refraction in this case.*

It is almost unnecessary to remark that the distance between the places of observation A and B should be known sufficiently near to give the contained arc SS true to a very few seconds of a degree. The refraction however, is generally too minute to be of consequence in the operations with a common Theodolite, which are usually confined to moderate distances.

Of Surveying.

237. Surveying is the Art of laying down the true positions of the principle features, and exhibiting an exact representation of the boundary of a country, or any part thereof, on a plane or paper, so that the dimensions, &c. may be readily measured by means of a scale of *miles*, *yards*, *chains*, &c. &c. When fields or other inclosures, and Gentlemen's estates are surveyed, not only a correct delineation of the boundaries is required, but the superficial content in *acres*, &c. must be computed. This is called Land Surveying, or Land Measuring.

238. To lay down or make a Map or Plan of any considerable extent of Country, a series of connected triangles should be carried in all directions to its boundaries from a long and well measured base as the foundation: For that purpose the most conspicuous points, as the summits of hills, roofs of

church-towers, &c. &c. must be chosen for stations; and all remarkable objects in view should be intersected at every place where the instrument for taking the angles is set up. When a high pointed spire, or the like, upon which the instrument cannot be conveniently placed, presents itself as a proper situation for carrying on the triangles, it should always be intersected from several stations in order to compare, or correct the connecting distances by a computation from independent triangles.

239. It will be adviseable to observe every angle of the principal triangles if the situations permit; then, as the sum of the three angles of each triangle ought to be very nearly equal to two right ones, the deviations will in some measure, enable us to judge of the accuracy of the work.

240. The sides of the principal triangles should be calculated. But objects situated within those triangles may be laid down by means of a protractor: these objects however, should if possible, be intersected from three stations.

241. The principal triangles and interior objects laid down on a large scale, suppose 5 or 6 inches to the mile, will be a sufficient ground work for Military sketches which are usually drawn by eye without any actual measurement. The method of adapting a scale to the Plan; and enlarging or diminishing it to any particular size is given in *Art.* 167.

242. But the most difficult and tedious operation connected with a Survey, is that of measuring a base-line accurately. We shall therefore recommend a perusal of the Account of the Trigonometrical Survey (236) to those who may engage in an undertaking of this kind when great exactness is required. A base for common surveys may be measured with a 20 *feet deal-rod*: for this purpose a rope not less than 100 *yards* should be stretched very tight along the ground; the rod must then be applied to the rope, and its extremity may be marked with a small pin stuck in the rope to preserve the distance while the rod is removed.

When the measurement is carried on to the extent of the rope, a peg should be driven in the ground and a notch cut on its top exactly under the end of the last rod. The rope must then be taken up and stretched again in the direction of the base, and the measurement continued as before.

When the measurement is carried over hollows or ditches, it may be necessary to support the rod in the middle; it should not however, be made very slender.

If rising grounds intervene, the slant distances must be measured separately as hypothenuses, and afterwards reduced to the corresponding horizontal lines (232): the elevations or depressions may be taken with a Theodolite which has a vertical arc.

It may be necessary to observe, that 20 *feet* should be transferred to the rod from a *standard measure*. And with respect to expansion and contraction, it is pretty well known that well seasoned deal is subject to very little alteration while it is kept dry.

243. If a measurement of this kind be performed with tolerable care, we may safely conclude there will not exist an error of more than $\frac{1}{10}$ of an *inch* in each rod of 20 *feet*, or $26\frac{1}{2}$ *inches* in a *mile*. Supposing however, the accumulated errors amount to 5 *feet* in a base of 2 *miles*, and that a series of triangles whose sides are about 3 *miles* to be determined from such a base, then combining the probable errors from observations made with a Theodolite, the uncertainty in a direct distance of 20 *miles* from the base cannot amount to 30 *yards*. Erroneous as this may be considered, we believe most of the County Maps have been laid down from operations less accurate.

244. If the variation of the Magnetical needle is known, the direction of the meridian may be drawn sufficiently near *for a Map* or Plan by means of the compass belonging to the *Theodolite*.

We shall now proceed to such trigo
...ally occur in the practice of Surveying

...45. Let AB be a base of 2 *miles* or
...e poles or flag-staffs are set up at the
...d that the angles at those stations take
... following;

namely, CAB	=	64° 29′
CBA	=	75 15
ACB	=	40 18
sum		180 2
BCD	=	53 41
CBD	=	64 8
BDC	=	62 14
sum		180 3

...t is required to find the distance of
...ion A?

...he error in the sum of the three observe
...es of the first triangle is 2′; in the second 3′
in the third 2″. The angle at P in the fourt
...ngle is supplemental.

...ut no certain rule can be given for correctin
observed angles: this must be left to th
...ment of the observer, who, from circum
...ces, will seldom be at a loss to point ou
...re the greatest uncertainty lies. To mak
...ill suppose the corrected angles

are CAB =	64° 28′		DCG
CBA =	75 14		CDG
ACB =	40 18		CGD
	180 0		
BCD =	53 40		DGP
CBD =	64 7		GDP
BDC =	62 13		GPD
	180 0		

Then (221).

ACB = 40° 18′	sin.	9·810763	
		0·189237	
AB = 3520..............	log.	3·546543	
CAB = 64° 28′	sin.	9·955368	
		3·691148	log. CB.
BDC = 62° 13′	ar. co. sin.	0·053196	
BCD = 53 40............	sin.	9·906111	
		3·650455	log. BD = 4471·5.
		3·741344	(223.)
CBD = 64° 7′............	sin.	9·954090	
		3·695434	log. CD.
CGD = 54° 34′	ar. co. sin.	0·088954	
DCG = 73 58............	sin.	9·982769	
		3·770157	log. GD.
GPD = 62° 3′	ar. co. sin.	0·053931	
DGP = 71 7	sin.	9·975974	
		3·800062	log. DP = 6310·5.

Now from the sides BA, BD, and the included angle 139° 21′ we get the angle BDA = 17° 48′, and AD = 7501·1 *yards*, (226).

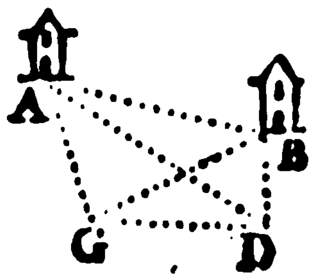
And if BDA be taken from 150° 32′ the angle BDP, there remains 132° 44′ the angle ADP, which, with the including sides AD = 7501·1, and DP = 6310·5 will give the distance from P to A = 12659 *yards*.

When triangles are carried on from the original base in all directions, the distances towards the extremities may, in some respect, be verified by independent calculations.

N. B. All the principal distances should be laid down from a scale of equal parts, because a triangle can be protracted more accurately with its sides than with the angles.

246. Suppose in making a Survey, the distance between the spires A and B has been determined equal to 6594 *yards*; and that G and D are two eminences conveniently situated for extending the triangles.

Now if we observe the angles



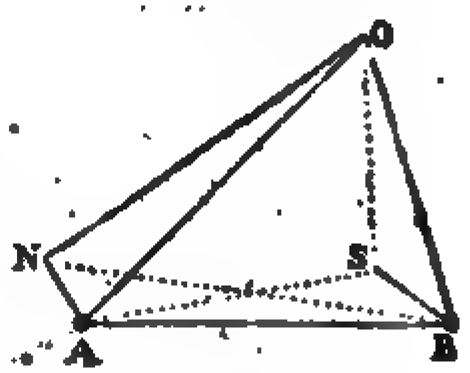
at G $\begin{cases} AGB = 85° 46′ \\ BGD = 23\ 56. \end{cases}$ at D $\begin{cases} ADG = 31° 48′ \\ ADB = 68\ \ 2. \end{cases}$

247. When the top of a Church steeple becomes a station in consequence of the wind-vane or a pinnacle having been intersected, the instrument is placed in the most convenient situation, and a reduction of the observed angles will in that case be necessary.

Let A and B represent the wind-vanes on two steeples, their distance having been determined equal to 2587 *yards* = AB; and suppose N is the place of the Theodolite when it is on the steeple A, and S its situation on the steeple B; also, suppose the observed angles at those stations are the following;

$$\text{at N} \begin{cases} \text{ONB} = 45^\circ\ 42' \\ \text{ONA} = 96\ \ \ 0 \end{cases} \qquad \text{at S} \begin{cases} \text{OSA} = 70^\circ\ 39' \\ \text{OSB} = 147\ \ \ 0. \end{cases}$$

And let the distance from N to the wind-vane A be 11½ *feet*, and that from S to B = 10½ *feet*. Hence it is required to find the angles OAB, OBA, or what the observed angles to the distant object O would be if the instrument was at the points A and B?



The angles 45° 42′, 70° 39′, and 63° 39′ their supplement, with the distance AB = 2587, will give 2066 and 2724 *yards*, the distances BO, AO, nearly.

Then (224) as AO : *sin.* 96° (ONA) :: NA, 3·83 *yards* : *sin.* 5′ nearly, the angle AON.

And 96° — 45° 42′ = 50° 18′, the angle BNA;

Hence, AB : *sin.* 50° 18′ :: NA : *sin.* 4′ nearly the angle ABN.

Therefore the sum of the two angles NOB, NBO is *greater* than the sum of the two angles AOB, ABO by the difference of AON, ABN; consequently ONB is *less* than OAB by 1′; therefore AOB is = 45° 43′.

Again, as BO : *sin.* 147° (OSB) :: 3½ *yards* (SB) : *sine* 3′ nearly, the angle SOB. And, AB : *sin.* 142° 21′ (ASB) :: 3½ (SB) : *sin.* 5′ nearly, the angle SAB.

Now the angles SAO, SOA together are *le*
…A by the sum of the angles SAB, SOB; th
…BO by that sum; hence the angle ABO = 7(
…, AO calculated with the corrected angl
…65·3 and 2720·2 *yards*.

It is not necessary that the angles
…ry accurately taken; but the distance
…lly measured.

248. If A, B, C, be three object
…ch other are AB = 4516, AC = 48(
…nd suppose at the station S we observe
…6′, BSA = 110° 12′; it is required t
…he station to the three objects.

Construction. If the triangle ABC be l…
…down with the three given distances, and …
…ments of circles described upon any two si…
…o contain the angles they subtend (172), …
…ntersection of the arcs will evidently be the …
…ion, whether it falls within, or without the …
…ngle. But the following method is rather m…
…imple.—About AB describe a circle so that …
…egment ABS shall contain the angle 110° 1…
…make the angle BAR = 62° 4′ the supplem…
…f 117° 56′ (CSB), join CR; and S, where …
…tation. For if AS, SB, BR are drawn, …
…′ by construction; and RSB being equal …
…ngle CSB which is its supplement, will be 117…

Calculation. The three sides 4516, 4809, …
…6° 26′ (72…).

Angle ABR (= ASR the supplement of A…
BAR ………………………………
ARB ………………………………
…le AB give BR = 4231·3.

The angle RBC = 48° 6′ + 76° 26′ = …
…luding sides, give RCB = 32° 47′, and C…

° 37'; therefore all the angles of the triangles

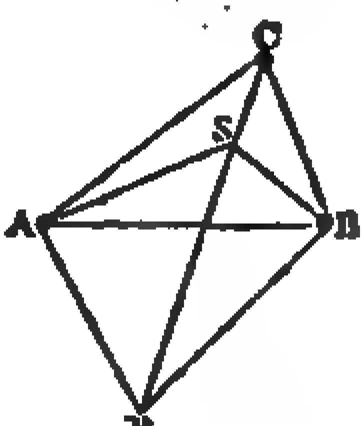
°37'	SCB = 32°47'
12	CSB = 117 56
11	SBC = 29 17

nces SA, SB, SC, are found to be 3530, 1851,

out the triangle (suppose at R) it is evident the that the outward segment ARB shall contain ARB; then if the angles ABS, BAS be made observed angles ARC, BRC, and CR drawn ation.

gle ARB should be equal to the supplement cle will pass through the point C; in which determinate; for the angles standing on the the same in all points of the arc ARB, (70.)

lem will be found useful in reconnoitring or plan; for the angles taken to any laid down, will determine the situation all pocket sextant is the most convenient ing the angles. And it appears from the that it is not necessary to describe a if the station be then the angles ade equal to the rved angles BSC, of AR and BR hen if the angle the angle ARC, the station. On place of observa- iangle, the angles



e equal to the observed angles ARC, en CR being drawn through S, and the ade equal to ASR, BSR, BR and AR station.

In this latter case however, when the point S
object C, the *construction* may give the point R
side of the truth.

250. In making a Survey we found two sp
conveniently situated for stations; and at S to
NSA = 52° 58′, NSB = 55° 4′, to the spires A
at N an intervening height hid the spire B; we
served the angle between the wind-mill W and s
found it = 38° 4′, and then took the angle SN
= 41° 46′. Now AW, AB, BW being respect
5232, 4490, 2678 *yards*, it is required to
tance SN?

Construction. With the three given sides lay down the triangle AWB. Then about AB and AW describe circles so that the segment ASB shall contain an angle of 108° 2′ (52° 58′ + 55° 4′); and the segment ANW an angle of 79° 50′ (38° 4′ + 41° 46′). Draw the chord AD to subtend an angle (AND) = 41° 46′, and the chord AG to subtend an angle (ASG) = 52° 58′; join DG; and the intersections S, N, will be the stations. For if SB, SA; NA, NW are drawn, the angles at S and N the three objects will be equal to the observed an struction, and *Art.* 70.

Calculation. Draw DW, GB. Then all the angles
DW, AGB are given;

viz. DNW = DAW = 38° 4′ . GSB = GA
DNA = DWA = 41 46 GSA = GB
ADW = 100 10 AG

As *sin.* ADW : 5232 (AW) :: *sin.* DWA : 3340
And *sin.* AGB : 4490 (AB) :: *sin.* GBA : 376

The sides of the triangle AWB give the angle WAB

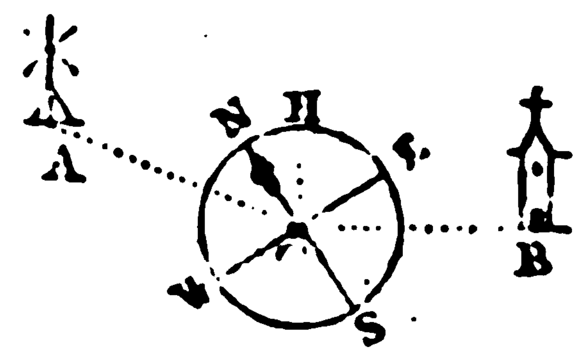
$$\begin{array}{llll} DAW = DNW & = & 38° \ 6' \\ WAB \ \ldots\ldots\ldots & = & 30 \ 47 \\ BAG = BSG & = & 55 \ \ 4 \\ \hline & DAG & = & 123 \ 55 \end{array}$$

with this angle and the including sides we get ADG = 29° = AWN; therefore in the triangle AWN all the angles and the side AW are given, whence AN = 2371; then, as the angles of the triangle ASN are also given, we get SN = 3217 *yards.*

And the method of construction and calculation will vary little from the preceding, howsoever posited the stations may be in respect of the three given objects.

Of Surveying with the Compass.

251. In this operation we do not measure the angles subtended by distant objects in the same manner as with a Theodolite, but take their angular distances or bearings from the *magnetical meridian.* Thus if NS represents the magnetic needle or meridian, W the west, and E the east; and suppose the sights on the Compass are directed to the wind-mill A: then if the angle ACN is 40°, for example, the wind-mill is said to bear NW 40°, or 40° westward from N the magnetical north. Or if the sights are directed to the spire B, and the angle SCB is 64° then the spire bears SE 64°.



If CH represents the direction of the *true meridian,* the angle NCH is called the *variation* of the magnetical needle; which, at this time, is about 23° or 24° westward at London.

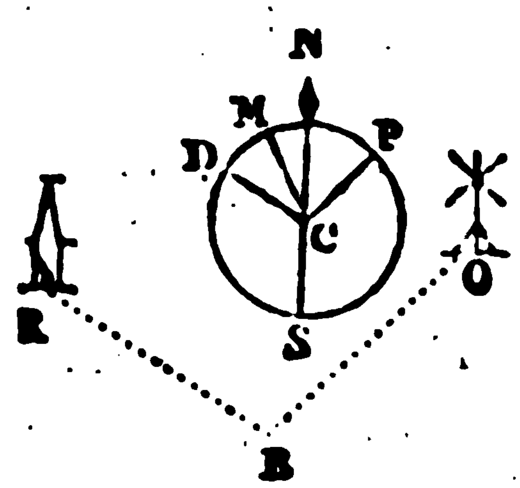
252. Let A and B be two stations bearing SW 61° and NE 61° from each other;

and suppose at *A the* objects $\left\{ \begin{array}{lll} G & \text{bears} & NW\ 29° \\ P & & SW\ 18 \\ R & & SW\ 54 \end{array} \right.$

may serve however, as the ground work of, or for correcting Military sketches.

For temporary use, it will sometimes be necessary to measure a distance by *pacing*, in order to adapt a scale to the plan or sketch (167).

254. The Compass will also be found useful in reconnoitring a country with a map or plan when the direction of the meridian is laid down, and we know the magnetical variation. Let SN be the direction of the true meridian on a map; and suppose the wind-mill O bears NE 68°, and the spire R, NW 36° by the compass; also let the variation be 23° W.

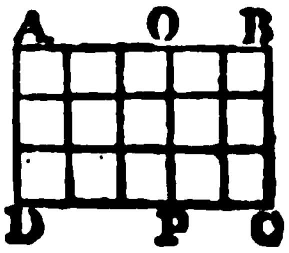


Make the angle MCN = 23°, then CM will represent the magnetical meridian: let the angle MCP = 68°, and MCD = 36°; then if OB, RB are drawn parallel to PC, DC, respectively, the intersection B will be the place of observation on the map or plan. If however, the intersection (B) is very acute or obtuse, the position thus determined may be considerably wide of the truth.

MENSURATION.

Of Right-lined Plane Figures.

255. The measure of the space or surface contained within the boundaries of any plane figure is called its Area or Superficial Content. This is estimated in acres, square yards, square feet, or some other fixed or determinate measure. Thus, if we suppose ABCD to represent the top of a rectangular table whose length DC is 5 feet, and breadth DA = 3; then the upper surface will contain 5 × 3 or 15 square feet (89, *corol.* 2): a square foot being the unit or integer by which the area is estimated.



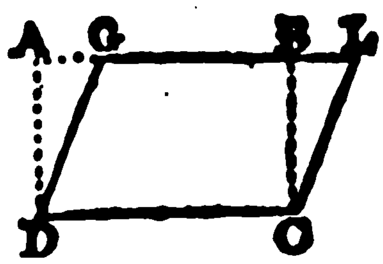
But if the dimensions are taken in yards, its length will be $1\frac{2}{3}$, and breadth 1 yard; and the superficial content = $1\frac{2}{3} \times 1$ = $1\frac{2}{3}$ square yards; for AOPD is the square yard, and the rectangle OC is $\frac{2}{3}$ of a square yard: in this case a square yard is the measuring unit. And when the length and breadth are denoted in inches, a square inch becomes the measuring unit or integer, and the area will be 60 × 36 = 2160 square inches.

To find the area of a Parallelogram of any kind.

256. Multiply the length by the perpendicular breadth, or the base by the height, and the product will be the area.

EXAMPLES.

1. What is the content of the parallelogram DGLC whose length DC is 5 *feet*, and breadth CB is 3?



Ans. 5 × 3 = 15 *square feet.*

For let DA be perpendicular to DC; then the parallelogram GLCD is equal to the rectangle ABCD (88); and the area of the latter is DC × CB.

2. What is the superficial content of a rectangular board, the length being 13*f.* 5*in.* and breadth 10½ *inches?*

Ans. 11*f.* 106½ *in.*

3. How many acres are contained in a square field, the side being 11 *chains*, 56 *links?*

Ans. 13*ac.* 58·1376 *poles.*

4. How many yards (in length) of matting that is ¾ of a yard wide will cover a floor 42½ feet long, and 26½ broad: And what will be the expence at 1*s.* 5¼*d.* per square yard?

Ans. 166$\frac{2}{3}\frac{2}{7}$ *yards* in length.
Expence 9*l.* 2*s.* 5$\frac{6}{7}$½*d.*

5. What length must be cut off a board which is 16½ inches broad, and 4½ feet long, so that the part remaining shall be equal to 5 square feet?

Ans. 10$\frac{4}{11}$ *inches.*

To find the area of a Triangle.

257. MULTIPLY the base by the perpendicular height, and half the product will be the area. Or multiply the base by half the height, or the height by half the base.

For a triangle is equal to half a parallelogram of the same base and altitude, (82, *corol.* 1).

EXAMPLES.

1. How many acres are contained in a triangular field, one side being 470 yards, and the perpendicular on that side = 396 yards?

Ans. 19¼.

Then (221).

ACB = 46° 18′	*sin.*	9·810763	
		0·189237	
AB = 3520	log.	3·546543	
CAB = 64° 28′	*sin.*	9·955368	
		3·691148	log. CB.
BDC = 62° 13′	*ar. co. sin.*	0·053196	
BCD = 53 40	*sin.*	9·906111	
		3·650455	log. BD = 4471·5.
		3·741344	(222.)
CBD = 64° 7′	*sin.*	9·954090	
		3·698431	log. CD.
CGD = 54° 34′	*ar. co. sin.*	0·088951	
DCG = 73 58	*sin.*	9·982769	
		3·770157	log. GD.
GPD = 62° 2′	*ar. co. sin.*	0·053931	
DGP = 71 7	*sin.*	9·975974	
		3·800062	log. DP = 6310·5.

Now from the sides BA, BD, and the included angle 139° 21′ we get the angle BDA = 17° 48′, and AD = 7501·1 *yards*, (226).

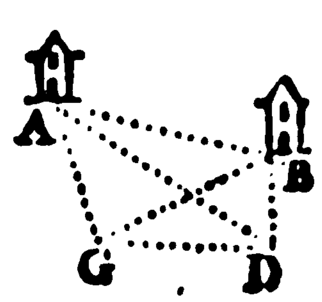
And if BDA be taken from 150° 32′ the angle BDP, there remains 132° 44′ the angle ADP, which, with the including sides AD = 7501·1, and DP = 6310·5 will give the distance from P to A = 12659 *yards*.

When triangles are carried on from the original base in all directions, the distances towards the extremities may, in some respect, be verified by independent calculations.

N. B. All the principal distances should be laid down from a scale of equal parts, because a triangle can be protracted more accurately with its sides than with the angles.

246. Suppose in making a Survey, the distance between the spires A and B has been determined equal to 6594 *yards*; and that G and D are two eminences conveniently situated for extending the triangles.

Now if we observe the angles



at G $\begin{cases} AGB = 85°\ 46′ \\ BGD = 23\ 56. \end{cases}$ at D $\begin{cases} ADG = 31°\ 48′ \\ ADB = 68\ \ 2. \end{cases}$

Then $\frac{1}{2}AD \times CP$ is the area
CO that of the triangle ACB
BCD.

But CP, CO, CR are equal,
her or half the perimeter of t
aus of the inscribed circle is th

About G in BG, the line bis
an arc of a circle is describ
touch BS, BT, and AD; an
the perpendiculars GS, GT
n BS and BT will each be equ
alf the perimeter of the triang
D: For AS = AI, and DT
(79, *corol.* 2), therefore B
BT together are equal to t
of three sides. And co
ently DT or DI is the differen
alf the perimeter (BT) and th
rence of half the perimeter (B

ut PD = RD, RB = OB,
refore 2PD + 2OB + 2OA is
OA = half the perimeter;
PD = AS = AI: now if P
PD, the remainders must be

nd since PD = AS, AD will
efore OB (the difference of B
the perimeter and the side A

nd because DT = DI =
rence of BT and BD, theref
rences between the half perim
ngle.

n the quadrilaterals ASGI,

and O and P are right ones, therefore OCP + OAP are equal to two right angles, and since OAP + SAI make two right angles, SAI = OCP, therefore SGI = OAP, and the two quadrilaterals are equiangular, and because GI = GS, and CO = CP, they are also similar. And since the triangles BOC, BSG are similar, we have

$$OA : OC :: SG : SA,$$

and $BO : BS :: OC : SG$, therefore (140, *Arith.*)

$$OA \times BO : OC \times BS :: SG \times OC : SA \times SG :: OC : SA :: OC \times BS : SA \times BS \ (87);$$

$$\text{or } AO \times BO : OC \times BS :: OC \times BS : SA \times BS;$$

Consequently $OC \times BS$ the area of the triangle, is a mean proportional between $OA \times BO$ and $SA \times BS$; that is, the square of the area $= OA \times BO \times SA \times BS$.

Corol. Hence the perimeter of the triangle ABD will always be equal to both the tangents BS, BT, whatever may be the position of the side AD, provided it is drawn to touch the circle whose centre is G.

Let BD = 42, AD = 30, and BA = 22, as in the preceding example;

$$\begin{array}{rrl} \text{Then} & 42 & \\ & 30 & \\ & 22 & \\ 2\,) & \overline{94} & \\ & \overline{47} & \text{half the perimeter} \\ & 5 & \\ & 17 & \text{the three remainders.} \\ & 25 & \end{array}$$

And $47 \times 5 \times 17 \times 25 = 99875$ the continued product; and the square root of 99875 is 316·03 nearly, the area of the triangle.

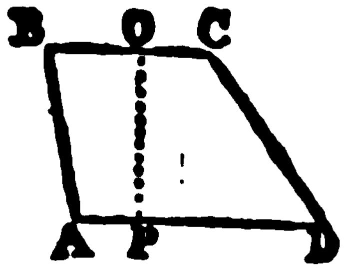
261. *To find the area of a Trapezoid.*

It is proved (*Art.* 81, *corol.* 1) that a trapezoid is equal to half a parallelogram whose base is the sum of the two parallel sides, and height equal to the distance of those sides: therefore,

Multiply the sum of the two parallel sides by their perpendicular distance, and half the product will be the area.

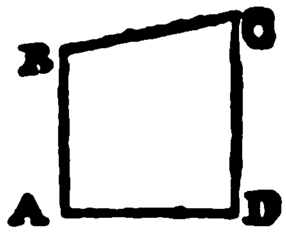
EXAMPLES.

1. What is the content of the trapezoid ABCD in yards, the parallel sides AD, BC being $24\frac{3}{4}$ and $16\frac{1}{2}$ feet, respectively, and the perpendicular distance QP = 18 feet?



$24\frac{3}{4}$
$16\frac{1}{2}$
$41\frac{1}{4}$ sum of the parallel sides, which multiplied by 18 gives $742\frac{1}{2}$, half of which is $371\frac{1}{4}$ the content in *feet*, equal to $41\frac{1}{4}$ *square yards.*

2. Suppose the parallel sides AB, DC of a field are 6 *ch.* 86 *links*, and 8 *ch.* 58 *links*, and their perpendicular distance AD = 9 *ch.* 7 *links*; what is the content?



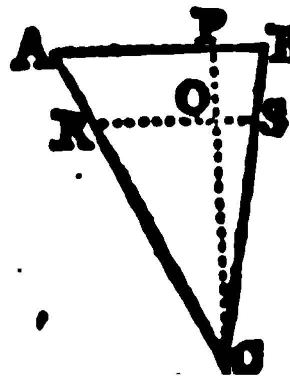
Ans. 7·00204 acres.

3. What cost 20 boards, each being $16\frac{1}{2}$ feet long, $14\frac{1}{4}$ inches broad at one end, and $12\frac{1}{2}$ at the other, at $5\frac{1}{2}d.$ the square foot?

Ans. 8*l.* 10*s.* $1\frac{1}{4}d.$

262. From a given triangle ACB to cut off a trapezoid ARSB of a given area.

Let CP be perpendicular to AB. Then $\frac{1}{2}$AB × PC is the area of the triangle ACB; from this area subduct the given area ARSB and the remainder is the area of the triangle RCS, which is equal to $\frac{1}{2}$RS × OC.



Now the triangles ACB, RCS being similar, we have

$$\tfrac{1}{2}AB \times PC : PC^2 :: \tfrac{1}{2}RS \times OC : OC^2 \text{ (101)};$$

And $\tfrac{1}{2}AB : PC :: \tfrac{1}{2}RS \times OC : OC^2$ (87).

Let AB = 40, and PC = 54; and suppose the area of the trapezoid to be 480. Then 1080 is the area of the triangle ACB, and 1080 — 480 = 600 the area of the triangle RCS = $\tfrac{1}{2}$RS × OC, therefore the last proportion will be, 20 : 54 :: 600 : 1620 = OC^2; and the square root of 1620 is 40·25 nearly, = OC; whence 54 — 40·25 = 13¾ = OP the breadth of the trapezoid.

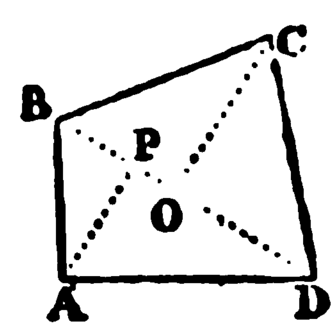
Examp. 2. What length must be cut off a board that is 12 feet long, 18 inches broad at one end, and 14 at the other, to make 10 feet square?

Ans. 7·1 *feet* at the greater end.

263. *To find the area of a Trapezium.*

Let the Trapezium be divided into two triangles by a diagonal, then the areas of the triangles added together will be the content of the Trapezium.

Examp. 1. What is the area of the trapezium ABCD, when the diagonal BD = 49·7, and the perpendiculars on BD are CO = 33·5, and AP = 22?



$$\frac{49{\cdot}7 \times 33{\cdot}5}{2} = 832{\cdot}475 \text{ area of triang. BDC.}$$

$$49{\cdot}7 \times 11 = 546{\cdot}7 \text{ area of triang. DBA.}$$

$$1379{\cdot}175 \text{ area of ABCD.}$$

2. Let the measured sides of the quadrangular field ABCD be

AB = 15*ch.* 24 *links*, CB = 18*ch.* 86 *links*,
AD = 11*ch.* 14 *links*, CD = 9*ch.* 90 *links*:

And suppose the angles at A and C taken with a Theodolite are DAB = 105° 28′, and DCB = 89° 54′,

What is the content in acres?

The sides AB and AD with the include
e area of the triangle BDA = 81·814 cha

And CB and CD with the included
·355 *chains* the area of CDB; the sum
ains = 17·5169 *acres*.

3. Having measured the side A
und it to be 311 *yards*, we observe

BAC = 44° 20′
CAD = 41 19

Hence the content of the field is re

ith AD and the given angles of th
, BDC, find the diagonal DB, an
).

en (258) BD and AD with the inclu
triangle ADB = 4·3335 *acres*: and B
, that of the triangle BDC = 4·317
content of the trapezium.

4. *To find the area of a regu*

ULTIPLY half the perimeter b
its centre on one of the sides,
of the polygon (106).

EXAMPLE

. If DB the side of a pentagon
be the area?

t C be the centre; then the angle
= 54°. Then (223) *radius* : *tang*. 5
perpendicular.

And $\frac{5}{2} = 2\frac{1}{2}$ half the perimeter, therefore $2\frac{1}{2} \times {\cdot}688191 = 1{\cdot}720477$ the area of the pentagon.

The area 1·720477 will serve as a *multiplier* for finding the content of any other regular pentagon whose side is given; Thus,

2. Suppose it is required to find the content of a pentagon whose side is 20: Then, similar plane figures being in the same proportion as the squares of their homologous sides (102), we have

$1^2 : 1{\cdot}720477 :: 20^2 : 400 \times 1{\cdot}720477$ or 688·19 the area required.

3. If the side of a regular hexagon be 1, what is the area?

The hexagon is composed of 6 equilateral triangles, each side being 1.

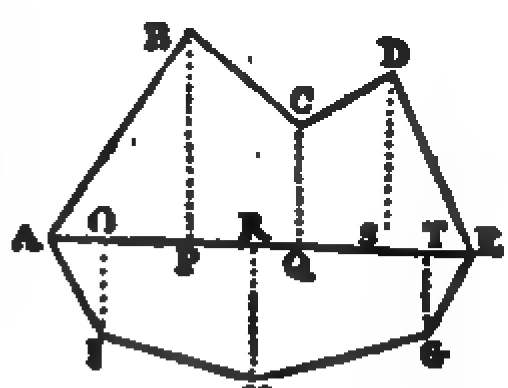
Now $1\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{16}$ is the square of the area of one of the triangles (260); and the square root of $\frac{3}{16}$ is ·433013 nearly, therefore $\cdot 433013 \times 6 = 2{\cdot}598078$, the content of the hexagon. And this area will be a *multiplier* for finding the content of any other regular hexagon whose side is given.

265. *To find the area of an irregular Polygon.*

Divide the polygon into triangles, or into triangles and trapezoids; then their areas added together will be the content of the polygon.

EXAMPLES.

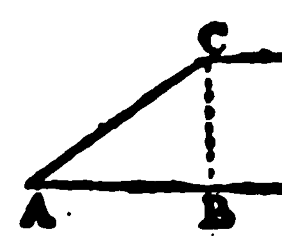
1. What is the content of the octangular figure BG, the lengths of the several parallels and perpendiculars being as follows;



AO = 44¾	OI =
OP = 124¼	RH =
PR = 80	TG =
RQ = 41	PB =
QS = 130¼	QC =
ST = 50	SD =
TE = 52¼	

In land-measuring, an instrument called
be very useful for finding the points O, P,
perpendiculars IO, BP, HR, &c. fall from
Field upon the base line AE. Or the sam
with a pocket Sextant, thus: Set the index
walk along the line AE (if towards E) di
object at E, then suppose you see the co
by reflection when you are at P, the a
right one.

2. Suppose the adjacent figure to represent the perpendicular section of a rampart; the several heights and breadths being as follows:

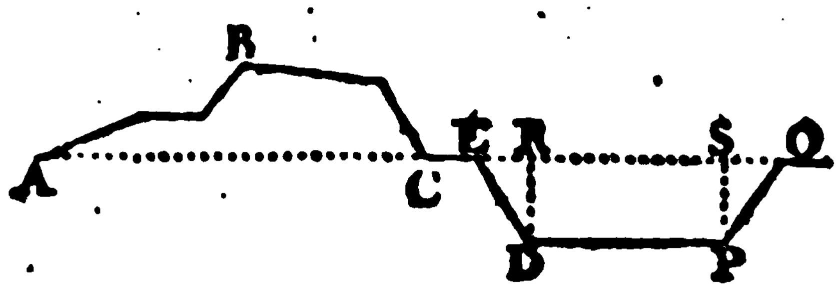


viz. AB = 16	BC =
BD = 18	DE =
DH = 2	HG =
HK = 3	KI =
KL = 2	LO =
LP = 12	PR =
PS = 10	

What is the superficial content of the se

The perpendiculars divide the figure into 2 trian
rectangle; and their areas added together mak
quired.

3. Let ABC be the profile or perpendicular section of a breast-work, and EP that of the ditch; now suppose the area of the section ABC is 88 *feet*, the depth of the ditch RD = 6 *feet*, and ER = 3 *feet*; what is the breadth of the ditch at top when the sections of the ditch and breast-work are equal, or when the earth thrown out of the ditch is supposed to make the breast-work?

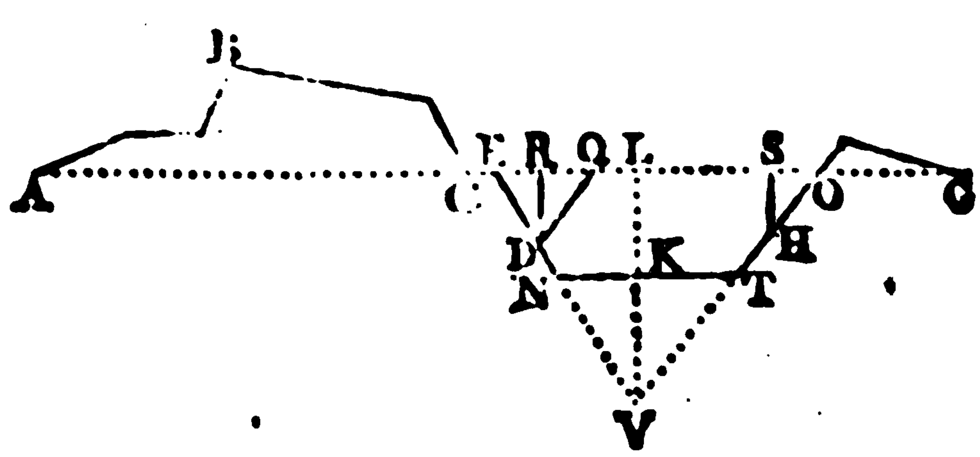


If the slope on each side of the ditch is the same, the areas of the triangles ERD, SPO together make 18 *feet*, which taken from 88 leaves 70, the area of the rectangle RP; this divided by the depth RD or SP gives $11\frac{2}{3}$ = RS, therefore EO the breadth of the ditch at top is $11\frac{2}{3} + 6 = 17\frac{2}{3}$ *feet*.

4. Let the section of the breast-work ABC be as in the preceding example, and EO the breadth of the ditch at top = 20 *feet*; also suppose the slopes of the ditch are unequal according to the following proportions, ER : RD :: 2 : 3, and SO : SH :: 2 : 4; RD and SH being perpendicular to EO: Now what must be the depth of the ditch, if the earth when thrown out is also to form a glacis whose height is 3 *feet*, and base OG = 14?

Area of ABC 88
of the glacis = $1\frac{1}{2} \times 14$ = <u>21</u>
of ENTO .... <u>109</u> section of ditch.

Let DQ be parallel to TO, and VL perpendicular to EO, V being the concourse of CN, OT produced.



Then the triangles RDQ, SHO are similar, whence

HS : SO :: DR : RQ
or 4 : 2 :: 3 : $1\frac{1}{2}$ = RQ; therefore EQ = $2 + 1\frac{1}{2} = 3\frac{1}{2}$.

and because the triangles EDQ, EVO are similar, we ha

$EQ : ED :: EO : LV$

or $3\frac{1}{2}$: 3 :: 20 : $17\frac{1}{7}$ = the perpendicular I

and $17\frac{1}{7} \times 10 = 171\frac{3}{7}$ the area of the triangle EVO.
ENTO = 109

$62\frac{3}{7}$ area of triangle NVT.

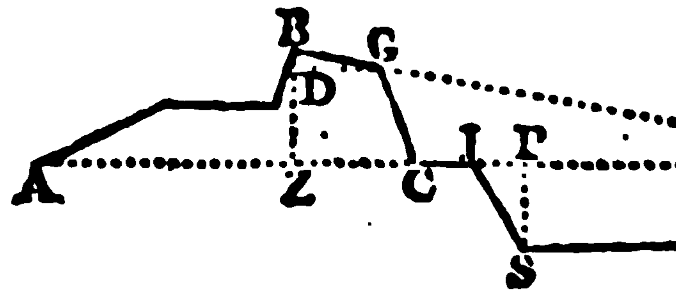
Therefore (262), $\frac{1}{2}EQ : ED ::$ triang. NVT : VK^2

or $1\frac{3}{4}$: 3 :: $62\frac{3}{7}$: $107\frac{1}{49}$, a

is 10·3 nearly, = VK; whence the depth LK = 1

feet.

Suppose the of the profile C = 100 ;



And BD = 1	ZP = 13
DG = 6	PS = 6 depth
BZ = 10	IP = 3 *feet.*

What must be the breadth of the ditch so th T shall be equal to the profile ABGC and on of the glacis) together, when the talus BG rior slope of the glacis are in the same plane TO being equal?

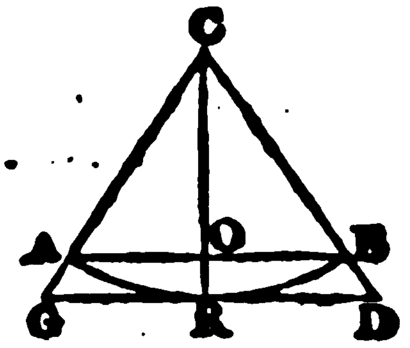
Ans. IT breadth at top =

86. *Of the Circle.*

:00051132692 be multiplied by 12288 318519296 the length of the perimeter of gon of 12288 sides when the radius of th . This perimeter must be very nearly equ ference of the circle, but somewhat less. It be worth while to calculate the perimeter of ing polygon of the same number of sides,

nce of the circle must be greater than the one, but he other.

The centre of the circle, AB the side ibed polygon, and GD that of the cor- circumscribing one; and suppose CR is ar to AB and GD.



is ·0002556346 or half of ·00051132692 : ıg = 1, we get CO = ·99999996728 nearly, (83, *corol.*).

use the perimeters of similar plane figures are in the same pro- heir homologous sides (103) we have

CR :: *perim. inscribed polyg.* : *perim. circums. polyg.*

99996728 : 1 :: 6·28318519296 : 6·2831854 nearly, the peri- e circumscribing polygon. We therefore conclude that the ce of the circle

is *less* than 6·2831854
but *greater* than 6·2831852

y half their sum or 6·2831853 must be very nearly the cir-

diameter being 2, the circumference of a circle whose diameter half of 6·2831853, or 3·14159265; which is correct in the last nd sufficiently near to give the circumference of the Earth than 2 *inches*, supposing it globular and the diameter 8000

uch accuracy is not required, the proportion of the diameter umference may be taken as 1 to 3·1416. Or that of 7 to 22 or common purposes. The ratio of 113 to 355 is a nearer tion than either.

To find the area of a Circle.

ULTIPLY the radius or half the diameter by half the ence, and the product will be the area (106, *corol.*).

Or the square of the diameter multiplied by ·7854 area.

III. Or multiply the square of the circumference by ·079577.

EXAMPLES.

1. What is the area of a circle whose diameter is 1?

Half the diameter is ·5, and half the circumference is 1·570796 &c.

$·5 \times 1·570796 = ·785398$ or ·7854 nearly, the area.

Now ·7854 is a common multiplier for finding the area of any other circle whose diameter is given: thus,

2. Let it be required to find the area of a circle whose diameter is 20?

Then circles being as the squares of their diameters (105, *corol.*) we have

$1^2 : ·7854 :: 20^2 : 400 \times ·7854 = 314·16$ the area sought (*rule* II).

3. Required the area of a circle whose circumference is 1?

As 3·1415926 : 1 :: 1 : ·31831 nearly, the diameter:

Therefore $\frac{1}{2} \times \frac{·31831}{2} = ·079577$ the content. Which is a multiplier for finding the area when the circumference is given (*Rule* III.).

4. How many square yards in a circle whose radius is $15\frac{1}{2}$ *feet*?

Ans. 81·1798, nearly.

5. What is the diameter of that circle whose area is an Acre?

Ans. $78\frac{1}{2}$ *yards*, nearly.

To find the area of the Sector of a Circle.

268. When the diameter and length of the arc are given, Multiply half the diameter by half the arc, and the product will be the *area*: (*this is* evident from 106, corol.).

[redacted]

EXAMPLES.

1. What is the area of the circular sector if the radius is $20\frac{1}{2}$, and length of the arc 36?

$20\frac{1}{2} \times 18 = 369$. *Ans.*

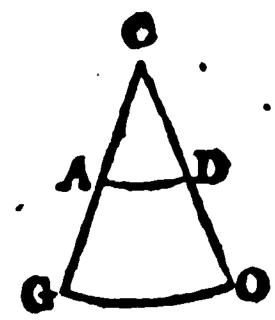
2. What is the area of a sector if the radius be 1, and the arc contains 40°?

When the radius is 1, the circumference is 6·2831853 (266):

Therefore $\frac{6 \cdot 2831853}{360} = \cdot 01745$ &c. length of the arc of 1°,

And $\cdot 01745 \times 40 = \cdot 698$ is the length of the arc of 40°.

And the area of the sector $= 1 \times \frac{\cdot 698}{2} = \cdot 349$ *Ans.*



269. Let CA, CG be the radii of two similar sectors CAD, CGO:

Then CA : AD :: CG : GO,

or $1 : \cdot 01745 \times 40^\circ :: CG : \cdot 01745 \times 40^\circ \times CG$ the length of the arc GO when the angle C is 40°:

Therefore if the number of degrees in a sector be multiplied by the radius and that product by the decimal ·01745 the result will be the length of the arc of the sector.

Since the area of the sector CAD is CA $\times \frac{1}{2}$AD, or $1 \times \frac{\cdot 01745 \times 40^\circ}{2}$ (if the angle C is 40°) it will be

$CA^2 : CG^2 :: 1 \times \frac{\cdot 01745 \times 40^\circ}{2}$: area of sector CGO (105, *corol.*).

or $1 : CG^2 :: \cdot 0087266 \times 40^\circ : CG^2 \times \cdot 0087266 \times 40^\circ$:

Consequently, if the square of the radius, the number of degrees in the sector, and the decimal ·0087266 are multiplied together, the product will be the area.

Examp. What is the area of a sector when the radius is 50, and its arc 94° 34½'?

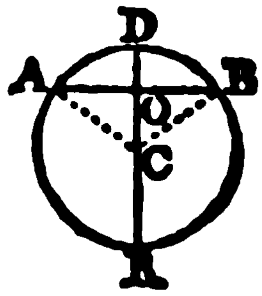
$94^\circ\ 34'\tfrac{1}{2} = 94^\circ{\cdot}575$

And $50^2 \times 94{\cdot}575 \times \cdot 0087266 = 2063{\cdot}2955$ the area sought.

270. *To find the area of a Segment of a Circle.*

Ex. 1. Let ADB be a segment whose chord AB = 36, and height or versed sine OD = 8; C being the centre of the circle.

Then DO : AO :: AO : OR (97, *corol.* 1.)
or 8 : 18 :: 18 : $40\tfrac{1}{2}$ = OR
$\quad$ 8 = OD
2) $48\tfrac{1}{2}$ = DR diam. of circle
$\quad 24\tfrac{1}{4}$ the radius.



Now OB being 18, we get (224.) the angle OCB = 47° 55½', therefore the angle of the sector ACB = 95° 51' = 95°·85.

And the area of the sector ADBC .... = 491·88 (268.)
area of the triang. ACB = (OB × OC)
= 18 × 16¼ .... = 292·5
Area of the segment ADB .... diff. 199·38

2. If the height or versed sine be 50, and the radius of the circle 40; what is the area of the segment?

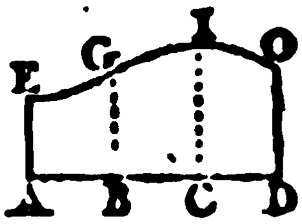
Ans. 3304·872.

Of mixt-lined Figures.

271. A mixt-lined figure is one bounded by both right and curved lines, as AO.

No general rule can be given for obtaining the exact contents of all figures of this description. The usual method of approximation is to divide the curved or crooked lines into short parts, and then consider each of those parts as the side of a right-lined figure.

Examp. 1. Suppose AD is divided into 3 equal parts, and let AE, BG, CI, DO, be perpendicular to AD; also suppose



AD = 21	CI = 10
AE = 6	DO = 9
BG = 8	

Then if we suppose EG, GI, IO, to be right lines, the figure will consist of 3 trapezoids having equal bases AB, BC, CD:

$$\text{And } \frac{6+8}{2} \times 7 = 49 = \text{trapezoid AG (261.)}$$
$$\frac{8+10}{2} \times 7 = 63 = \ldots\ldots\ldots\ldots \text{ BI}$$
$$\frac{10+9}{2} \times 7 = 66\tfrac{1}{2} = \ldots\ldots\ldots\ldots \text{ CO}$$
$$\text{sum } \overline{178\tfrac{1}{2}} \text{ the whole content.}$$

But the same result is obtained by multiplying the arithmetical mean breadth by the base or length AD. Thus, take half the sum of the extreme breadths AE and DO for one breadth, to which add BG and CI, and divide the whole by 3 (the number of parts into which AD is divided) for the mean breadth.

For the sum of the 3 fractions having the common denominator 2 is $\frac{6+8+8+10+10+9}{2}$, or $7\tfrac{1}{2}+8+10$.

Therefore the sum $7\tfrac{1}{2}+8+10 \times 7$; or $25\tfrac{1}{2} \times \frac{21}{3}$ or $\frac{25\frac{1}{2}}{3} \times 21$,

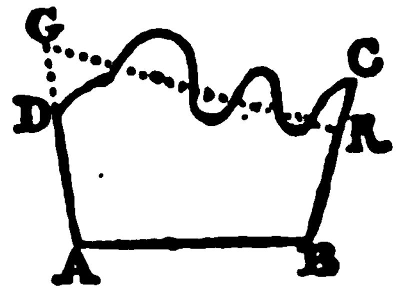
or $\frac{7\frac{1}{2}+8+10}{3} \times 21 = 178\tfrac{1}{2}$, is equal to the 3 trapezoids: where $7\tfrac{1}{2}$ is half the extreme breadths AE, DO; and $\frac{7\frac{1}{2}+8+10}{3}$ or $8\tfrac{1}{2}$ is the mean breadth.

It is evident that the area 49 is *too great*, because the curved side EG is convex towards the opposite side AB: but GI and IO are bent the contrary way, and consequently 63 and 66½ are both *too little*. Hence it appears, that the greater the number of equal parts into which the base (AD) is divided, the more accurate will be the result.

Examp. 2. The length or base of an irregular figure being 37·6, and the breadths at 9 equi-distant places 0, 4·4, 6·5, 7·6, 5·4, 8, 5·2, 6·1, 6·5; what is the area?

Ans. 218·315.

272. The following method of reducing a crooked boundary to a straight line is sometimes practised in land-measuring. Suppose ABCD is a field protracted from a survey, the side DC being very irregular: Then to reduce this side to a straight line, lay a fine thread GR across it, and guess by the eye when the parts of the surface excluded on one side of the thread are equal to those taken in on the other; then draw



the line GR with a pencil; and the surface of the field will be reduced to the quadrilateral ABRG. A fine silk thread, or horse hair, stretched after the manner of a bow-string, will be found very convenient for this purpose.

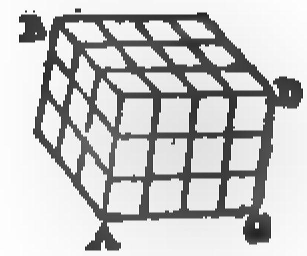
Mensuration of Solids.

273. By the mensuration of solids we understand that of their superficies, as well as the capacities or solid contents.

If a solid is bounded by planes they must be right-lined figures (125); and their areas added together will give the whole surface of the solid.

EXAMPLES.

1. To find the superficies of the rectangular prism or parallelopiped BC, its height being 3, breadth 3, and length 4.



$3 \times 4 = 12$ area of AD one of the 4 equal sides.
$\quad\quad 4$
$\quad\quad \overline{48}$
$\quad\quad 18$ twice the area of AB.
$\quad\quad \overline{66}$ area of the 6 sides or whole superficies.

2. What is the superficies of a cube whose sides is 7?

$7 \times 7 = 49$ area of one of the 6 equal faces.
$49 \times 6 = 294$ the whole surface.

3. What will be the expense of lining a rectangular cistern with sheet lead, its length being $5\frac{1}{3}$, breadth 4, and depth $2\frac{1}{4}$ *feet*, at 9*d*. the square foot?

Ans. 2*l*. 6*s*. $4\frac{1}{2}$*d*.

4. What is the area of the inner surface of a ditch surrounding a square fort, the slope on each side being equal, and the breadth at top 30, at bottom 26, and depth 6 *feet*; and the side of the inner square 300 *feet*?

Half the difference of 30 and 26 is 2, therefore (83.) the square root of 40 (or $36 + 4$) will be the slant depth of the ditch.

Upper side of inner slope of $\frac{1}{4}$ of the whole ditch 300 *feet*.
Lower side 304
Upper side of outer slope........................ 360
Lower side 356

$4 \times \frac{300 + 304}{2} \times \sqrt{40}$, or $4 \times 302 \times \sqrt{40}$ area of inner slope all round.

$4 \times \frac{356 + 360}{2} \times \sqrt{40}$, or $4 \times 358 \times \sqrt{40}$ area of outer slope all round.

The sum is $4 \times 660 \times \sqrt{40}$ .... $= 16697$ nearly.

And $4 \times \frac{30 + 356}{2} \times 26$ area of bottom all round $= 34320$

Area sought $\overline{51017}$ feet.

5. What is the superficies of a tetraedron or pyramid contained by 4 equilateral triangles, each side being 6?

Ans. 62·3538 nearly.

To find the convex surface of an upright Cylinder.

274. If an upright hollow cylinder of paper or other thin material be cut in a direction perpendicular to its ends, and then opened flat, it will form a rectangular parallelogram: Therefore to find the convex surface, multiply the length of the cylinder by its circumference.

Examp. What is the whole superficies of a cylinder, its length being 10 feet, and diameter 3?

The circumference is = 3 × 3·1416 = 9·4248
And 9·4248 × 10 (the convex surface) .... = 94·248
Contents of both ends, add .... = 14·1372
Ans. .... 108·3852 *feet.*

To find the convex surface of an upright Cone.

275. Conceive the surface to be opened out in a plane, and the circumference of the base will then become the arc of a sector of a circle whose radius is the slant height: Hence, half the circumference of the base multiplied by the slant height will give the curve superficies (268).

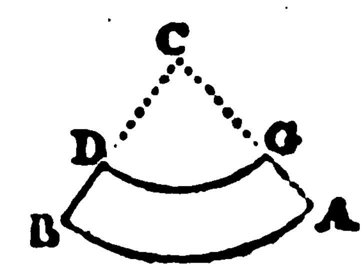
Examp. What is the convex surface of a cone, whose height is 20 feet, and the diameter of its base 10?

The slant height is 20·6155. Circumference of the base 31·416.

And $\frac{31\cdot416}{2} \times 20\cdot6155 = 323\cdot828$ feet. *Ans.*

To find the convex surface of a Conic Frustum, or a part cut from the bottom by a plane parallel to the base.

276. If the sector BCA represents the curve surface of the whole cone, BDGA will be that of a frustum: therefore the difference of the sectors BCA, DCG, is the surface of the frustum.



Examp. Suppose the circumferences of the two ends of the frustum are 24 and 15, and the slant height 6; what is the curve surface?

Because the sectors BCA, DCG are similar, we have

$$BA : DG :: BC : DC$$

And $BA - DG : DG :: BC - DC : DC$ (94, *schol.*)

or $24 - 15 : 15 :: 6\,(BD) :: 10 = DC$:

Therefore $\frac{15}{2} \times 10 = 75$ area DCG

and $\frac{24}{2} \times 16 = 192$ area BCA.

diff. 117 area BDGA.

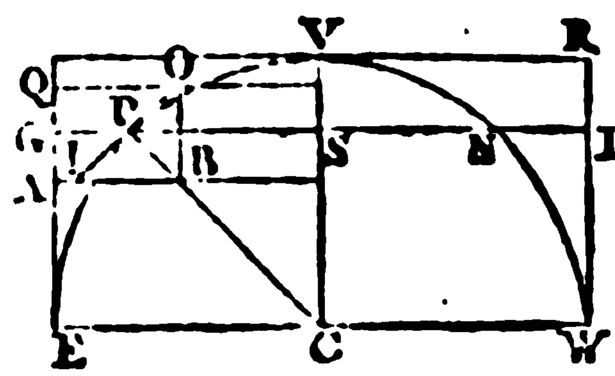
Or thus:—Since the segment BDGA is analogous to a trapezoid, if half the sum of the parallel sides DG, BA be multiplied by their perpendicular distance BD, the product will be the area.

$$\frac{15 + 24}{2} \times 6 = 117 \text{ the curve surface, as before.}$$

To find the surface of a Globe or Sphere.

277. Multiply the diameter of the sphere by its circumference, and the product will be the superficies.

Let the semi-circle EVW be circumscribed by the rectangle ER, and suppose IO, which is drawn to touch the circle in P, to be bisected by the radius CP; also, let CV be perpendicular, and AB, GD, QO parallel to EW.



Then if the semi-circle and rectangle revolve about the axis CV, the former will describe a hemisphere, and the latter a cylinder; and IO will describe the curve surface of a conic frustum.

Let OB be perpendicular to AB: then the triangles PCS, OIB are similar, whence PC : PS :: OI : OB; but QA = OB, and GS = CP, therefore GS : PS :: OI : QA:

But the circumferences of circles are as their diameters (103, *corol.*); and because GD is double GS, and PN double PS, we have (from the last proportion)

As *circumf. circle* GD : *circumf. circle* PN :: OI : QA;

whence QA × *circumf.* GD = OI × *circumf.* PN,

But QA × *circumf.* GD is the curve surface of the cylinder, whose height is QA (274). And because PO = PI, the circumference described by the point P will be half the sum of the circumferences described by the points O and I, and therefore the slant height OI × *circumf.* PN is the curve surface of the conic frustum described by OI 276;) whence it appears that the convex surfaces of the cylinder, and conic frustum described by QA and OI are equal.

Now if the points Q and A nearly coincide with G, the corresponding points O and I will nearly coincide with the point P, and in that case, we may consider the indefinitely small conical surface as coinciding with, and equal to the indefinitely small portion of the spherical surface, and as this will hold in every part of the quadrant EV, the sum of all the conic surfaces must be equal to the whole spherical surface, which therefore, will be equal to the corresponding surface of the cylinder:—Hence the surface of the hemisphere is equal to that of the cylinder ER, or the surface of a sphere equal to that of its circumscribing cylinder, or equal to 4 times the area of the circle whose diameter is that of the sphere.

Corol. 1. Hence also, the convex surface of any spherical *segment*, or *zone*, is equal to the circumference of the sphere multiplied by the height of the said segment, or zone.

Corol. 2. And because the areas of circles are as the squares of their diameters, or circumferences, the surfaces of spheres will be as the squares of their diameters, or circumferences.

EXAMPLES.

1. What is the superficies of a globe whose diameter is 4 inches?

$4 \times 3{\cdot}1416$ the circumference; and $4 \times 4 \times 3{\cdot}1416 = 50{\cdot}2656$ inches, *Ans.*

2. What would be the cost of gilding a globe 10 feet in diameter, at 6*d.* the superficial foot?

Ans, 7*l.* 17·08*s.*

3. At what height above the earth must a person be to see one fourth of its surface, supposing the earth to be perfectly spherical, and its diameter 8000 miles?

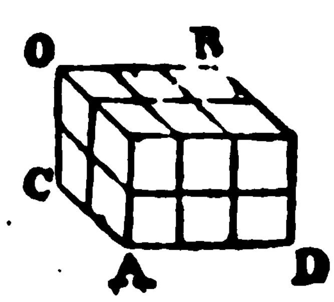
Ans. 4000 *miles.*

To find the solid or cubic contents of a Prism or Cylinder.

278. MULTIPLY the area of the base by the height, and the product will be the solid or cubic contents (131, and *corol.*).

EXAMPLES.

1. How many cubic inches are contained in the rectangular prism or parallelopiped AB, the length AD being 3 inches, breadth $AC = 2$, and height $CO = 2$?



$2 \times 3 = 6$ area of the base;
And $6 \times 2 = 12$ inches the cubic contents.

This is called *cubic measure* because the capacity or magnitude is estimated in cubic integers, as cubic yards, cubic feet, or cubic inches: Thus, in the present example, a cubic inch is the measuring integer or unit, the whole prism containing 12 of these units or inch cubes.

2. How many gallons of water will a cubic cistern contain, its depth being 4 feet?

$$4 \times 4 \times 4 = 64 \text{ cubic feet, the capacity;}$$

$$\text{And } \frac{64 \times 1728}{231} = 478 \tfrac{174}{231} \text{ gallons, } \textit{wine measure.}$$

3. What is the value of a cylindric stone pillar whose diameter = $3\frac{1}{2}$ feet, and height 20 feet, at 2*s*. 10*d*. the cubic foot?

Ans. 27*l*. 5*s*. $2\frac{1}{2}$*d*.

4. If the velocity of water through a cylindrical pipe $1\frac{3}{4}$ inches in diameter, be 13 inches per second, what quantity would it supply in 24 hours?

Ans. 8592 gallons, *wine measure*.

5. If the depth of an oblique parallelopiped be 4 feet, the acute angle of the base 42°, and the including sides $7\frac{1}{2}$ and 3 feet, what is the content in cubic yards?

Ans. 3·7174.

To find the solid contents of a Pyramid or Cone.

279. MULTIPLY the base by the perpendicular height, and $\frac{1}{3}$ of the product will be the area.—Or, multiply the base by $\frac{1}{3}$ of the height (133).

EXAMPLES.

1. How many cubic feet in a triangular pyramid, the sides of the base being 7, 8, and 9 feet, and the perpendicular height 17?

Ans. 132·03 nearly.

2. Required the number of cubic yards in an upright pyramid, the base being a regular heptagon, whose side is 10 feet, and the slant height from the middle of the side of the base = 20 feet?

Ans. 126·27.

3. How many cubic yards in an upright cone, the circumference of the base being 70 feet, and the slant height 30?

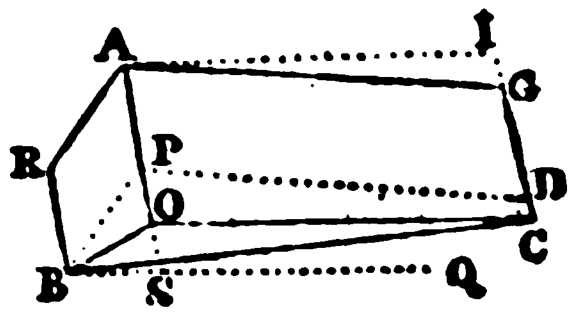
Ans. 134·09.

4. What is the content of an oblique cone, the greatest slant height being 20 feet, the least 16, and the base a circle whose diameter is 8 feet?

Ans. 234·636 *feet.*

280. *To find the contents of a Cuneus or Wedge.*

A WEDGE is a solid having one of its ends flat, and the other an edge made by the concourse of two opposite plane sides. Thus the trapezoid ARBO is the flat end; and GC the concourse of the planes AOCG and RBCG, the other end or edge of the wedge BG.



When the planes AOCG, and RBCG are rectangular and equal, the end ARBO will also be a rectangle, and the wedge is of the common form, or half a parallelopiped having the rectangle AC for its upper side, and OS (which is perpendicular to BQ) for the depth or thickness; BQ being parallel to OC.— Or the wedge is a prism having the triangle BOC for its base, and OA the height; and the content, in that case, is = OA × *area triang.* BOC.

Suppose the planes or sides AOCG and RBCG, and also the end ARBO are trapezoids, and the latter any how inclined

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to the two former; and let the plane B... side RAG. Then the whole wedge BG... triangular prism ARBPDG, and the p... latter having B for its vertex, the trapezoi... and OS the perpendicular height of B abo...

Then if AI is the perpendicular distance... area of the parallelogram AD will be AP ×...

And AP × AI × ½ OS is the content of t... OS being the depth of the parallelopiped.

And $\frac{PO + DC}{2}$ × AI is the area of the... base of the pyramid (261).

And $\frac{PO + DC}{2}$ × AI × ⅓OS the conten... (279).

But AP × AI × ½OS the content of the pri... as 3AP multiplied by the rectangle or product A...

And $\frac{PO + DC}{2}$ × AI × ⅓OS the content o... the same as PO + DC multiplied by the rectangl...

Therefore the sum of both or 3AP + PO + I... by the rectangle AI × ⅙OS is the content of the ...

But AP + PO is equal to AO:
And AP + DC equal to GC:
Also AP is equal to RB;

Therefore AO + GC + RB is equal to 3AP + I... And consequently the *sum* AO + GC + RB *mu*... *the product* AI × ⅙OS is the content of the wedge: ... the perpendicular distance of RB from the face AC...

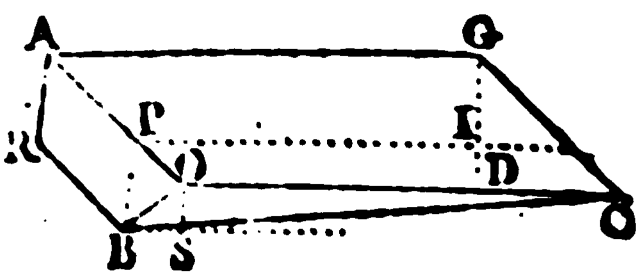
But AO + GC + RB multiplied by AI × ⅙OS is the same as $\frac{AO + GC + RB}{3}$ multiplied by AI × ½OS, (because ½ × ⅓ = ⅙):

Now AI × ½OS is the area of the perpendicular triangular section of the wedge; that is, if we suppose CO to be perpendicular to AO and GC, and OB at right angles to AO, then OC = AI, and the triangle BOC is that triangular section; and considering OC as the base, OS will be the altitude of the triangle: Hence to find the content of a wedge,—*Add the edge and those two sides of the opposite end that are parallel to the edge together, and multiply ⅓ of the sum by the area of that section of the wedge which is perpendicular to those three lines; and the product is the content* (Rule 1.).

Examp. 1. Let AO = 4, GC = 3, RB = 2½, the perpendicular AI = 12, and OS the perpendicular distance of BR from the face AC (produced) = 3½ *feet*;

Then $\frac{4 + 3 + 2\frac{1}{2}}{3} \times 12 \times \frac{3\frac{1}{2}}{2} = 66\frac{1}{2}$ cubic *feet*, the contents.

Examp. 2. Suppose the depth of a waggon road is 5½ feet below the common surface of the ground, and that another road leading out to the surface is to be cut obliquely through the bank or side: now if the length of the new cut at top is 51 feet, the perpendicular breadth at top 9 feet, and the narrowest breadth at bottom 6 feet; what will be the content of the excavation?



If the trapezoid ARCO is the opening in the bank or entrance of the new cut, CBO will be one of its sloping sides, and the parallelogram AOCG the top whose length is 51 feet, and perpendicular breadth GD = 9, and if the plane BPI is parallel to the side RAG (as in the preceding example), then GI (6 feet) will be the narrowest or perpendicular breadth at the entrance RB.

Now because AG × GI is the area of the parallelogram PG, and OC × ID that of the parallelogram PC; therefore, if instead of AO and GC and their perpendicular distance (as in the foregoing example) we make use of the other sides AG and OC and their perpendicular distance GD, the content of the wedge or excavation BG

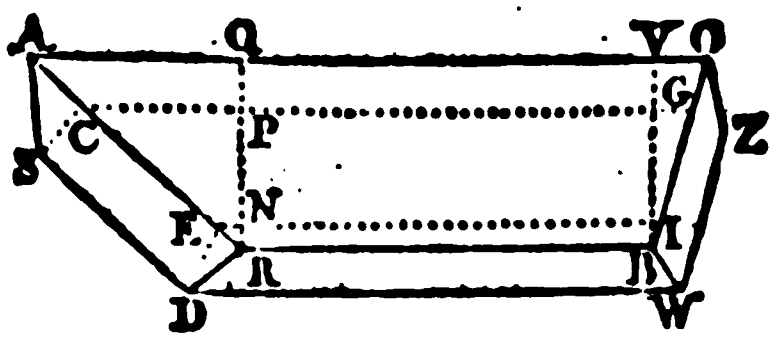
will be GD + GD + GI or 2GD + GI *multiplied by the product* AG × $\frac{1}{6}$ OS (*Rule* 2):

$$\text{or } 24 \times 51 \times \frac{5\frac{1}{2}}{6} = 204 \times 5\tfrac{1}{2} = 1122 \text{ cubic } \textit{feet}, \text{ the } \textit{Answer}.$$

Hence it appears that whatever may be the obliquity or direction of the new cut with respect to the other road, the cubic contents of the excavation will remain the same.

281. *To find the contents of the Frustum of a Wedge, or a part cut off the end opposite the edge by a plane parallel to that end.*

Solids of this kind are sometimes called Prismoids.



LET the trapezoids ARBO, and SDWZ represent the greater and less ends of the frustum, and suppose the sides ASDR, and OBWZ are perpendicular to the ends.

If the frustum is cut through S and D by the planes SCGZ, DEIW perpendicular to the ends, it will be divided into two wedges AZ and EW, and the prism CDG.

The content of the prism is the trapezoid CEIG multiplied by the height DE or SC, or $\frac{CG + EI}{2} \times PN \times DE$; RQ being perpendicular to RB and AO,

The content of the wedge DEBW is $\frac{RB + EI + DW}{3} \times \frac{RN \times ED}{2}$ (260, *Rule* 1.) or $\frac{RB + 2EI}{3} \times \frac{RN \times ED}{2}$ (because DW = EI); DW being the edge, the trapezoid EIBR the opposite end, and $\frac{RN \times ED}{2}$ the content of the triangle which is the section of the wedge perpendicular to EI, RB, DW:

And the content of the wedge AZ is $\frac{AO + 2CG}{3} \times \frac{PQ \times CS}{2}$; CG being equal to the edge whose extremities are S and Z.

Consequently those three results added together will be the content of the frustum.

Examp. 1. What is the capacity of a ditch surrounding a square Fort whose side is 100 yards, when the breadth of the ditch at top is 10 yards, at bottom 8, and depth 3, and the bottom of the inner slope ½ a yard from the perpendicular?

Here the frustum AW represents ¼ of the ditch;

And RB =	100	the inner side
AO =	120	the outer
RQ =	10	the breadth at top
DE =	3	the depth
RN =	½	
NP =	8	the breadth at bottom
PQ =	1½	
CQ =	117	
EI =	101.	

Then $\frac{117 + 101}{2} \times 8 \times 3$ = ... 2616 content of prism SG.

$\frac{RB + 2EI}{3} \times \frac{RN \times ED}{2} = \frac{302}{3} \times \frac{1\frac{1}{2}}{2}$ = ... 75½ ...... of wedge EW.

$\frac{AO + 2CG}{3} \times \frac{PQ \times CS}{2} = \frac{334}{3} \times \frac{4\frac{1}{2}}{2}$ = ... 265½ ...... of wedge AZ.

sum 2957 the frustum AW.

And 2957 × 4 = 11828 cubic *yards*, the whole excavation.

282. When the opposite faces DRBW, and SAOZ are equally inclined to the ends, RN and PQ are also equal, and the content is equal to half the sum of the ends or top and bottom multiplied by the depth. The same thing however, appears from a different consideration; for if BV be a perpendicular section parallel to RQ, then the solid VW cut off by the plane BV is equal to half a prism whose breadth is VO and depth ED, and therefore its content is the perpendicular section VB × $\frac{1}{2}$VO: In like manner the content of the solid DQ is = the section RQ × $\frac{1}{2}$AQ; therefore the frustum AW is QV + $\frac{1}{2}$VO + $\frac{1}{2}$AQ or $\frac{AO + RB}{2}$ or the length along the middle of the top or bottom, multiplied by the perpendicular section, or half the sum of the trapezoids AB and SW multiplied by the depth ED:

Viz. $\frac{120 + 100}{2} \times \frac{10 + 8}{2} \times 3 = 2970$ the frustum AW when the slopes are equal.

Examp. 2. What is the capacity of a ditch surrounding a regular pentangular Fort whose side is 150 yards: the breadth of the ditch at top being 10 yards, at bottom 8, and depth 3; supposing the slope on each side to be equal?

Let the preceding figure represent $\frac{1}{5}$ of the ditch, the planes AD, OW through the angular points A, R, B, O, being perpendicular to the top of the ditch, as in the foregoing example.

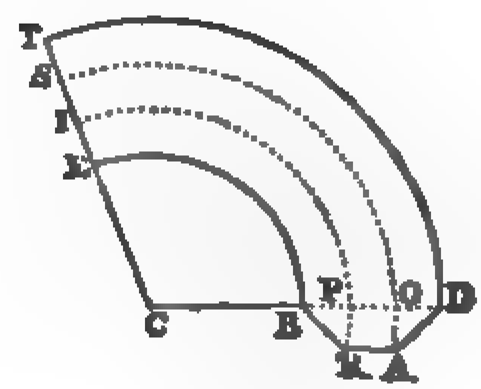
Then RQ being 10 *yards*, and the angle RAQ = 54°, we get (221) AQ = 7·2654 *yards*; therefore AO = 150 + 14·5308 = 164·5308 *yards*, the outer side of the ditch; and $\frac{10 + 8}{2} \times 3 = 27$ the perpendicular section:

Whence $\frac{164 \cdot 5308 + 150}{2} \times 27 = 4246 \cdot 1658$ *yards* the cubic contents of $\frac{1}{5}$ of the ditch; and 4246·1658 × 5 = 21230·829 the whole excavation.

[redacted]

And the solid contents of a rampart having salient angles may be found in a similar manner; for if the angles are bisected by perpendicular sections, the parts of the ramparts between those sections are readily divided into prisms and wedges.

Examp. 3. If BRAD be the perpendicular section, and ETDB the top of a ditch next the rounded angle ECB of a fortification:



And BD = 16 *yards*, the breadth at top.
RA = 13 breath at bottom.
PR or OA = 4 depth.
BP = 1
OD = 2
Angle ECB = 120°,
radius CB = 50 *yards*. What is the cubic capacity?

The ditch being the frustum of a *circular wedge*, if the circular planes RPI and AOS are perpendicular to the bottom RA, or parallel to BE, the frustum or excavation will be divided into a circular prism, and two circular wedges.

And because CBD is a right line, we have

The *radii*	CB = 50	and the arcs	BE = 104·7
	CP = 51		PI = 106·794
	CO = 64		OS = 134·016
	CD = 66		DT = 138·204 *yards*.

The content of the prism is the area ISOP multiplied by the depth PR, or the area of the bottom of the ditch multiplied by the depth, or half the sum of the arcs PI, OS multiplied by the section RPOA;

viz. $\frac{106·794 + 134·016}{2} \times 52 = 6261·06$ *yards*.

The edge of the inner wedge is equal to the arc PI, therefore the *sum* BE + 2PI *multiplied by* PR × $\frac{1}{6}$PB is the content of the wedge, (220, *Rule* 2);

or $318·288 \times 4 \times \frac{1}{6} = 212·192$ *yards*.

And the content of the outer wedge = the *sum* DT + 2OS *multiplied by* OA × $\frac{1}{6}$OD:

$$\text{or } 406{\cdot}236 \times 4 \times \tfrac{1}{3} = 541{\cdot}648 \textit{ yards.}$$
$$212{\cdot}198$$
$$6261{\cdot}06$$
$$\text{sum } \underline{7014{\cdot}9} \text{ the whole excavation.}$$

If the slopes BR, DA are equal, the content will be half the sum of the extreme arcs BE and DT multiplied by the perpendicular section BRAD:

$$\text{or } \frac{104{\cdot}7 + 138{\cdot}204}{2} \times 58 = 7044{\cdot}2 \text{ cubic } \textit{yards.}$$

In the same manner the solid contents of the circular part of the rampart are found, for it may be divided into circular prisms and wedges.

Examp. 4. How many gallons, wine measure, will a cistern contain, if its length and breadth at top are 5 and 4 *feet*, respectively, and at bottom 4 and 3 *feet*; the perpendicular depth being $3\frac{1}{4}$ *feet*?

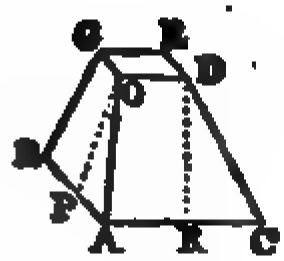
Ans. $414\frac{6}{11}$.

Examp. 5. Suppose a bank of earth 40 feet thick at bottom, 12 at top, and each of its sloping sides 18 feet; now if a road 6 feet broad at bottom and 10 at top be cut directly through the bank, what will be the content of the excavation.

Ans. 2247·7 *cubic feet.*

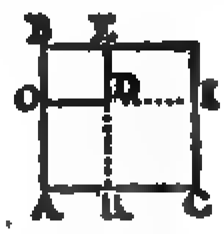
283. *To find the content of the Frustum of a Pyramid.*

Let GC be the frustum of a pyramid, the ends GD and BC being squares; also suppose the face GA is perpendicular to the ends.



If the plane EDR is parallel to the face GA it will divide the frustum into a wedge RDEC, and the frustum of a wedge RG.

The content of RG is half the sum of the opposite faces BR and DG multiplied by the height OP (282), or $\frac{AB \times AR + DE^2}{2} \times OP$; $AB \times AR$ being the face BR, and ED^2 the top DG.



Now let the square BC represent the base of the frustum, and the square BD its top; then EC is the base of the wedge.

And the content of the wedge is $2DR^2 + 3DE \times DR$ *multiplied by* $\frac{OP}{6}$ (280.) or $\frac{2DR^2 + 3DE \times DR}{6} \times OP$.

But the rectangle $AB \times AR$ is $= DE^2 + DE\,(AR) \times DR$; therefore $\frac{AB \times AR + DE^2}{2} \times OP = \frac{2DE^2 + DE \times DR}{2} \times OP$,

or $\frac{6DE^2 + 3DE \times DR}{6} \times OP$ the content of GR: and the sum of both solids.

or $\frac{6DE^2 + 6DE \times DR + 2DR^2}{6} \times OP$, or $\frac{3DE^2 + 3DE \times RD + DR^2}{3}$ $\times OP$, is the content of the frustum GC.

But the two squares BD and DC together with the two equal rectangles AD and EI or twice the rectangle AD make the square BC, or $DE^2 + 2DE \times DR + DR^2$ is the area of the base BC, and DE^2 is the area of the top GD; also $DE + DR$ is the side of the base, and DE the side of the top, and their rectangle or product is $DE^2 + DE \times DR$; now those three areas, namely, $DE^2 + 2DE \times DR + DR^2$ the base, DE^2 the top, and $DE^2 + DE \times DR$ together make $3DE^2 + 3DE$

$\times DR + DR^2$; but the product of two numbers is a mean proportional between their squares (Arith. 188, *Examp.* 7), therefore the *sum* DE + DR *multiplied by* DE is a mean proportional between the square of DE + DR and the square of DE, or a mean proportional between the ends of the frustum:

Therefore, if the two ends of the frustum be added to the mean proportional between them, and $\frac{1}{3}$ of the sum multiplied by the height, the product will be the content of the frustum.

Now it is evident (132) that the frustum GC is equal to the frustum of any other pyramid having an equal base, whatever may be its figure, provided the heights, and also the opposite ends, are respectively equal: And therefore the same rule will also give the content of the frustum of a Cone.

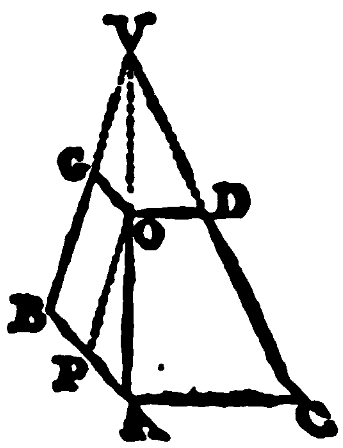
This however, may be obtained without comparing it with a frustum of a pyramid having plane sides, if we conceive the conic frustum to be composed of a cylinder and a circular wedge.

EXAMPLES.

1. Let $AC = 7$, $OD = 5$, and the height $OP = 6$:

Then $7 \times 7 = 49$ the area of the base; and $5 \times 5 = 25$ that of the top GD; and the mean proportional between 49 and 25 is the square root of 49×25 or $7 \times 5 = 35$; therefore $\frac{49 + 25 + 35}{3} \times 6 = 218$ the content.

Or the content of the frustum may be found thus:—Let V be the vertex of the pyramid when completed: then the difference of the contents of the whole pyramid BCV and the upper pyramid GDV will evidently be that of the frustum GC.



Let the face GA be perpendicular to the ends of the frustum, and OP (perpendicular to BA) its height, as above; then by similar triangles,

As the difference of the sides AB and OG, to OP, so is OG, to the height of the upper pyramid or the distance of V from the base GD; this added to OP will give the height of the whole pyramid BCV.

Suppose AC = 7, OD = 5, and OP = 6, (as before):

Then 7 — 5 : 6 :: 5 : 15 the altitude of the pyramid GDV, which added to 6 (OP) is 21 the altitude of the pyramid BCV:

Therefore 49 × $\frac{21}{3}$ = 343 the content of the pyramid BCV (133):
And 25 × $\frac{15}{3}$ = 125 that of GDV:
diff. 218 content of the frustum, as before.

2. Required the solid contents of the frustum of a triangular pyramid, the sides of the base being 6, 6, and 10; and of the top 3, 4, and 5, supposing the height 30?

Ans. 420.

3. How many cubic feet in a squared piece of Timber, the areas of the two ends being 504. and 372 *inches*, and its length $31\frac{1}{2}$ *feet*?

Ans. 95·4.

4. If the length of a tapering round piece of Timber or body of a tree be 26 *feet*, and the diameters of the ends 22, and 18 *inches*, respectively; what is the solid content?

$22^2 \times \cdot 7854 = 380 \cdot 134$ *inches*, area of greater end
$18^2 \times \cdot 7854 = 254 \cdot 47$ .... of the less;

And the square root of their product is 311·018 the mean proportional between the areas of the ends:

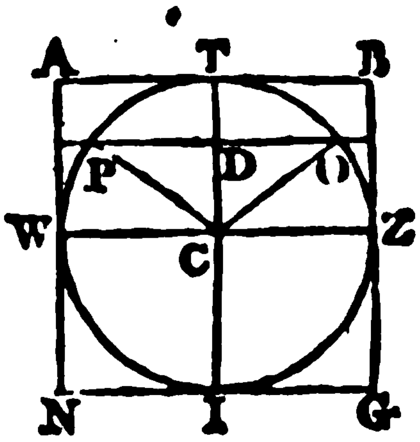
Then $\frac{254 \cdot 47 + 380 \cdot 134 + 311 \cdot 018}{3} = 315 \cdot 207$ which multiplied by 26 × 12 gives 98345 *cubic inches*, or 56·9 *feet*, the content.

3. If a cask or barrel in the form of two conic frustums joined at the greater ends, has its bung or middle diameter 18, head diameter 14, and length 30 *inches*; how many pounds of gunpowder will it contain, supposing 30 cubic inches to the pound?

Ans. 209.

284. *To find the solid content of a Globe or Sphere.*

MULTIPLY the superficies by $\frac{1}{6}$ of the diameter, and the product will be the content.



Let C be the centre of a sphere, and ABGN its circumscribing cylinder: then the diameters of the sphere and cylinder are equal; and the former is $\frac{2}{3}$ of the latter, (134).

The base of the cylinder is $\frac{NG}{2} \times \frac{circumf.}{2}$ (106, *corol.*)

And its content $\frac{NG}{2} \times \frac{circumf.}{2} \times NA$ or NG, or $\frac{NG^2 \times circumf.}{4}$;

Therefore the content of the sphere is $\frac{2}{3} \times \frac{NG^2 \times circum.}{4}$,

or $NG \times circumf. \times \frac{NG}{6}$; that is, the surface ($NG \times circum.$) *multiplied by* $\frac{1}{6}NG$.

EXAMPLES.

1. What is the solid content of a sphere whose diameter is 1?

The circumference is 3·14159 &c.

And the superficies 1 × 3·14159 &c.

Therefore the solid content is 1 × 3·14159 &c. × $\frac{1}{6}$, or 0·5236.

2. Required the content of a sphere whose diameter is 20?

Since spheres are as the cubes of their diameters (135, *corol.* 3) we have

As 1^3 : ·5236 :: 20^3 : 8000 × ·5236 = 4188·8 the *answer*.

Therefore the cube of the diameter of a globe or sphere multiplied by the decimal ·5236 gives the content.

3. The diameter of a 9*lb.* iron shot being 4 *inches* nearly, then what is the weight of a cubic inch of cast iron?

$4^3 \times \cdot 5236 = 33 \cdot 5104$ cubic inches the content;

And $\dfrac{9 \times 16}{33 \cdot 5104} = 4 \cdot 297$ *ounces* nearly, the answer.

4. If the gilding of a Globe cost 3*l.* at 6*d.* the superficial foot, what is its cubical content?

Ans. 123·6 *feet.*

5. If the Earth be a sphere 8000 miles in diameter, what is its cubic content?

Ans. 268082572800 *miles.*

283. *To find the solid content of a Segment of a Sphere.*

Let PDOT (*see the preceding figure*) be a spherical segment, its base PO being parallel to the diameter WZ, and also to the ends of the circumscribing cylinder.

Then 3·1416 × AB × DT is the convex surface of the segment (277, *corol.*).

And because the solid content of the sphere is the surface multiplied by $\frac{1}{6}$ of the diameter, therefore the content of the conical solid CPTO having the convex surface of the segment for its base, and C the vertex, will be *that* surface multiplied by $\frac{1}{6}$ of the sphere's diameter (or $\frac{1}{3}$ of the height TC), or 3·1416 $\times$ AB $\times$ DT $\times \dfrac{AB}{6}$, or $\dfrac{3 \cdot 1416}{12} \times AB^2 \times 2DT$.

And PO being the diameter of the base of the cone PCO, its area will be $\frac{3\cdot1416 \times PO^2}{4}$, therefore the content of the cone is $\frac{3\cdot1416 \times PO^2}{4} \times \frac{DC}{3}$, or $\frac{3\cdot1416}{12} \times PO^2 \times DC$.

And the difference of the cones or solids CPTO and CPO, or the *difference* $AB^2 \times 2DT - PO^2 \times DC$ *multiplied by* $\frac{3\cdot1416}{12}$ or the decimal $\cdot2618$, is the segment PTO: therefore,

Multiply the square of the sphere's diameter by twice the height of the segment,

And the square of the diameter of the segment's base by the difference between its height and the radius of the sphere;

Then the difference of the products multiplied by the decimal $\cdot2618$ is the solid content of the segment.—A shorter practical rule however, may be derived algebraically.

EXAMPLES.

1. If PO the diameter of the base is 8, and its height DT $= 2$, what is the content of the segment?

Because PD is a mean proportional between TD and DI (97, *corol.* 1) we have $DI = \frac{PD^2}{TD} = \frac{16}{2} = 8$; therefore the diameter $TI = 10$, and $DC = 3$;

$$\begin{array}{rl} \text{And } 10^2 \times 4 = & 400 \\ 8^2 \times 3 = & \underline{192} \\ \text{diff.} & \overline{\underline{208}} \end{array}$$ which multiplied by $\cdot2618$ gives $54\cdot4544$ the content of the segment PTO.

2. If the diameter $WZ = 6$, and $PO = 8$, what is the content of the frustum or zone WPOZ?

$$
\begin{aligned}
& CO^2 = 9 \\
& DO^2 = 6{\cdot}25 \\
\text{(23, corol.)}\ & DC^2 = \overline{2{\cdot}75} \text{ and } DC = 1{\cdot}6583 \\
& \qquad\qquad\qquad CT = 3 \\
& \qquad\qquad\qquad DT = \overline{1{\cdot}3417}
\end{aligned}
$$

Whence the content of the segment PTO $= 14{\cdot}437$
hemisphere WTZ $= 56{\cdot}549$
zone WPOW $= \overline{42{\cdot}112}$ diff. *Ans.*

3. If a segment 3 inches high be cut from a globe 9 inches in diameter, what is its cubic content?

Ans. 98·96 *inches.*

4. Suppose the muzzle of a 32 pounder is stopt with a 42*lb.* ball; required the content of the part within the bore, if $\frac{1}{18}$ of an inch has been allowed for windage?

Ans. 36·3 *cubic inches.*

We recommend the use of *models* for all the solids having plane sides. The planes may be cut in stiff *paste-board*; and when folded up, the edges are easily fastened together with slips of thin paper and *gum-water.*

$$\begin{array}{rl} CO^2 &= 9 \\ DO^2 &= 6\cdot25 \\ \text{(83, corol.) } DC^2 &= \overline{2\cdot75} \text{ and } DC = 1\cdot6583 \\ & \qquad\qquad CT = 3 \\ & \qquad\qquad DT = \overline{1\cdot3417} \end{array}$$

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We recommend the use of *models* for all the solids having plane sides. The planes may be cut in stiff *paste-board*; and when folded up, the edges are easily fastened together with slips of thin paper and *gum-water*.

16. If the radius of a circle be 10; what are the sides of the regular inscribed trigon, tetragon, pentagon, hexagon, octagon, and decagon?

Ans. 17·32—14·142—11·756—10—7·654—6·18, nearly.

17. A plan of a fortified town has a scale of 100 toises which is 1·6 inches in length; the plan is 30 inches long, and 24 broad; now what will be the size when it is copied to a scale of 6 inches the English mile?

Ans. 13·6 *in.* long, and 10·9 broad.

18. If the length of a pair of proportional compasses be 7 inches; how far from the ends is the centre answering to the division 3 on the line of Lines?

Ans. 1$\frac{3}{4}$ and 5$\frac{1}{4}$ *inches.*

19. Suppose the length of a pair of proportional compasses to be exactly 9 inches; how far from the ends must the centres be for enlarging or diminishing a plane surface twice, and a solid three times?

Ans. 3·728 and 5·272 *in.* in the *former case.*
3·685 and 5·315 *in.* in the *latter.*

20. If the length of a cannon be 8 *f.* 10 *in.* its diameter at the breech 19$\frac{1}{2}$ *in.* at the mouth 14$\frac{1}{7}$ *in.* at what distance would the outer surface meet the axis of the bore supposing both were produced?

Ans. 23$\frac{4}{11}$ *feet*, from the muzzle.

21. How many degrees, &c. are contained in that arc of a circle whose length is equal to the radius?

Ans. 57°·295779 nearly.

22. If the line of numbers from 1 to 10 on a *logarithmic* or *Gunter's Scale* is a foot; required the distance from 1 to 3.—And what is the distance from 10 on the line of numbers to 40° on the *line of tangents*?

Ans. 8·3876 &c. and 0·914 &c. *inches.*

[redacted]

7. In the preceding example, what is the length of the tangent to the circle drawn from the given point?

Ans. 28·284 &c. *in.*

8. To what extent on the surface of the sea (exclusive of the effect of refraction) can a person see from the top-mast-head of a man of war, his height above the water being 30 *yards*, and the earth's diameter 7960 *miles*?

Ans. 11·6 *miles*, nearly.

9. If a line 10 inches long be cut according to mean and extreme proportion; what are the lengths of the two parts?

Ans. 6·18 and 3·82 *in.* nearly.

10. If the base of a triangle be 40, and the other two sides 30 and 20; what is the length of its perpendicular?

Ans. 14·52 &c.

11. If the base of a triangle be 40, and the two sides 30 and 20; what are the segments of the base made by a line bisecting the vertical angle?

Ans. 24 and 16.

12. If the diameter of a circle be 30; what is the side of the inscribed equilateral triangle?

Ans. 25·98 nearly.

13. If the side of an equilateral triangle be 10; what are the radii of the inscribed, and circumscribing circles?

Ans. 2·8868 and 5·7736 nearly.

14. The side of a square being 10; then what is the radius of its circumscribing circle?

Ans. 7·071 &c.

15. If the side of a regular pentagon be 10; what are the radii of its inscribed, and circumscribing circles?

Ans. 6·882 and 8·506 nearly.

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Ans. 8·3876 &c. and

23. The length of a line of chords of 90° being 4½ inches; [t]hen what is the length of 45° on the same line?

Ans. 2·3 in. nearly.

24. If the radius of a circle be 20; what are the lengths [o]f the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* [o]f 30°?

Ans. 10—17·32—11·547—34·641—23·094—40.

25. If the base of a right-angled triangle be 4, and the per[p]endicular 3: what are the lengths of the *sine*, *cosine*, *tangent*, [a]nd *cotangent* of the least angle, if the radius be 1?

Ans. 0·6 — 0·8 — 0·75 — 1·333 &c.

26. If the base of a right-angled triangle be 0·28, and the [a]djacent acute angle 59° 11′; what are the other sides?

Ans. 0·5466, and 0·4694.

27. The base of a right-angled triangle being 74·7 *yards*, [a]nd its opposite angle 21° 13′; what are the other sides?

Ans. 192·4, and 206·4 *yds.*

28. The hypotenuse of a right-angled triangle being 5479 *feet*, and one of the acute angles 29° 51′; then what are the other sides?

Ans. 4746 and 2723·5 *feet.*

29. If the three angles of a plane triangle are 106° 41′, 46° 24′, and 26° 55′, and the side opposite the greatest angle = 297·6 *yds.* then what are the other sides?

Ans. 225, and 140·7 *yards.*

30. Suppose the angles of a plane triangle to be as in the preceding example, and the side opposite the least angle 297·6 *feet*; required the other sides?

Ans. 476·1 and 629·7 *feet.*

31. The hypotenuse of a right-angled triangle being

14 *f.* 10 *in.* and the base 10 *f.* 7 *in.* then what is the perpendicular?

Ans. 10 *f.* 4·7 *in.*

32. Two sides of a triangle being 311 and 397 yards, and the angle opposite the greater of those sides = 38° 33′; then what is the third side?

Ans. 589·7 *yds.*

33. Suppose two sides of a triangle are 311 and 221 yards, and the angle opposite the least of those sides is 38° 33′; required the third side?

Ans. 349·4, or 137·04 *yds.*

34. If two sides of a triangle are 179·8 and 121·6 feet, and the included angle 79° 51′; what is the third side?

Ans. 198·5 *feet.*

35. The base and perpendicular of a right-angled triangle being 1139, and 1074 yards; required the acute angles, and hypotenuse?

Ans. 43° 19′ — 46° 41′ — hypot. = 1565·5 *yds.*

36. If an angle of a triangle be 129° 34′; and the ratio of the including sides as 4 to 7; what are the other two angles?

Ans. 32° 32′ 7″—17° 53′ 53″.

37. How many inches subtend an angle of 1″ at the distance of 7 miles?

Ans. 2·1 nearly.

38. Suppose the sides of a triangle are 14272, 13141, and 11799 yards; required the angles?

Ans. 69° 34′½ — 59° 38′½ — 50° 47′.

39. If the sides of a triangle have the proportion of ½, ⅓, and ¼; what are the angles?

Ans. 117° 16′ 46″ — 36° 20′ 10″ — 26° 23′ 4″.

40. Let the base of a right-angled triangle be 30, and the ratio of the other two sides as 1 to 2; what are those sides?

Ans. 17·32, and 34·64 nearly.

41. If the hypotenuse of a right-angled triangle be 100, and the other sides as 1 to 2; what are those sides?

Ans. 44·72 and 89·44 nearly.

42. If the hypotenuse of a right-angled triangle be 40, and the sum of the other two sides 50; what are those sides?

Ans. 11·771 — 38·229 nearly.

43. Suppose the hypotenuse of a right-angled triangle to be 40, and the difference of the other sides 10; required the sides?

Ans. 22·839 — 32·839 nearly.

44. If the base of a right-angled triangle be 40, and the sum of the other sides 80; what is the perpendicular?

Ans. 30.

45. If the perpendicular of a right-angled triangle be 40, and the difference of the other sides 10; what are those sides?

Ans. 75 and 85.

46. Suppose a regular pentagon whose side is 170 fathoms, to be fortified; and that the salient angle of the bastion is 71°, and its face 47 fathoms; required the flank, and curtain, supposing the line of defence is perpendicular to the flank?

Ans. Flank 23·63
Curtain 64·57.

47. If a square whose side is 170 fathoms is regularly fortified, and the salient angle of the bastion 61°; what are the principal dimensions if the length of the face of the bastion, is to that of the flank, as 7 to 3; the line of defence being perpendicular to the flank?

Ans. Face of bastion 46·6—Flank 20—Curtain 69·8.

48. If at the top of a mountain the true depression of the horizon of the sea is found to be 1° 31′; what is the mountain's height, supposing the earth a sphere whose diameter is 8000 miles?

Ans. 1·4 *miles*, nearly.

49. In surveying with a compass an object bore NE 50°; and when we had gone 170 paces in a SE 55° direction, its bearing was NE 6°. Required its distance from each station?

Ans. 214, and 237 *paces*.

50. Wanting to know the breadth of a river, we measured a straight base of 30 chains along the bank, and at its extremities took the horizontal angles 64° 11′, and 78° 38′ to an object on the opposite shore. Hence the breath is required?

Ans. 964 *yards*.

51. From the top of a hill I observed two mile stones in the same direction on level ground; the depression of the nearest was 14° 3′; and that of the other 3° 56′ below the horizontal line: hence the height of the hill is required?

Ans. 301 *feet*.

52. Having observed the elevation, of an object on the top of a distant hill, and found it 2° 27′, we measured a base of 520 yards on sloping ground directly towards the object, and at that end the object was elevated 3° 4′. Now the farthest extremity of the base was found to be 10 *feet*, higher than the other. Hence the height, and distance of the hill are required?

Ans. Height above the lowest end of the base 127 *yds*.
Distance from that end 2371 *yds*.

53. To find the height, and distance of an object on the top of a hill, we measured a base of 470 yards on sloping

ground which was inclined to the horizon in an angle of 4° 44′; and then observed the horizontal angles between the base and object at the lower and upper ends of the base, and found them to be 91° 12′, and 72° 37′, respectively; also at the lower end of the base, the object was elevated 4° 3′. Hence the height and distance of the hill are required?

Ans. Horizontal dist. from the lower end of the base 1640 *yds.*
Height above that end 116 *yds.*

54. At the top and bottom of a tower which stood on a hill near the sea shore, we observed the depressions of a ship at anchor to be 1° 39′, and 1° 9′, respectively: hence the height of the hill, and also its distance from the vessel are required; the tower itself being 72 feet high?

Ans. Bottom of tower above the sea 166 *feet.*
Horizontal distance of ship 8246.

55. To obtain the height, and distance of an object on the summit of a hill I measured a base of 450 *yards* on level ground, and set up marks at its extremities equal to the height of the eye. At one end of the base the angle between the other end and the object was found with a *sextant* to be 74° 35′; and at the other end 77° 41′ where the elevation of the object was observed = 6° 29′. Hence the height of the hill, and its distance from each extremity o the base are required?

Ans. Height of the hill 105·3 *yds.*
Distances 926·2.
938·8.

56. In surveying with a *compass*, a spire bore NE 18°, distant 2 miles; and the bearing of a wind-mill was NW 20°, now the distance of the wind-mill from the spire was known to be 1½ miles: hence its distance from the station is required?

Ans. 2395, or 3153 *yards.*

57. A ladder 28 feet long will reach from one side of a ditch which is 20 feet broad, to the top of a wall on the other side: what is the height of the wall?

Ans. 19·6 *feet.*

58. From the top of a work 15 feet high, a point-blank shot struck an object on the ground at the horizontal distance of 120 yards. What was the depression of the piece?

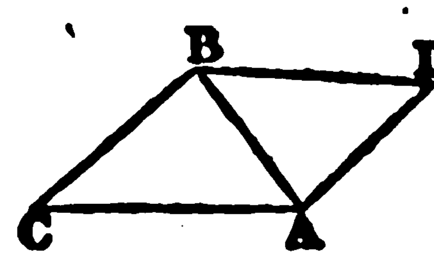
Ans. 2° 23′.

59. Two forts commanding the mouth of a harbour bore SE 16°, and SW 24°$\frac{1}{2}$, distant 1$\frac{3}{4}$ and 2$\frac{1}{4}$ miles, respectively: required the distance from one to the other, and also their bearing?

Ans. Distance 2870 *yds.*
Bearing 68° 41′ NE and SW.

60. At the extremities of the base AB of 40 *chains*, we took the following angles with a theodolite to the elevated objects C and D:

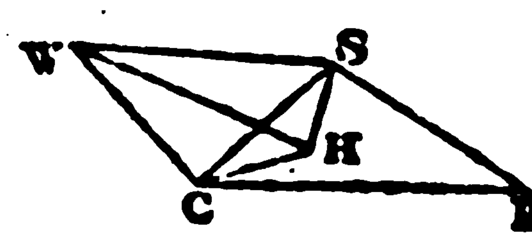
At A $\begin{cases} CAB = 51° 6′ \\ DAB = 83 \;\; 5 \\ C \textit{ elevated } 4° \; 17′ \\ D \textit{ elevated } 3 \;\; 8 \end{cases}$ At B $\begin{cases} CBA = 90° 56′ \\ DBA = 48 \;\; 3 \end{cases}$



Hence the distance from C to D; and also their heights are required?

Ans. Dist ...... 2129 *yds.*
Height of C 107·1
of D 47·6

61. Let W be West Wycombe church, H High Wycombe church, and P Penn beacon-pole: Now at the stations C and S we took the following angles with a theodolite.



WCS = 108° 14′		WSC = 42° 42′	
SCH = 28 20	at S	CSH = 25 26	
SCP = 33 51.		CSP = 126 20.	

evious operation the distance WH (between the was found to be 4646 *yards*. Hence the distance beacon to West Wycombe church is required?

Ans. 9144 *yds.*

n reconnoitring a county by the help of a map, we d two spires A and B in the same direction, A being est; we then observed the angle subtended by A and a pire C and found it 41° 52′: now the distance of A and asured on the scale to the map, was 3640 *yards*, of A 4280, and of B and C 5460. Required the distances to ires A and C?

Ans. From A 4527 *yds.*
From C 6403.

3. In surveying with a *pocket-sextant* I observed the angle tended by two churches A and B = 45° 30′, and that been A and another church C = 25° 40′, all in the horiontal plane nearly: The distance from A to B was $2\frac{1}{2}$, from to C $2\frac{1}{2}$, and from B to C $4\frac{1}{4}$ miles, the church A being the earest. Hence the place of observation is required?

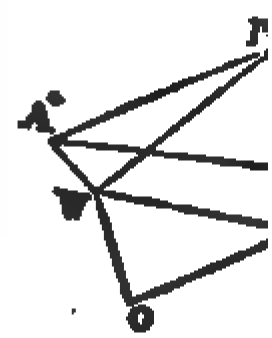
Ans. 3314 *yards* from A.
5500 .... from B.
7146 .... from C.

64. From the top of the tower A we observed the angle BAW between the wind-mill W and the spire B, and found

it 23°; but at W the tower A could not be seen from the ground, we therefore took the angle BWC subtended by the spires B and C, which was 123¼°. Now the three distances AB, AC, BC were known to be 5430, 4600, 4850 *yards*, respectively. Hence the situation of the wind-mill is required?

Ans. Wind-mill from A 3
from B 2
from C 2

65. The distance (WE) of the stations W, E, and situation of the object O became necessary in carryi survey: now A, B, and C, were three known objects, the distances being AC = 4060, AB = 3200, and CB = 1840 *yards*; but at the station W the object C could not be seen; and an intervening height hid the object A at the other station E; we therefore set up marks at W and E and took the following angles: namely,



At W $\begin{cases} AWB = 96° \; 10' \\ BWE = 48 \;\; 30 \\ OWE = 38 \;\; 44 \end{cases}$ At E $\begin{cases} BEC = 50° \; 4' \\ BEW = 70 \;\; 56 \\ WEO = 88 \;\; 50. \end{cases}$

Hence WE, EO, and WO are required?

Ans. WE = 2697 *yds.*
EO = 2306
WO = 1463

66. In walking along a straight road directly west, I observed two spires A and B both bearing NE 22½°, the nearest being A; an hour afterwards a third spire C and the spire B appeared in one direction; and the next hour brought C and A in a right line; the distance of A from B (on a

map) was $1\frac{3}{4}$ miles, of A from C 2 miles; and that of B from C $2\frac{1}{4}$ miles. How far did I walk *per* hour, supposing the rate equable?

Ans. 7867 *yds.* the whole distance walked, or $3933\frac{1}{2}$ *per* hour.

67. The base of a parallelogram being 61, and its perpendicular $37\frac{1}{2}$ *feet*; what is the content in yards square?

Ans. $254\frac{1}{6}$.

68. The length and breadth of a rectangular field are 13 *chains*, 61 *links*, and 11 *ch.* 9 *lin.* Required the content in *acres*?

Ans. 15·12676.

69. The parallel sides of a trapezoid are 37 *f.* 10 *in.* and 16 *f.* 8 *in.* and their perpendicular distance 11 *f.* 6 *in.* What is the area?

Ans. $313\frac{3}{8}$ *feet.*

70. If the base of a triangle be $17\frac{1}{4}$ yards, and its perpendicular $11\frac{1}{7}$ yards; what is the area in *feet*?

Ans. $862\frac{1}{3}\frac{6}{7}$.

71. If the side of a rhombus is $29\frac{1}{4}$ feet, and the acute angle 62°; what is the content in *yards*?

Ans. 85·38 nearly.

72. The sides of a triangular field being 171, 161, and 145 yards; then what is the area in *acres*?

Ans. 2·2527 nearly.

73. The sides of a quadrangular field being successively 26, 20, 16, and 10 poles, and the angle (taken with a theodolite) included by the two longest sides = 56°. Required its content?

Ans. 287·676 *poles*, or 1 *ac.* 127·676 *pol.*

74. The breadth of a ditch at top being 72, at bottom $38\frac{3}{5}$,

the sloping sides $26\frac{1}{4}$ and 20 feet, and the top and bottom of the ditch horizontal. Required the area of the perpendicular section?

Ans. $883\frac{1}{3}$ *feet.*

75. The area of the perpendicular section of a ditch being 135 feet, the breadth at top 30, and at bottom 15 feet. What is the depth?

Ans. 6 *feet.*

76. If the area of the perpendicular section of a ditch be 154 feet, its depth $5\frac{1}{2}$ feet, and the breadth at top, to that of the bottom, as 9 to 5: what are those breadths?

Ans. 36 and 20 *feet.*

77. The area of a right-angled triangle being 605, and the ratio of the base to the perpendicular as 2 to 5: what are those sides?

Ans. 22 and 55.

78. What is the side of that equilateral triangle whose area is 100?

Ans. 15·197 nearly.

79. If the side of an equilateral triangle be 10; what will be the side of another equilateral triangle whose area is *one-fourth* of the former?

Ans. 5.

80. If the area of a triangle is 1000, and the sides are in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$; what are those sides?

$$\textit{Ans.}\ \left.\begin{array}{r} 40{\cdot}074 \\ 50{\cdot}093 \\ 66{\cdot}791 \end{array}\right\} \text{nearly.}$$

81. If the hypotenuse of a right-angled triangle be 17, and its area 60; what are the base and perpendicular?

Ans. 8 and 15.

82. How many acres would be contained within the boundaries of the pentangular fortification, *Examp*. 46, supposing it completed?

Ans. 126089 *yds.* or 26·051, &c. *acres.*

83. If the equal sides of an isosceles triangle are each 17, and its area 120; what is the base?

Ans. 16.

84. If the diameters of two concentric circles are 20 and 30; what is the content of the *annulus* or space contained by the circumferences?

Ans. 392·7.

85. If the area of a circle be 100; what is the area of its inscribed square?

Ans. 63·66 nearly.

86. If the base and perpendicular of a right-angled triangle are each 1; what is the area of a circle having the hypotenuse for its diameter?

Ans. 1·5708 nearly.

87. If the circumference of a circle be 1000; what is its area?

Ans. 59577.

88. If the area of the sector of a circle be 100, and the length of its arc 20; what is the angle of the sector?

Ans. 114° 35·5′ nearly.

89. If the centre of a circle whose diameter is 20, is in the circumference of another circle whose diameter is 40; what are the areas of the three included spaces?

Ans. 173·652.
140·308.
1116·332.

90. How many square feet of board are required to m[ake] a rectangular box whose length shall be 3½ feet, breadth 2 f[eet] and depth 20 *inches*?

Ans. 32⅓

91. What quantity of canvas is necessary for a conical ten[t] whose height is 8 *feet*, and the diameter at bottom 13 *feet*?

Ans. 210½ *feet* square.

92. What would a circular reservoir whose diameter at top is 40 *yards*, at bottom 38⅔ *yards*, and the side or slant depth 11 *feet*, cost the lining with brick-work at 3*s*. 10*d*. the square yard?

Ans. 311*l*. 10*s*. 2*d*.

93. The inside of an hemispherical dome cost 100*l*. the gilding at 8*d*. the foot; what was its diameter?

Ans. 43·7 *feet*.

94. If the diameter of a globe be 8 inches; what is the diameter of another globe three times as big?

Ans. 11·538 *in.* nearly.

95. If the area of the perpendicular section of a rivulet is 4½ feet, and the velocity of the water 30 feet per minute; how much would it supply in 24 hours?

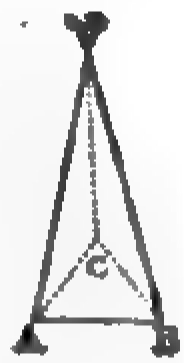
Ans. 1454213 *gall.* wine measure.

96. Suppose a sack when laid flat is 2 feet broad, and 3 feet long; how many gallons, dry-measure, will it contain if it has a circular bottom, and 9 inches is left for tying the top?

Ans. 34·8 *gall.* nearly.

97. The outer and inner circumferences of the ring of an anchor being respectively 30 and 25 inches; what is its weight, supposing 3·81648 cubic inches of iron weigh a pound avoirdupois?

98. Suppose the triangle BCA is the base of a pyramid, V its vertex, the side BC = 80 *feet*, 10 *inches*, and the angles



VAC = 20° 30'	VAB = 60° 18	VBC = 16° 4'
VCA = 140 6	VBA = 73 44	VCB = 149 10

What is the cubic content?

Ans. 1709·8563 *cubic feet*, nearly.

99. If a cask which is two equal conic frustums joined together at the bases, has its bung diameter 34, head diameter 27, and depth 50 inches; how many gallons, ale measure, will it contain?

Ans. 130, nearly.

100. What is the difference between a bushel, *running measure*, when measured with a Winchester bushel which is 18½ inches in diameter, and measured with another bushel only 12 inches in diameter, supposing the *cop* or *cap* or conical part is ½ of the diameter in height?

Answer. The buyer loses 301 cubic inches, or upwards of a gallon in every bushel by the narrowest measure.

101. If a piece of squared timber be 25 feet long, the side of the greater end 20 inches, and that of the less 16; what length must be cut off the less end to make 10 cubic feet?

Ans. 5 *f.* 4 *in.*

102. If the depth of a vessel in the form of a conic frustum, be 16 inches, and the top and bottom diameters in the proportion 5 to 3; what are those diameters, supposing the vessel holds 20 wine gallons?

Ans. 23·722, and 14·233 *inches.*

103. Suppose the following are the dimensions of the bed of a waggon,

viz. length ....
depth ....
breadth at top behind ....
——— at bottom ....
breadth at top in front ....
——— at bottom ....

How many bushels, dry measure, will it c
Ans. 405 *gall.* or 5

104. If the salient angle of a bastion be 71°
faces 50 fathoms: required the number of cub
part of the rampart next the faces, supposing
Examp. 2, is the profile or section perpendicular
angle of the shoulder?

105. Suppose the breadth of a circular ditch
bottom 19½, the outer slope 10, and inner slope
tively; required its capacity in cubic yards; th
inner circle or edge of the ditch being 600 feet,
bottom of the ditch horizontal?

END OF THE FIRST VOLUME

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Errata in Vol. I.

Page	Line	*for*	*read*
63	19	$\frac{3}{4}$	$\frac{43}{8}$
107	10	·03802	·03082
112	23	4	·4
118	9	$\frac{19}{2}$	$\frac{16}{2}$
185	14	*ae*	*are*
233	5	but RO[a]	but RC[a]
237	16	with same	with the same
244	24	*castramentation*	*castrametation*
246	6 from bott.	ORC	ORA

In the Logarithms.

Log. of 6241 *for* 4254 *r.* 5254
of 6481 *for* 0612 *r.* 1612
Log. tang. 18° 56′ *for* 9·3 &c. *r.* 9·5 &c.
cosine 22° 15′ *for* 969 &c. *r.* 966 &c.
cotang. 45° *for* 1· *r.* 10·

Errata in Vol. II.

Page	Line	*for*	*read.*
7	13	$d^2 + 8d$	$c^2 + 8c$
64 examp. 3.		$x^2 \quad 1$	$x^2 + 1$
75	1	$l(a\sqrt{c})^{\frac{1}{2}}$	$b(a-\sqrt{c})^{\frac{1}{2}}$
77	3	$(a-)^{\frac{1}{2}}$	$(a-x)^{\frac{1}{2}}$
79	2	$-\frac{1}{10}\sqrt{6}$	$=\frac{1}{10}\sqrt{6}$
	12	$\times z^2$	$+z^4$
96	20	$\frac{1}{2}a^2$	$\frac{1}{4}a^2$
100	20	$2-3$	$z-3$
203	20	$-\frac{1}{5}a^2b^3+\frac{1}{10}ab^4$	$+\frac{1}{5}a^2b^3+\frac{1}{10}ab^4-b^5$
204	1	$+a^3b^2$	$+50a^3b^2$
	14	$\frac{1}{3}xy$	$\frac{1}{2}xy$
	18	$\frac{2}{11}xy$	$\frac{9}{11}xy$
205	4	$9x$	$9x^2$
206	5	$10a$	$15a$
207	4	$5ax$	$5ax^2$
209	14	$\frac{z}{a}+\frac{z^2}{a^2}$	$z+\frac{z^2}{x^2}$ in the *Ans.*
	25	$-\frac{1}{4}xy$	$+\frac{1}{4}xy$
237	12	$-$AD	$=$AD
270	1	*Every* &c.	*Every circumscribing parallelogram having its sides parallel to two conjugate diameters is equal, &c.*
347 bott. line		of $\frac{1}{2}m^2$ &c.	of $\frac{1}{2}m^2$, or m is the least possible, &c.

A

COURSE

OF

MATHEMATICS,

DESIGNED FOR THE USE

OF THE

OFFICERS AND CADETS,

OF THE

ROYAL MILITARY COLLEGE.

By ISAAC DALBY,

Professor of Mathematics in the said College.

VOL. II.

LONDON:

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NO. 25, HATTON GARDEN.

1806.

PREFACE TO VOL. II.

THE subjects composing this Volume are *Algebra, Conic Sections, Mechanics, Hydrostatics,* and *Pneumatics.* But the particulars are enumerated in the table of contents which renders a detail in this place unnecessary. We therefore have little more than a few detached observations to offer by way of preface.

After Fractions, the arrangement in Algebra is not exactly similar to that usually found in more extensive treatises. Some reasons are given for a particular deviation (*p.* 82). And it is from considerable experience in teaching that we were induced to prefer the order in which the several rules or parts follow one another. Learners however, generally consider the management of radical quantities as the most difficult task in Algebra: a master therefore may sometimes perceive the necessity of bringing a student forward to a particular extent in Quadratic Equations before he enters upon Surds.

The examples of extracting roots, Art. 109, may not be thought sufficient to illustrate the general rule; but the algebraist will remember that the two right hand figures of the dividends in extracting the cube root, three in extracting the 4th root, and the four right hand figures in those of the 5th root, &c. are always neglected in making the division. But the

method derived from Dr. Halley's formula is preferable to all others.

After the Algebra was drawn up, we found an investigation of the Binomial Theorem in M. Dubourguet's Algebra by means of the same artifice as that in the work quoted p. 149; the author has extended the demonstration to fractional and negative indices.

The different series to which Art. 172—181 are an introduction, may be reckoned among the speculative parts of mathematics. The principal theorem in the Arithmetic of Infinites however, is deduced from the Differential Method, (Art. 179); the application of this formula has been of considerable use in the subsequent part of the volume.

In treating of the Conic Sections, the fundamental property or the equation of each curve is derived from the solid: afterwards they are considered in plano; and as the expressions for the ellipse and hyperbola differ in nothing but the signs + and —, the same demonstration frequently answers for both sections by only changing those signs; for which reason the enunciations of some properties of the hyperbola are thought sufficient.

That part of Mechanics which relates to the Centre of Gravity is given at some length on account of its extensive use. In Art. 386, 387, 388, different methods of computing the thickness of walls or revetments are compared. The results, as might be expected from different hypotheses, vary considerably. By adopting the method however, in the first work refered to (p. 383, we evidently are led to the following conclusion (p. 386) which is correct only in the case of a fluid,—namely, that the lateral pressure of a body of loose earth depends on its height without any regard to thickness.—But as all the computations are founded upon uncertain data, no correction of principle is attempted: *and the only alteration* is that of giving a more convenient form

to M. Belidor's solution, which, as it nearly agrees with the practice of Vauban, seems the least liable to exception. All theories however, respecting the strength of walls, and also that of *timber*, must necessarily be imperfect. On the latter subject, see an account of the very extensive and laborious experiments of M. de Buffon in *Mem. Acad. des Sciences*, 1740.

The speculative mechanician therefore will seldom find an exact agreement between his conclusions and the results from experiment; particularly in what relates to the working of machinery, because no theory of Friction has yet been discovered by which its effects can be calculated; for that reason the subject is not considered in the following pages.

As a work of this kind must unavoidably consist of abridgements, considerable care was bestowed in selecting what appeared the most useful to students who have not an opportunity of perusing the separate and more diffuse treatises on the different subjects. Some new solutions are introduced: but the mathematical reader cannot expect much new matter in any form.

To conclude.—The experience of two or three years proves that it will not be necessary to extend the Course beyond this Volume for the use of the College. Those Officers or Cadets who may gain a thorough knowledge of the principal matters contained in both volumes during their stay, and are inclined to continue the study of mathematics after quitting the Institution, will consult books professedly written on the higher branches, and pursue their researches without the assistance of a master.

High Wycomb,
June 6, 1803.

[redacted]

CONTENTS OF VOL. II.

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is a science which treats of the properties of numbers in general, by means of the numeral figures and other symbols; it is therefore called *Universal Arithmetic*: and sometimes *Analysis*, or, the *Analytic Art*.

2. In the operations by common Arithmetic we employ the numeral figures only, but in Algebra it is usual to represent quantities of every kind, both known and unknown, by letters of the Alphabet; and in this consists its peculiar excellence, because the reasoning is carried on with the letters or symbols, whose values are to be found, in the same manner as with those which denote given quantities.

3. The initial letters of the alphabet, a, b, c, d, &c. are commonly put for given quantities; and the final letters, x, y, z, v, &c. for those that are unknown, or required.

4. The leading marks or characters of abbreviation are given in the Arithmetic; but a more particular explanation will be necessary.

+ *plus*, signifies *addition*: this is called the *affirmative* or *positive* sign.

− *minus*, signifies *subtraction*; the *negative* sign.

× *(into)*, signifies *multiplication*.

÷ *division*.

= *equal to*, the mark for *equality*.

Thus $a + b$ is read *a* plus *b*, and denotes the sum of *a* and *b*.
Let $a = 8$, $b = 3$, and $c = 11$:
then $a + b = c$ (*a* plus *b* equal to 11) or $8 + 3 = 11$.

$a - b$ is read *a* minus *b*, and shews that the quantity represented by *b* is to be subtracted from that denoted by *a*.
If $a = 8$, $b = 3$, and $d = 5$:
then $a - b = d$ (*a* minus *b* is equal to *d*) or $8 - 3 = 5$.

$a \times b$ (*a* into *b*) denotes the product of *a* and *b*.
If $a = 8$, $b = 3$, and $c = 24$:
then $a \times b = c$, or $8 \times 3 = 24$. Here *a* and *b* are the factors of the product *ab*.

But simple factors, as *a*, *b*, *c*, &c. are usually placed without any mark or sign between them, to denote their product: sometimes however, a full stop is used:

Thus $a \times b$, or ab, or $a.b$, signify the product of *b* and *a*.
And $4abc$ the continued product of the factors 4, *a*, *b*, *c*.
If $a = 2$, $b = 3$, and $c = 5$; then $4abc = 4 \times 2 \times 3 \times 5 = 120$.

$a \div b$ shews that *a* is to be divided by *b*. But the usual form of setting down the quotient is that of a vulgar fraction; thus $\frac{a}{b}$, which denotes the quotient of *a* divided by *b* (37. Arith.)

Also $\frac{a+b}{a-c}$ denotes the quotient of $a+b$ divided by $a-c$.

Let $a=4$, $b=3$, and $c=2$; then $a \div b$, or $\frac{a}{b} = \frac{4}{3}$.

And $\frac{a+b}{a-c} = \frac{4+3}{4-2} = \frac{7}{2}$.

5. An Equation is known by the symbol $=$ (equal to):
Thus $x=a+b$ is an equation, shewing that x is equal to the sum of a and b.

6. The character $\backsim$ denotes the *difference* of two quantities when it is not known which is the greatest:
Thus, $a \backsim b$ denotes the difference of a and b.

7. Proportions are set down as in Arithmetic (92. Arith.)
Thus $a : b :: c : d$, are read, as a is to b, so is c to d.

8. $>$ denotes *greater*; and $<$ *less*:
Thus $a > b$ signifies that a is greater than b:
and $c < a$, that c is less than a.

9. A bar or line drawn over several quantities, denotes that they are to be taken collectively; this is called a *vinculum*: a parenthesis or brackets are also used for the same purpose:

$$\text{Thus } \overline{a+b} \times \overline{c-d}$$
$$\overline{a+b} \cdot \overline{c-d}$$
$$(a+b) \times (c-d)$$
$$(a+b) \cdot (c-d)$$
$$(a+b)\ (c-d),$$
all denote the same thing; namely, that the sum of a and b is to be multiplied by the difference $c-d$.

And $(a+b)-(-d+c)$ denotes that $-d+c$ is to be subtracted from $a+b$.

10. The *Co-efficient* of any term is the number or known quantity prefixed to it, and denotes how many times it is taken:

Thus 4 is the co-efficient in the quantity $4a$.
And in the quantity $3bx$, if b is given, and x is unknown, then $3b$ is the co-efficient of the unknown quantity x.
Also in $10\,(a+b)$, the co-efficient of $a+b$ is 10.
And 1 is the co-efficient of a, for $1 \times a$ is a.
Also $\frac{1}{4}$ is the co-efficient of $\frac{a}{4}$, for $\frac{1}{4}a$ is the same as $\frac{a}{4}$.

11. *Powers* are denoted by a small figure, called the index or exponent (111. Arith.)

Thus a^2 (the square of a) is the same as $a \times a$, or aa.
a^3 the same as $a \times a \times a$, or aaa.
a^n denotes a raised to the nth. power.
$(b+c-d)^5$ denotes the 5th. power of the compound quantity $b+c-d$: and $(b+c-d)^n$ its nth. power.

12. $\surd$ is the *radical* sign, and shews that the root is to be taken.

Thus $\sqrt{81}$ is 9, viz. the square root of 81 is 9.
$\sqrt[3]{27} = 3$, or the cube root of 27 is 3.
$\sqrt[4]{a^4} = a$, or the 4th root of a^4 is a.
$\sqrt[3]{(a+b)}$ denotes the cube or 3d. root of $a+b$.
$\sqrt[n]{(a^2+b^2)}$, the nth root of a^2+b^2.

But fractional indices are rather more commodious;

Thus $(a+b)^{\frac{1}{2}}$ is the same as $\sqrt{(a+b)}$, each denoting the square root of $a+b$.
$(a+b)^{\frac{1}{3}}$ is the same as $\sqrt[3]{(a+b)}$.
$(a^2+b^2)^{\frac{1}{n}}$, the same as $\sqrt[n]{(a^2+b^2)}$.

13. A *rational quantity* is that which has no radical sign ($\surd$) or index annexed to it, as b, or $3ca$.

14. A *surd* or *irrational quantity* is that which has not an exact root, as $\sqrt{3}$, or $\sqrt{a^3}$, or $(a+b)^{\frac{1}{3}}$.

15. The *reciprocal* of any quantity is 1 divided by that quantity:

Thus, the reciprocal of b is $\frac{1}{b}$.

and the reciprocal of $\frac{a}{c}$ is $\frac{c}{a}$.

16. *Like* or similar quantities are such as differ only in their co-efficients, as $5a$ and a, or $2bc^2x$ and $-3bc^2x$.

17. *Unlike* quantities are those which consist of different letters, or different powers, as $2b$ and a, or $-3a$ and $4a^2$, or $5ab$ and $5ab^2$.

18. *Like signs* are all affirmative ($+$), or all negative ($-$):

Thus, a, b, $a+c$, $4a^2$, are all affirmative, or supposed to have the sign $+$; these are called positive quantities. And $-2ax$, $-3b$, $-a$, have all the negative sign; these are called negative quantities.

19. *Unlike signs* are when some are positive and some negative, as $-4ab+3bc$, or $3x-4b$.

20. *Simple quantities* are those which consist of one term only, as $3ab$, or $\frac{4a}{b}$, or $-7ac^2$.

21. *Compound quantities* consist of several terms, as $a+b$, or $3b-2a$, or $(b+c)\times(d-a)$.

22. A *Binomial* consists of two terms; a *trinomial* of three; a *Quadrinomial* of four, &c.

Thus $a+b$ is binomial:

$a-b+c$, a trinomial:

$2a-2x+c-4d$, a quadrinomial.

23. A *Residual quantity* is a binomial having one of its terms negative, as $2a-c$.

24. *Composite quantities* are those produced by the multiplication of two or more terms called its *factors:*

Thus $3abc$ is a composite quantity, its factors being 3, a, b, c.

And $3abc$ is the common multiple of 3, a, b, c.

25. *Given quantities* are those whose values are known.

26. *Unknown quantities* are those whose values are unknown, or required.

N. B. By *quantities*, in these Definitions, we understand such magnitudes as can be represented by numbers.

27. *Examples* illustrating the method of representing, or combining numbers or quantities algebraically.

Let $a=5$ $\quad c=1$

$b=4$ $\quad d=0$.

Then $4a^2-6abc+ab^2=100-120+80=60$.

$(a^2-b^2)\times(a-b)=\overline{25-16}\times\overline{5-4}=9\times 1=9$.

$7(b-c)\,(2a-2bc)=21\times 2=42$.

$7(b-c)\times 2a-2bc=21\times 10-8=210-8=202$.

$\frac{b^3-a^2}{b+d}\times(c-d)+\frac{b}{a}=\frac{64-25}{4+0}\times 1+\frac{4}{5}=9\frac{3}{4}+\frac{4}{5}=10\frac{11}{20}$.

$(a-b\times c^2+d^2)^3=1$.

$\frac{a^2b^2}{ab}\times cd+\frac{a}{b}\times\frac{b}{a}=\frac{400}{20}\times 0+\frac{5}{4}\times\frac{4}{5}=0+\frac{20}{20}=1$.

$(a^2-b^2)^{\frac{1}{2}}\times(a+b)=(25-16)^{\frac{1}{2}}\times 9=3\times 9=27$.

$(a^2-b^2)^{\frac{1}{2}} \times a + b = (25-16)^{\frac{1}{2}} \times 5 + 4 = 3 \times 5 + 4 = 19.$

$ab - (a - b - c + d) = 20 - 0 = 20.$

$ab - (b^2 + a - c) = 20 - 20 = 0.$

$\frac{3abc}{\sqrt{(ab+b^2)}} \times (a + b + c) = \frac{60}{6} \times 10 = 100.$

$(\overline{ab^2+c} \cdot \overline{b-c})^{\frac{1}{3}} = (91 \times 3)^{\frac{1}{3}} = \sqrt[3]{213} = 7.$

$(a + b) \cdot (a - b) - (a + b + c) = 9 \times 1 - 10 = -1.$

$\sqrt{ab} \times \sqrt{ab} = \sqrt{20} \times \sqrt{20} = 20.$

$\sqrt{bc} \times \sqrt{ab} = \sqrt{4} \times \sqrt{20} = 2\sqrt{20}.$

$(3ab + b)^{\frac{2}{3}} = 64^{\frac{2}{3}} = \sqrt[3]{64^2} = 4096^{\frac{1}{3}} = 16.$

$b \backsim a = 1.$

$a \pm b$, or a *plus* and *minus* $b = 5 \pm 4 = 9$, and 1.

$x = ab - c$, or $x = 20 - 1$, or $x = 19.$

$[\sqrt{(a^2-b^2)} \times (d^2 + 3d)]^{\frac{1}{3}} = (\sqrt{(25-16)} \times 9)^{\frac{1}{3}} = (3 \times 9)^{\frac{1}{3}} = 3.$

ADDITION.

28. The Addition of Algebraic quantities is performed by collecting those that are alike or similar into one sum, and setting down that sum, together with the unlike quantities, all with their proper signs.

29. When the quantities are alike, and have like signs, add their co-efficients together, and prefix the sum to the letter or letters common to each term.

Thus, suppose the sum of the affirmative quantities $2a$, $5a$, $7a$, a, and $3a$ is required:

$$\begin{array}{rl} & 2a \\ & 5a \\ & 7a \\ & a \\ & 3a \\ \hline \text{Sum} & 18a \end{array}$$

Here it is evident, whatever be the value of a, that the sum $2a + 5a + 7a + a + 3a$ is $18a$; where 18 is the sum of the co-efficients.

$$\begin{aligned} \text{Let } a = 4: \text{ then } 2a &= 8 \\ 5a &= 20 \\ 7a &= 28 \\ a &= 4 \\ 3a &= 12 \\ \hline 72 = 18a &= 18 \times 4. \end{aligned}$$

If the quantities were negative, or $-2a$, $-5a$, $-7a$, $-a$, $-3a$, the sum would be denoted by $-18a$.

30. When quantities are alike, but have different signs, take the sums of the affirmative and negative co-efficients, respectively, and subtract the less from the greater, then prefix the sign of the greater to the difference, and subjoin the common quantity.

Let the sum of $2ab - ab$, $4ab - 2ab$, $7ab - 5ab$, and $ab - ab$, be required:

$$\begin{aligned} 2ab &- ab \\ 4ab &- 2ab \\ 7ab &- 5ab \\ ab &- ab \\ \hline 14ab - 9ab &= 5ab \text{ the sum.} \end{aligned}$$

Now $14 - 9 = 5$, therefore, the sum is $+ 5ab$ or $5ab$.

Let $a = 2$, $b = 1$:

$$\begin{aligned} \text{Then } 2ab - ab &= 4 - 2 \\ 4ab - 2ab &= 8 - 4 \\ 7ab - 5ab &= 14 - 10 \\ ab - ab &= 2 - 2 \\ \hline 28 - 18 = 10 &= 5ab \text{ the sum.} \end{aligned}$$

This process, however, in which subtraction is blended with addition, is evidently nothing more than adding together several differences:

$$\begin{aligned} \text{For } 2ab - ab &= ab \\ 4ab - 2ab &= 2ab \\ 7ab - 5ab &= 2ab \\ ab - ab &= 0 \\ \hline &5ab \text{ the sum of the differences, as before.} \end{aligned}$$

31. If the negative coefficients together exceed the affirmative ones, the sum will be negative:

$$\begin{array}{rl} \text{Thus, the sum of} & -2ab + ab \\ & -4ab + 2ab \\ & -7ab + 5ab \\ & -ab + ab \\ \hline \text{is} & -14ab + 9ab, \text{ or } -5ab. \end{array}$$

32. When the positive and negative coefficients are equal, the sum becomes $= 0$:

Thus, let the quantities be $2x + a - 3x$, $x - 4x + a$, and $5x - 2a - x$:

$$\begin{array}{rl} & 2x - 3x + a \\ & x - 4x + a \\ & 5x - x - 2a \\ \hline \text{sum} & 8x - 8x + 2a - 2a = 0. \end{array}$$

For $8x - 8x = 0$; and $2a - 2a = 0$.

33. Sometimes the terms may be collected mentally without setting them down one under another.

Thus, suppose the quantities to be added or abridged are $3\sqrt{ax} + 7$, $2\sqrt{ax} - 20$, and $18 - \sqrt{ax}$; then the order of setting down the sum may be

thus, $5\sqrt{ax} - \sqrt{ax} + 7 + 18 - 20$

or, $4\sqrt{ax} + 25 - 20$, which is $4\sqrt{ax} + 5$, the sum.

34. When quantities are unlike, and have like, or different signs, collect those that are similar together, as in the foregoing examples, then set the whole down with their proper signs.

Thus, if the terms are $-ba$ and $+ca$, they may be set down

thus, $-ba + ca$

or thus, $ca - ba$

or thus, $(c - b)\,a$.

Let $a = 4$, $b = 2$, $c = 5$:

then, $ca - ba = 20 - 8 = 12$

or, $(c - b)\,a = (5 - 2) \times 4 = 12$.

It therefore appears, that when the quantities are all unlike, the number of terms cannot be abridged, which is also evident from the following example:

$$\begin{array}{rl} & ax - bx \\ & -cx + dx \\ & gx - hx \\ \hline \text{sum} & ax + dx + gx - bx - cx - hx \\ \text{or,} & (a + d + g - b - c - h)\,x. \end{array}$$

35. If however, a, d, g, b, c, h (the coefficients of x) are given quantities, the whole may be reduced to one term:

Thus, let $a = 6$, $d = 10$, $g = 5$, $b = 12$, $c = 4$, $h = 3$:
then $(a + d + g - b - c - h)\,x$ will be $(21 - 19)\,x$ or $2x$.

Other Examples.

$$\begin{array}{l} -2\,xy + c - 7 \\ -4\,c - xy + 9 \\ axy + \frac{1}{2}xy - 3 \\ \hline (a - 2\frac{1}{2})\,xy - 3c - 1 \end{array} \qquad \begin{array}{l} \sqrt{a} - \sqrt{x} + 2a \\ b\sqrt{x} - \frac{1}{2}\sqrt{a} - 2ba \\ \hline (b - 1)\sqrt{x} + \frac{1}{2}\sqrt{a} + (2 - 2b)\,a. \end{array}$$

$$\begin{array}{l} axy - 14 - \sqrt{z} \\ -2xy + 2z^{\frac{1}{2}} + 15 \\ \frac{1}{2}xy - z^3 - 1 \\ \hline (a - 1\frac{1}{2})\,xy + \sqrt{z} - z^3 \end{array} \qquad \begin{array}{l} 42 - a\sqrt{(x^2 - z^2)} + \sqrt[3]{m} \\ b\sqrt{(-z^2 + x^2)} + 10 - am^{\frac{1}{3}} \\ \hline (b - a)\,(x^2 - z^2)^{\frac{1}{2}} + (1 - a)\,m^{\frac{1}{3}} + 52 \end{array}$$

SUBTRACTION.

36. CHANGE the signs of all the terms to be subtracted, and then collect the several quantities together as in Addition.

Thus, if $5a - 3a$ is to be subtracted from $4a$:
then $5a - 3a$ when the signs are changed
will be $-5a + 3a$, to this add $4a$ and
we have $-5a + 3a + 4a$
or $7a - 5a$ or $2a$, the required difference:

Therefore the truth of the rule is manifest, because $5a - 3a$, or $2a$ taken from $4a$ leaves $2a$.

37. Again, if $2b - 4b$ (or $-2b$) is to be taken from $3b$: then if the signs of $2b - 4b$ are changed, and $3b$ added, we have $-2b + 4b + 3b$, or $7b - 2b$, or $5b$ the required difference.

Therefore, subtracting $-2b$ is the same as adding $2b$: consequently subtracting a *negative* quantity gives the same result as adding an equal *positive* one.

38. When the quantities are alike, and have numeral coefficients, the operation may be performed as in common arithmetic, if those to be subtracted are the least:

$$\begin{array}{rl} \text{Thus, from} & -7x + 3z - 4y \\ \text{take} & -5x + z - 3y \\ \hline \text{rem.} & -2x + 2z - y \end{array}$$

Other Examples.

$$\begin{array}{rl} \text{From} & 2ax - bx + 3cx \\ \text{take} & -dx + 2fx - 3gx \\ \hline \text{diff.} & 2ax - bx + 3cx + dx - 2fx + 3gx \\ \text{or} & (2a + 3c + d + 3g - b - 2f)\,x. \end{array} \qquad \begin{array}{r} 5\sqrt{ax} - 3x^2 \\ -\sqrt{ax} + 2x^2 \\ \hline 6\sqrt{ax} - 5x^2 \end{array}$$

$$\begin{array}{rl} \text{From} & (ax+y)^2 + 5mx - \sqrt{(a^2 - x^2)} \\ \text{take} & 3a\,(ax+y)^2 + 6mx + a\sqrt{(a^2 - x^2)} \\ \hline \text{diff.} & (1-3a)\,(ax+y)^2 - mx - (1+a)\sqrt{(a^2 - x^2)} \end{array}$$

By uniting the coefficients, as in the last example, the results are frequently simplified.

MULTIPLICATION.

39. The general rule for the signs in the product is, that like signs produce *plus* (+), and unlike signs *minus* (−).

Thus $+a \times +b$, or $a \times b$ gives $+ab$ or ab.
And $-a \times -b$ is also $+ab$ or ab.
But $+a \times -b$ produces $-ab$.

40. **Simple factors, as a, b, c, &c. may be set down in any order to denote their product:**

Thus $a \times b \times c$ is the same as abc or bca or cba, &c.
For suppose $a = 2$, $b = 3$, $c = 4$:
Then $abc = 24$ the continued product of 3, 4, and 2, any how varied.

41. **When the factors have numeral coefficients, prefix their product to the letters with their proper signs:**

Thus $3a \times 2b$ is $6ab$; this is manifest, if it be admitted that ab denotes the product of a and b:
for the factors of $3a \times 2b$ are 3, 2, a, and b,
and therefore $3 \times 2 \times a \times b$ is the same as $6 \times ab$, or $6ab$.

42. **When a factor is multiplied into itself, the product becomes a power whose root is that factor:**

Thus $a \times a \times a$ or aaa or a^3 is the third power or cube of a; where the small figure 3 is the index or exponent (11). And the cube root of a^3 is a.

And $aaaa$, &c. repeated to n times is a^n.

And the nth root of a^n is a.

Also $ab \times ab$ is $abab$ or $aabb$ or a^2b^2; therefore $(ab)^2$ is a^2b^2.

And $(cx)^n$ is c^nx^n.

Other Examples.

	$7a$	$9bc$	$-axy$
	$4b$	$-3d$	$-2bz$
Product	$28ab$	$-27bcd$	$+2abxyz$

	$-5x^2z$	$3z^3x$	$-\frac{1}{2}xyz$
	axz	$\frac{1}{2}ax^3$	$-\frac{1}{3}yzx$
Product	$-5ax^3z^2$	$1\frac{1}{2}az^3x^4$	$+\frac{1}{6}x^2y^2z^2$

43. **When one of the factors is a compound quantity, the product is found by multiplying each of its terms by the other factor:**

Thus $a + b$ multiplied by c is $ca + cb$,
And $a - b$ multiplied by c is $ca - cb$.

That $ca + cb$ is the product of the factors $a + b$ and c, will be manifest, if we consider that the whole is equal to all its parts taken together:

For let mc be the whole product of the factors c and m; then this whole (mc) is made up of several products, as $\frac{1}{2}mc + \frac{1}{2}mc$, or $\frac{1}{3}mc + \frac{1}{3}mc + \frac{1}{3}mc$, &c. &c.; therefore, whatever be the number of parts into which m is divided, the products of those parts, and the factor c, taken together, will be equal to the whole product mc. If, therefore, we consider a and b as the *parts* of a *whole*, the two products $ca + cb$ will denote the product of $a + b$ and c.

Let $a = 6$, $b = 8$, and $c = 3$:

Then $ca + cb = 18 + 6 = 24$, the same as the sum $6 + 8$ multiplied by 3.

Again, let $m - \frac{1}{4}m$, and c, be two factors; then their product will be equal to the difference of the products mc and $\frac{1}{4}mc$, or equal to $mc - \frac{1}{4}mc$:

For $mc - \frac{1}{4}mc = \frac{3}{4}mc$, which is the same as $(m - \frac{1}{4}m)c$, or $\frac{3}{4}m \times c$.

Here it is evident that instead of m and $\frac{1}{4}m$, we may make use of any other two quantities whose difference can be expressed in a simple term. It therefore appears that when a and b are quantities which can be compared, the product $(a - b)c$ is equal to $ca - cb$.

The same conclusion however, will be manifest, if we consider, that in order to have the product of $a - b$ and c, the product ca must be diminished by c times b, because a is greater than $a - b$ by b.

Hence $+c \times -b$ or $-b \times c$ is $-cb$, therefore, *unlike signs* give *minus* ($-$) in the product.

Corol. Therefore a compound expression, as $axyz - bxyz + 2cxyz$, where one of the factors (xyz) is common to all the terms, may be set down thus: $(a - b + 2c)\ xyz$.

44. When both the factors are compound quantities, multiply all the terms of the multiplicand by each of the terms in the multiplier; then collect the several products, as in Addition.

Thus, let $a - c$ be multiplied by $b - d$;

$$\begin{array}{rl} & a - c \\ & \underline{b - d} \\ \text{Product} & \underline{ab - bc - da + dc} \end{array}$$

By the preceding article, the product of $a - c$ by b is $ab - bc$, which would be the true result, provided b was the only multiplier; but the multiplier is *less* than b by d, and therefore the product $ab - bc$ is d times $a - c$ *greater* than what ought to be produced by the multiplier $b - d$, consequently d times $a - c$ should be subtracted from $ab - bc$ to have the true product:

Now d times $a - c$ is $da - dc$, which subtracted from $ab - bc$, gives $ab - bc - da + dc$ (36); the same result as by the rule. Therefore, $-c \times -d$ is $+ dc$; and consequently *like signs* give *plus* (+) in the product.

Other Examples.

$$\begin{array}{rlll} & 5xy - ab + z & -2\tfrac{1}{2}z + x & 3a^3 + 2x^3 - y^3 \\ & \underline{4a} & \underline{-14} & \underline{3ax} \\ \text{Product} & \underline{20axy - 4a^2b + 4az} & \underline{35z - 14x} & \underline{9a^4x + 6ax^4 - 3axy^3} \end{array}$$

$$\begin{array}{l} x + 1 \\ \underline{x + 1} \\ x^2 + x \\ \underline{\quad + x + 1} \\ \underline{x^2 + 2x + 1} \end{array} \text{ the square of } x + 1. \qquad \begin{array}{l} x + y \\ \underline{x - y} \\ x^2 + xy \\ \underline{\quad - xy - y^2} \\ x^2 \quad\quad - y^2 \end{array} \text{ or } x^2 - y^2,$$

viz. the *product of the sum and difference of two numbers is equal to the difference of their squares.*

$$\begin{array}{ll}
x^3+x^2y+xy^2+y^3 & \qquad x^4-x^3+x^2-x+1 \\
x-y & \qquad x+1 \\
\hline
x^4+x^3y+x^2y^2+xy^3 & \qquad x^5-x^4+x^3-x^2+x \\
\quad -x^3y-x^2y^2-xy^3-y^4 & \qquad \quad +x^4-x^3+x^2-x+1 \\
\hline
x^4 \qquad\qquad -y^4 \text{ or } x^4-y^4 & \qquad x^5 \qquad\qquad +1 \text{ or } x^5+1.
\end{array}$$

$$\begin{array}{l}
x^2-ax+b \\
x+c \\
\hline
x^3-ax^2+bx \\
\quad +cx^2-cax+cb \\
\hline
x^3+cx^2-ax^2+bx-cax+cb
\end{array}$$

$$\left.\begin{array}{l}
\text{or, } x^3+(c-a)\,x^2+(b-ca)\,x+cb \\
\text{or, } x^3-(a-c)\,x^2-(ca-b)\,x+cb
\end{array}\right\}\text{ by uniting the coefficients } a, c, b.$$

In this example let $a=4$, $c=2$, $b=3$: then the two last expressions will be

$$x^3+(2-4)\,x^2+(3-8)\,x+6, \text{ or } x^3-2x^2-5x+6.$$
$$x^3-(4-2)\,x^2-(8-3)\,x+6, \text{ or } x^3-2x^2-5x+6.$$

But if c is greater than a, and b greater than ca, then x^2 and x with their coefficients will be affirmative.

43. Because $a\times a$ is a^2, $a\times a\times a$ is a^3, $a\times a\times a\times a\times a$ is a^5, &c. (42); it follows, that the addition of the indices answers to the multiplication of the factors (111. Arith.) for $a^2\times a^3=a^5=a^{2+3}$, &c. Therefore, when powers of the same quantity are to be multiplied together, add the indices together for the index of the product:

$$\begin{array}{ll}
\text{Thus } a\times a=a^{1+1} \text{ or } a^2 & \qquad a^2x^3\times a^3x^2=a^{2+3}x^{3+2} \text{ or } a^5x^5. \\
a^2\times a=a^{2+1} \text{ or } a^3 & \qquad a^n\times a^m=a^{n+m} \\
a^3\times a^3=a^{3+3} \text{ or } a^6 & \qquad a^nb^m\times ab=a^{n+1}b^{m+1}:
\end{array}$$

And $(a^2-x^2)^n\times(a^2-x^2)^m\times(a^2-x^2)^r$ is $(a^2-x^2)^{n+m+r}$

DIVISION.

46. DIVISION in Algebra, as in common Arithmetic, consists in finding a quantity which multiplied by the divisor, shall produce the dividend.

Therefore, the rule for the signs will be the same as in Multiplication; namely, like signs give *plus* in the quotient, and unlike signs *minus*.

Thus $+ab$ divided by $+b$ gives $+a$; for $a \times b$ is ab the dividend.

Also $-ab$ divided by $-b$ gives $+a$; because $-b \times a$ is $-ab$ the dividend.

But when ab is divided by $-b$, the quotient will be $-a$; for $-b \times -a$ is $+ab$ or ab.

47. When the divisor and dividend are simple quantities, the quotient, in most cases, may be discovered by inspection only, if we make a fraction of the terms, and consider it as the result of the division (37. Arith.)

Thus if 3 be divided by 9, the quotient is $\frac{3}{9}$ or $\frac{1}{3}$:

In like manner when $3abc$ is divided by bc, the quotient may be denoted by $\frac{3abc}{bc}$:

And dividing the numerator and denominator by the factors (bc), which are common to both, we have $\frac{3abc}{bc} = \frac{3a}{1} = 3a$ the quotient:

For bc (the divisor) $\times\ 3a$ gives $3abc$ the dividend.

Other Examples.

Divide $15axy$ by $-3ay$

$\frac{15axy}{-3ay} = -5x$ the quotient: for $-3ay \times -5x = 15axy$.

Divide $-7a^2x^2$ by $-14ax$.

$\frac{-7a^2x^2}{-14ax} = \frac{1}{2}ax$ the quotient: for $-14ax \times \frac{1}{2}ax = -7a^2x^2$.

Divide $-a^3x^3$ by $5ax^2$.

$\frac{-a^3x^3}{5ax^2} = -\frac{1}{5}a^2x$ the quotient: for $-\frac{1}{5}a^2x \times 5ax^2 = -a^3x^3$.

Divide $4axy$ by $4bxy$.

$\frac{4axy}{4bxy} = \frac{a}{b}$ the quotient.

Divide $-5ax$ by $-10axx$.

$\frac{-5ax}{-10axx} = \frac{1}{2x}$ the quotient.

The preceding operations are evidently analogous to that of reducing Fractions to their lowest terms in Arithmetic.—(39. Arith.)

48. Divide $3xy$ by $5az$. Here the divisor and dividend have no common factor, and therefore the quotient is $\frac{3xy}{5az}$.

49. When powers of the same quantity are to be divided one by the other, subtract the index of the divisor from that of the dividend, and the difference will be the index of the quotient.

Thus a^5 divided by a^2 gives a^{5-2} or a^3 the quotient;

For $a^2 \times a^3 = a^5$ (45).

Or denoting the quotient by the fraction $\frac{a^5}{a^2}$, and reducing it to its lowest terms,

$$\frac{a^5}{a^2} = \frac{a^2 \times a^3}{a^2} = \frac{a^3}{1} = a^3 \text{ the quotient.}$$

Also $x^4 \div x^3$ is $x^{4-3} = x^1 = x$ the quotient.

And x^m divided by x^3 is x^{m-3}.

$x^m \div x^n$ is x^{m-n}.

Also $(c-z)^m \div (c-z)^{\frac{1}{2}}$ or $\frac{(c-z)^m}{(c-z)^{\frac{1}{2}}}$, is $(c-z)^{m-\frac{1}{2}}$.

30. **When the index of the divisor is greater than that of the dividend, the quotient will have a negative index.**

Thus x^3 divided by x^5 will give x^{3-5} or x^{-2}; for $3-5$ is -2. (36. 31.)

But $x^3 \div x^5$ may be denoted by $\frac{x^3}{x^5}$ which reduced to its lowest terms is $\frac{1}{x^2}$ therefore $\frac{1}{x^2}$ is the same as x^{-2}.

31. **If the dividend be a compound quantity, and the divisor a simple one, each term of the former must be divided by the latter, as in the foregoing examples.**

Thus, let $ab-ac$ be divided by a:

Then the quotient may be set down thus $\frac{ab-ac}{a}$, which reduced is $\frac{b-c}{1}$ or $b-c$, the quotient.

For $(b-c)\ a$ is $= ab-ac$.

Or the quotient may be denoted thus $\frac{ab}{a}-\frac{ac}{a}$, and these fractions reduced are $\frac{b}{1}-\frac{c}{1}$ or $b-c$ the quotient, as before.

Other Examples.

Divide $6acx-8a^2c-10a^2cx$ by $2ac$.

$$\frac{6acx-8a^2c-10a^2cx}{2ac}=3x-4a-5ax \text{ the quotient.}$$

Divide $5x^2-14ax+16z^3$ by 7.

$\frac{5x^2}{7}-\frac{14ax}{7}+\frac{16z^3}{7}$, or $\frac{5}{7}x^2-2ax+2\frac{2}{7}z^3$ the quotient.

Divide $9ax^2 - 12a^3x^2$ by $-3ax$.

$$\frac{9ax^2 - 12a^3x^2}{-3ax} = -3x + 4a^2x \text{ the quotient.}$$

Divide $acx + cx - ac$ by ac.

$\frac{acx + cx - ac}{ac} = \frac{ax + x - a}{a}$, the quotient: this is found by cancelling c, the only factor common to the whole numerator, and denominator: but if we denote the quotient by three fractions, it may be reduced to a more simple form:

Thus $\frac{acx}{ac} + \frac{cx}{ac} - \frac{ac}{ac} = x + \frac{x}{a} - 1$, the quotient. This, however, is only the former quotient reduced: for $\frac{ax + x - a}{a}$ is the same as $\frac{ax}{a} + \frac{x}{a} - \frac{a}{a}$, or $x + \frac{x}{a} - 1$.

Divide $(x^2 + z)^2 + ay$ by bz.

Here the divisor and dividend have no common factor, and, therefore, $\frac{(x^2 + z)^2 + ay}{bz}$ is the quotient.

52. When the divisor and dividend are both compound quantities: Arrange their terms according to the powers of some one letter in both, the higher powers being to the left.

Find how often the first term of the divisor is contained in the first term of the dividend, and set the result in the quotient.

Multiply the whole divisor by the result thus found, and subtract the product from the dividend: to this remainder bring down as many other terms of the dividend as are necessary for the next operation; then divide as before, and so on, till all the terms are brought down.

Thus to divide $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{r|l} a - b) & a^2 - 2ab + b^2\ (a - b \text{ quotient.} \\ & \underline{a^2 - ab\qquad\ } \\ & \quad - ab + b^2 \\ & \quad \underline{- ab + b^2} \\ & \qquad 0 \end{array}$$

Here a, the left hand term of the divisor, is contained a times in a^2, the left hand term of the dividend, therefore a is the first term of the quotient; and the divisor $a-b$ multiplied by a is a^2-ab, which taken from a^2-2ab in the dividend leaves $-ab$; to this bring down $+b^2$, and $-ab+b^2$ is the second dividend.

Next, a in the divisor is contained $-b$ times in $-ab$ (the left hand term of the dividend $-ab+b^2$); therefore $-b$ is the second term in the quotient: now the divisor $a-b$ multiplied by $-b$ gives $-ab+b^2$ the second dividend; therefore $a-b$ is the quotient without a remainder.

For $(a-b)\times(a-b)=a^2-2ab+b^2$; the proof, as in common arithmetic.

Other Examples.

Divide x^5+1 by $x+1$.

$$
\begin{array}{l}
x+1)\ x^5+1\ (x^4-x^3+x^2-x+1 \text{ quotient.} \\
\quad\ \ \underline{x^5+x^4} \\
\qquad\ -x^4+1 \\
\qquad\ \underline{-x^4-x^3} \\
\qquad\qquad +x^3+1 \\
\qquad\qquad \underline{+x^3+x^2} \\
\qquad\qquad\quad -x^2+1 \\
\qquad\qquad\quad \underline{-x^2-x} \\
\qquad\qquad\qquad +x+1 \\
\qquad\qquad\qquad \underline{+x+1} \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

In this example, x is contained x^4 times in x^5, and the divisor $x+1$ multiplied by x^4 gives x^5+x^4, which subtracted from x^5+1 the dividend and $-x^4+1$ remains, the second dividend.

Next, x is contained $-x^3$ times in $-x^4$, therefore $-x^3$ is the second term in the quotient; and $(x+1)\times -x^3$ gives $-x^4-x^3$, which taken from $-x^4+1$, and the remainder is $+x^3+1$. And so on.

$(x^4-x^3+x^2-x+1)\times(x+1)=x^5+1$. See the multiplication, Art. 44.

Divide $6x^2-x-12$ by $2x-3$.

$$
\begin{array}{l}
2x-3)\ 6x^2-x-12\ (3x+4 \text{ quotient.} \\
\qquad\ \ \underline{6x^2-9x} \\
\qquad\qquad +8x-12 \\
\qquad\qquad \underline{+8x-12} \\
\qquad\qquad\quad 0
\end{array}
$$

$$x+c)\ x^3+cx^2-ax^2-cax+bx+cb\ (x^2-ax+b \text{ quotient.}$$
$$\begin{array}{l} \underline{x^3+cx^2} \\ \qquad -ax^2-cax \\ \qquad \underline{-ax^2-cax} \\ \qquad\qquad +bx+cb \\ \qquad\qquad \underline{+bx+cb} \\ \qquad\qquad\quad 0 \end{array}$$

$$x+z)\ x^2-z^2+y\ \left(x-z+\frac{y}{x+z} \text{ the quotient.}\right.$$
$$\begin{array}{l} \underline{x^2+xz} \\ \quad -xz-z^2 \\ \quad \underline{-xz-z^2} \\ \qquad\quad +y \end{array}$$

Here y is the remainder, therefore $\frac{y}{x+z}$ is a fractional part of the quotient, as in the division of whole numbers. In these cases however, the quotient is usually set down thus, $\frac{x^2-z^2+y}{x+z}$.

$$2x^2-3ax+a^2)\ 4x^4-9a^2x^2+6a^3x-a^4\ (2x^2+3ax-a^2 \text{ quotient.}$$
$$\begin{array}{l} \underline{4x^4-6ax^3+2a^2x^2} \\ \quad +6ax^3-11a^2x^2+6a^3x \\ \quad \underline{+6ax^3-9a^2x^2+3a^3x} \\ \qquad\qquad -2a^2x^2+3a^3x-a^4 \\ \qquad\qquad \underline{-2a^2x^2+3a^3x-a^4} \\ \qquad\qquad\qquad 0 \end{array}$$

$$x+y)\ x^2-y^2\ (x-y \text{ quotient.}$$
$$\begin{array}{l} \underline{x^2+xy} \\ \quad -xy-y^2 \\ \quad \underline{-xy-y^2} \\ \qquad 0 \end{array}$$

53. By the last example it appears, that the difference of two squares is divisible by the sum, and also by the difference of their roots.

Again, $\frac{x^3+y^3}{x+y} = x^2-xy+y^2$ the quotient.

$\frac{x^3-y^3}{x-y} = x^2+xy+y^2$ quotient.

$\frac{x^4-y^4}{x+y} = x^3-x^2y+xy^2-y^3$ quotient.

$$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3 \text{ quotient.}$$

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4 \text{ quotient.}$$

$$\frac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4 \text{ quotient.}$$

&c. &c.

34. Hence we conclude, that x^n+y^n is divisible by the sum of the roots $x+y$, and x^n-y^n by their difference, when the index n is an odd number;

And that x^n-y^n is divisible by the sum, and also by the difference when n is an even number.

ALGEBRAIC FRACTIONS.

35. THE learner should perfectly understand the theory and practice of Vulgar Fractions in Arithmetic, before he attempts this part of Algebra, because he will perceive that the same rules answer equally in both.

To reduce a Fraction to its lowest Terms.

36. DIVIDE the numerator and denominator by the factor or factors common to both, and the result will be the answer.

The rule is derived from this obvious principle, that, if the terms of a fraction are multiplied, or divided by the same number or quantity, its value is not altered.

Thus, let $\frac{6a}{8b}$ be the fraction:

then $\frac{6a}{8b}=\frac{2\times 3a}{2\times 4b}$, therefore 2 is a factor common to the numerator and denominator, and because it is the greatest common measure of $6a$ and $8b$, the fraction in its lowest terms is $\frac{3a}{4b}$.

And $\frac{ab}{ac}$ reduced is $\frac{b}{c}$; for a is the factor common to b...

Also $\frac{-5a^2xy}{10axz}$ reduced is $\frac{-ay}{2z}$; where $5ax$ is the ...

For $\frac{-ay \times 5ax}{2z \times 5ax} = \frac{-5a^2xy}{10axz}$.

...gain, $\frac{4abz - 7aby}{2abx + 6abx}$ reduced is $\frac{2z - 3\frac{1}{2}y}{x + 3x}$.

...7. The simple divisors are readily found, as i... ...mples. But to discover the compound divisor... ...he fraction be resolved into their factors:

...hus, to reduce the fraction $\frac{4a^2x^2 - 4a^2y^2}{bx + by}$ to its lowe...

$\frac{\ldots - 4a^2y^2}{\ldots x + by} = \frac{4a^2\,(x^2 - y^2)}{b\,(x + y)} = \frac{4a^2\,(x - y)\,(x + y)}{b\,(x + y)}$ ...

...refore both terms of the fraction are divisible by x ... ...tly their greatest common measure; and the frac... ...ms is $\frac{4a^2\,(x - y)}{b}$ or $\frac{4a^2x - 4a^2y}{b}$.

58. But when the numerator and denominato... ...terms, the usual method of proceeding is to fi... ...mmon measure thus:

Reject the simple divisors in both terms of the...

Divide the greater by the less, and the last di... ...mainder, and so on till nothing remains; the... the greatest common measure, as in Arithme...

Thus, to reduce $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^4b + 2a^3b^2 + 2ab^3 + b^4}$ to its low...

Then $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^4b + 2a^3b^2 + 2ab^3 + b^4} = \frac{(a^2 + 2ab + \ldots}{(a^3 + 2a^2b + \ldots}$

...or the simple divisors are $5a^3$ and b;

Therefore we have to find the greatest common measure of $a^2+2ab+b^2$ and $a^3+2a^2b+2ab^2+b^3$;

$$
\begin{array}{l}
a^2+2ab+b^2)\ a^3+2a^2b+2ab^2+b^3\ (a \\
\qquad\qquad\qquad\ \underline{a^3+2a^2b+ab^2} \\
\qquad\qquad\qquad\qquad\qquad ab^2+b^3)\ a^2+2ab+b^2 \\
\qquad\qquad\qquad\qquad\qquad \text{or}\quad a+b)\ a^2+2ab+b^2\ (a+b \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{a^2+ab} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ab+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{ab+b^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

The first remainder is ab^2+b^3, therefore the next operation is that of finding the greatest common measure of ab^2+b^3, and the last divisor $a^2+2ab+b^2$, for which reason we reject b^2 the simple divisor of ab^2+b^3, and then it is reduced to finding the greatest common measure of $a+b$ and $a^2+2ab+b^2$, which is $a+b$ the last divisor.

$a+b)\ \dfrac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}\left(=\dfrac{5a^4+5a^3b}{a^2b+ab^2+b^3}\right.$ the fraction in its lowest terms.

The reason that $a+b$ measures the terms of the proposed fraction is evident from this consideration, that if a divisor measures a quantity, it must also measure any multiple of that quantity.

To reduce $\dfrac{8a^2b-10ab^2+2b^3}{9a^4-9a^3b+3a^2b^2-3ab^3}$ to its lowest terms.

The simple divisors are $2b$ and $3a$, hence the fraction becomes

$$\frac{(4a^2-5ab+b^2)\,2b}{(3a^3-3a^2b+ab^2-b^3)\,3a};$$

Therefore we have to find the greatest common measure of the terms between the parentheses:

Now, that the less may divide the greater, let the latter be multiplied by 4:

$$
\begin{array}{lr}
 & 3a^3-3a^2b+ab^2-b^3 \\
 & \underline{4\qquad\qquad\qquad\qquad} \\
4a^2-5ab+b^2) & 12a^3-12a^2b+4ab^2-4b^3\ (3a \\
 & \underline{12a^3-15a^2b+3ab^2}\qquad\quad \\
 & +3a^2b+ab^2-4b^3 \\
\text{again, multiply by} \ldots\ldots\ldots\ldots & \underline{4\qquad\qquad\qquad} \\
 & +12a^2b+4ab^2-16b^3\ (3b \\
 & \underline{+12a^2b}-15ab^2+3b^3 \\
\textit{divide by } 19b^2 \ldots\ldots\ldots\ldots\ldots\ldots & +19ab^2-19b^3 \\
 & +a-b)
\end{array}
$$

$$
\begin{array}{l}
+a-b)\ 4a^2-5ab+b^2\ (4a-b \\
\qquad\quad \underline{4a^2-4ab} \\
\qquad\qquad\ -ab+b^2 \\
\qquad\qquad\ \underline{-ab+b^2} \\
\qquad\qquad\qquad 0
\end{array}
$$

therefore the last divisor $a-b$ is the greatest common measure.

$a-b\Big)\dfrac{8a^2b-10ab^2+2b^3}{9a^4-9a^3b+3a^2b^2-3ab^3}\Big(\dfrac{8ab-2b^2}{9a^3+3ab^2}$ the fraction in its lowest terms.

59. The multiplication of the dividends (as in the last example) cannot affect the common-measure, because the dividends thus increased, are only multiples of the former dividends: Sometimes however, the necessary factor or factors for that purpose, are not discoverable at first sight; for example,

Let it be required to reduce the fraction $\dfrac{3bcz+5mxz+30mx+18bc}{4adz-7vrz+24ad-42vr}$ to its lowest terms:

Here it appears that a numeral multiplier will not answer the purpose, and therefore one term of the fraction must be multiplied by some factor or factors of the other before a division can take place: but in the present case, the shortest method of reduction is that of resolving the numerator and denominator into their factors:

thus $\dfrac{3bcz+5mxz+30mx+18bc}{4adz-7vrz+24ad-42vr}=\dfrac{(3bc+5mx)z+30mx+18bc}{(4ad-7vr)z+24ad-42vr}$

$=\dfrac{(3bc+5mx)z+(5mx+3bc)6}{(4ad-7vr)z+(4ad-7vr)6}=\dfrac{(3bc+5mx)(z+6)}{(4ad-7vr)(z+6)}$; therefore $z+6$ is a common divisor; and the fraction is reduced to $\dfrac{3bc+5mx}{4ad-7vr}$, which is in its lowest terms.

To reduce an improper Fraction to its equivalent whole or mixed quantity.

60. This is nothing more than a division; therefore, divide the numerator by the denominator, and the quotient will be the answer. (53.)

Thus, if the fraction to be reduced is $\frac{6ab^2}{2ab}$:

Then $\frac{6ab^2}{2ab} = 3b$ the quotient, or the fraction reduced to its equivalent *whole*. Consequently when there is no remainder after division, the operation is that of reducing a fraction to its lowest terms: for $\frac{6ab^2}{2ab} = \frac{3b}{1}$, the fraction in its lowest terms.

Also $\frac{x^3+y^3}{x+y}$ reduced is $x^2 - xy + y^2$. (53.)

Suppose the fraction to be reduced is $\frac{3ac-4bc}{a-b}$:

$$a-b)\ 3ac-4bc\ (3c+\frac{-bc}{a-b}, \text{ or } 3c-\frac{bc}{a-b}, \text{ the quotient.}$$
$$\frac{3ac-3bc}{-bc}$$

Here the remainder is $-bc$, therefore $\frac{-bc}{a-b}$ is the fractional part of the required mixed quantity, but this fraction is negative (46); and since $+\frac{-bc}{a-b}$ and $-\frac{bc}{a-b}$ denote the same thing, the quotient may be set down either way.

Again, let the proposed fraction be $\frac{ac-cb-m-n}{a-b}$.

$$a-b)\ ac-cb-m-n\ (c+\frac{-m-n}{a-b}, \text{ or } c-\frac{m+n}{a-b}, \text{ the quotient.}$$
$$\frac{ac-cb}{-m-n}$$

In this example the remainder is $-m-n$, and the fraction is $\frac{-m-n}{a-b}$; but $-m-n$ is the same as $-(m+n)$, namely, the sum of m and n is negative; therefore $+\frac{-m-n}{a-b}$ and $-\frac{m+n}{a-b}$ are expressions for the same quantity.

To reduce a mixed quantity to an equivalent Fraction.

61. This is the reverse of the operation in the preceding article; therefore,

Multiply the integral part by the denominator of the fraction; and add its numerator to the product; then the sum placed over the said denominator will form the fraction required.

Thus, let the mixed quantity be $a + \frac{x-y}{a+b}$.

$$\begin{array}{rl} \text{then } a \times (a+b) = & a^2 + ab \\ \text{add} \ldots\ldots & x - y \\ \hline \text{sum} & a^2 + ab + x - y \end{array}$$

and $\frac{a^2+ab+x-y}{a+b}$ is the fraction sought.

Reduce $3c - \frac{bc}{a-b}$ to an equivalent fraction.

$$\begin{array}{rl} 3c \times (a-b) = & 3ac - 3cb \\ & -bc \text{ add} \\ \hline & 3ac - 4bc \text{ the sum;} \end{array}$$

therefore the fraction is $\frac{3ac-4bc}{a-b}$.

Reduce $c + \frac{-m-n}{a-b}$ $\left(\text{or } c - \frac{m+n}{a-b}\right)$ to an equivalent fraction;

$$\begin{array}{rl} c \times (a-b) = & ac - bc \\ & -m-n \text{ add} \\ \hline & ac - bc - m - n \text{ the sum,} \end{array}$$

and $\frac{ac-bc-m-n}{a-b}$ is the fraction required.

If the quantity were given in this form, $c - \frac{m+n}{a-b}$, the thing to be done is evidently that of subtracting the fraction $\frac{m+n}{a-b}$ from the integer c.

Also, $cx - ax - \frac{b-c}{-a}$ reduced to a fraction is $\frac{-cax+a^2x-b+c}{-a}$, which denotes the difference of $cx - ax$ and $\frac{b-c}{-a}$.

To bring Fractions with different denominators to equivalent Fractions having a common denominator.

62. MULTIPLY each numerator into all the denominators, except its own, for the new numerator of that fraction; and all the denominators together for the common denominator.

The rule may be investigated exactly as in Arithmetic. (45. Arith.)

Let the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{m}{n}$, be brought to equivalent fractions having a common denominator.

$$\left.\begin{matrix} adn \\ cbn \\ mdb \end{matrix}\right\}\text{ the three new numerators.}$$

bdn the common denominator.

And the three fractions are $\frac{adn}{bdn}$, $\frac{cbn}{bdn}$, $\frac{mdb}{bdn}$.

Reduce $\frac{a}{b}$, $\frac{b}{c}$, $\frac{a}{c}$, to a common denominator.

$$\left.\begin{matrix} acc \\ bbc \\ acb \end{matrix}\right\}\text{ the numerators.}$$

bcc the common denominator.

Hence the fractions are $\frac{acc}{bcc}$, $\frac{bbc}{bcc}$, $\frac{acb}{bcc}$, or $\frac{ac}{bc}$, $\frac{bb}{bc}$, $\frac{ab}{bc}$, in their lowest terms.

63. When the denominator of one fraction is a multiple of the denominator of another, divide the greater denominator by the less, then multiply the terms of that fraction which hath the least denominator by the quotient, and the two fractions will be reduced to a common denominator. (47. Arith.)

Thus, if the fractions are $\frac{m}{a}$ and $\frac{2x}{ab}$;

Then ab is a multiple of a; and ab divided by a gives the quotient b; therefore if the terms of the fraction $\frac{m}{a}$ are multiplied by b, we have $\frac{mb}{ab}$ which has the same denominator as $\frac{2x}{ab}$.

In like manner the fractions $\frac{x}{a}$, $\frac{2z}{ac}$, $\frac{y}{acd}$, when brought to a common denominator are $\frac{cdx}{acd}$, $\frac{2dz}{acd}$, $\frac{y}{acd}$: For the terms of the first $\left(\frac{x}{a}\right)$ are multiplied by cd, and those of the second by d.

64. Let $a+x$ and $\frac{nx}{ac}$ be brought to fractions having a common denominator.

Making 1 the denominator of $a+x$ gives the fraction $\frac{a+x}{1}$, and if its terms are multiplied by ac we get $\frac{a^2c+acx}{ac}$: and the two required fractions are $\frac{a^2c+acx}{ac}$ and $\frac{nx}{ac}$.

Hence, an integral quantity $(a+x)$ is brought to an equivalent fraction, having a given denominator (ac), by multiplying the former by the latter, and placing the product over that denominator.

To Add fractional quantities together.

65. BRING the fractions to a common denominator, then add the numerators together, and place the sum over the common denominator, as in Vulgar Fractions.

Examples.

1. Required the sum of the fractions $\frac{ax}{b}$, $\frac{cx}{b}$, and $\frac{dx}{b}$?

$$\frac{ax}{b}+\frac{cx}{b}+\frac{dx}{b}=\frac{ax+cx+dx}{b} \text{ or } \frac{(a+c+d)x}{b} \text{ the sum.}$$

2. Required the sum of $\frac{5x}{9}$, $\frac{5x}{6}$, and $\frac{-3x}{4}$?

...denominators to equivalent
...mon denominator.

...or into all the denominators,
...ator of that fraction; and all
...common denominator.

...ed exactly as in Arithmetic.

...rought to equivalent fractions having

...merators.

...nator.

...are $\frac{adn}{bdn}, \frac{cbn}{bdn}, \frac{mdb}{bdn}$.

...mon denominator.

...rs.

...minator.

...$\frac{acc}{bcc}, \frac{bbc}{bcc}, \frac{acb}{bcc}$, or $\frac{ac}{bc}, \frac{bb}{bc}, \frac{ab}{bc}$, in their lowest

...minator of one fraction is a multiple of
...ther, divide the greater denominator by
...the terms of that fraction which hath
...by the quotient, and the two fractions will
...on denominator. (47. Arith.)

...ions are $\frac{m}{a}$ and $\frac{2x}{ab}$?

[redacted]

The fractions brought to a common denominator are

$$\frac{20x}{36},\ \frac{30x}{36},\ \frac{-27x}{36}.$$

therefore $\frac{20x}{36}+\frac{30x}{36}-\frac{27x}{36}=\frac{20x+30x-27x}{36}=\frac{23x}{36}$ the answer.

Or the fractions $\frac{5}{9}, \frac{5}{6}$, and $\frac{3}{4}$ may be considered as the coefficients of x;

then $\frac{5}{9}+\frac{5}{6}-\frac{3}{4}=\frac{23}{36}$, and the answer is $\frac{23}{36}x$, as before.

3. Let $\frac{a}{a-c}$ and $\frac{a-c}{a}$ be added together.

$$\frac{a}{a-c}=\frac{a^2}{a^2-ac},\ \text{and}\ \frac{a-c}{a}=\frac{a^2-2ac+c^2}{a^2-ac};$$

then $\frac{a^2}{a^2-ac}+\frac{a^2-2ac+c^2}{a^2-ac}=\frac{2a^2-2ac+c^2}{a^2-ac}=2+\frac{c^2}{a^2-ac}$ the sum.

4. Suppose the fractions to be added together are $\frac{a-b}{c+d}$ and $\frac{a+b}{c-d}$.

Reducing the fractions to a common denominator, they take this form,

$$\frac{(a-b)\ (c-d)}{(c+d)\ (c-d)}\ \text{and}\ \frac{(c+d)\ (a+b)}{(c+d)\ (c-d)}.$$

Then $\frac{(a-b)\ (c-d)+(c+d)\ (a+b)}{(c+d)\ (c-d)}=\frac{2ac+2bd}{c^2-d^2}$ the sum required.

5. To add $\frac{x-y}{2}$, $\frac{x-y}{5}$, and $\frac{13x-13y}{10}$ together.

The three fractions may be denoted thus, $\frac{1}{2}(x-y)$, $\frac{1}{5}(x-y)$, and $\frac{13}{10}(x-y)$.

Then $\frac{1}{2}(x-y)+\frac{1}{5}(x-y)+\frac{13}{10}(x-y)$, or $(\frac{1}{2}+\frac{1}{5}+\frac{13}{10})(x-y)=(\frac{20}{10})(x-y)=2x-2y$ the sum of the three fractions.

To Subtract one fractional quantity from another.

66. Bring the fractions to a common denominator; then set the difference of the numerators over the common denominator for the answer, as in Vulgar Fractions.

Examples.

1. From $\frac{5ax}{16}$ take $\frac{3ax}{16}$?

$$\frac{5ax-3ax}{16}=\frac{2ax}{16}=\frac{ax}{8} \text{ the remainder.}$$

2. From $\frac{dx}{a-c}$ take $\frac{dx}{a+c}$?

The fractions with a common denominator are $\frac{dx\,(a+c)}{a^2-c^2}$ and $\frac{dx\,(a-c)}{a^2-c^2}$;

$$\text{then } \frac{dx\,(a+c)-dx\,(a-c)}{a^2-c^2}=\frac{dx\,(2c)}{a^2-c^2} \text{ the required difference.}$$

3. Let $\frac{a-z}{a\,(b-z)}$ be subtracted from $\frac{c+z}{d\,(b-z)}$?

$\frac{d\,(a-z)}{da\,(b-z)}$ and $\frac{a\,(c+z)}{da\,(b-z)}$ are the fractions with a common denominator.

$$\text{Therefore } \frac{a\,(c+z)-d\,(a-z)}{da\,(b-z)}=\frac{ac-da+z\,(a+d)}{da\,(b-z)} \text{ the remainder.}$$

4. Let the fraction $\frac{m+n}{a-b}$ be subtracted from the integer c. (61.)

The integer c brought to an equivalent fraction having the denominator $a-b$, is $\frac{ca-cb}{a-b}$ (64).

Then $\frac{ca-cb}{a-b}-\frac{m+n}{a-b}=\frac{ca-cb-m-n}{a-b}$ is the fraction denoting the difference.

To Multiply fractional quantities together.

67. MULTIPLY the numerators together for the numerator of the product, and the denominators together for its denominator, as in Vulgar Fractions.

Examples.

1. Required the product of $\frac{a}{b}$, $\frac{d}{c}$, and $\frac{m}{n}$?

$$\frac{a\times d\times m}{b\times c\times n} \text{ or } \frac{adm}{bcn} \text{ the product required.}$$

2. What is the product of $\frac{3x}{4}$, $\frac{3x-3}{5}$, and $\frac{10}{9}$?

$$\frac{3x\,(3x-3)\,10}{4\times5\times9}=\frac{90x^2-90x}{180}=\tfrac{1}{2}x^2-\tfrac{1}{2}x \text{ the product.}$$

3. What is the product of $\frac{ax}{by}$ and $\frac{b^2y^2}{ax^2}$?

$$\frac{ax\times b^2y^2}{by\times ax^2} \text{ or } \frac{ax\times by\times by}{by\times ax\times x}=\frac{by}{x} \text{ the product in its lowest terms.}$$

68. By resolving the terms of the fractions into their factors, the operation is frequently abridged:

4. Thus, to find the product of $\frac{a^2-x^2}{a+b}$, $\frac{ma^2-mb^2}{ax+x^2}$, and $\frac{bx^3}{mna-mnb}$?

$$\frac{a^2-x^2}{a+b}=\frac{(a+x)\,(a-x)}{a+b}.$$

$$\frac{ma^2-mb^2}{ax+x^2}=\frac{m\,(a+b)\,(a-b)}{x\,(a+x)}.$$

$$\frac{bx^3}{mna-mnb}=\frac{bx^2\times x}{mn\,(a-b)}.$$

Then, $\frac{(a+x)\,(a-x)\times m\,(a+b)\,(a-b)\times bx^2\times x}{(a+b)\times x\,(a+x)\times mn\,(a-b)}=\frac{(a-x)\times bx^2}{n}$ $=\frac{abx^2-bx^3}{n}$, the product, by rejecting the like factors in the numerator and denominator, as in reducing fractions to their lowest terms. (56.)

69. The product of an integral and fraction is found by multiplying the numerator of the fraction by the integral, as in Vulgar Fractions.

Thus $(a+x)\times x\times\frac{5}{x-4}=\frac{5ax+5x^2}{x-4}$ the product.

70. Powers of the same fraction are multiplied together by the addition of their exponents, in the same manner as integral quantities. (45.)

Thus $\left(\frac{b}{a}\right)^m\times\left(\frac{b}{a}\right)^n=\left(\frac{b}{a}\right)^{m+n}$, the product.

And $\frac{x^m}{c(a+x)^r} \times \frac{ax^{-n}}{b(a+x)^v} = \frac{ax^{m-n}}{cb(a+x)^{r+v}}$.

Also $\left(\frac{xz}{y}\right)^{n-1} \times \frac{xz}{y} = \left(\frac{xz}{y}\right)^n$ the product.

Division of fractional quantities.

71. INVERT the divisor, then proceed as in Multiplicat
This rule is the same as that for Vulgar Fractions in Arithm

Examples.

1. Divide $\frac{9ax}{16by}$ by $\frac{3a}{4b}$?

$\frac{4b}{3a} \times \frac{9ax}{16by}$, or $\frac{4b \times 3a \times 3x}{3a \times 4b \times 4y} = \frac{3x}{4y}$ the quotient (by rejecting th
factors in the numerator and denominator).

Or thus, $\frac{3a}{4b}\Big)\ \frac{9ax}{16by}\ \Big(\frac{3x}{4y}$ the quotient, as before.

2. Let $\frac{2ax+x^2}{c^3-x^3}$ be divided by $\frac{x}{c-x}$?

$\frac{2ax+x^2}{c^3-x^3} \times \frac{c-x}{x} = \frac{c-x}{c^3-x^3} \times \frac{(2a+x)x}{x} = \frac{c-x}{c^3-x^3} \times \frac{2a+x}{1}$
quotient.

Now $\frac{c-x}{c^3-x^3}$ in its lowest terms is $\frac{1}{x^2+cx+c^2}$;

therefore $\frac{1}{x^2+cx+c^2} \times \frac{2a+x}{1} = \frac{2a+x}{x^2+cx+c^2}$ is the quoti
its lowest terms.

3. Divide $x+\frac{2x}{x-3}$ by $x-\frac{2x}{x-3}$?

$$x+\frac{2x}{x-3}=\frac{x^2-x}{x-3};\ \text{and}\ x-\frac{2x}{x-3}=\frac{x^2-5x}{x-3}.$$

Then $\frac{x^2-x}{x-3} \times \frac{x-3}{x^2-5x} = \frac{x-1}{1} \times \frac{1}{x-5} = \frac{x-1}{x-5}$ the quotient.

4. Let $\frac{3a(c-x)^{\frac{1}{3}}}{2x-2z}$ be divided by $\frac{5x(c-x)^{\frac{1}{3}}}{2x+2z}$.

$$\frac{3a(c-z)^{\frac{1}{2}}}{2x-2z} \times \frac{2x+2z}{5x(c-z)^{\frac{1}{2}}} = \frac{3a(x+z)}{5x(x-z)} \text{ the quotient.}$$

5. Divide $\frac{ax^2}{c}$ by $\frac{dx^5}{b}$.

$$\frac{ax^2}{c} \times \frac{b}{dx^5} = \frac{ab}{cd} \times \frac{x^2}{x^5} = \frac{ab}{cd} \times \frac{1}{x^3} \text{ the quotient.}$$

But $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$ (50); therefore $\frac{ab}{cd} \times x^{-3}$ also denotes the quotient.

6. Let $\frac{a^{-m}}{b^{-m}}$ be divided by $\frac{a^m}{b^m}$.

$$\frac{a^m}{b^m}\Big)\, \frac{a^{-m}}{b^{-m}} \Big(\frac{a^{-2m}}{b^{-2m}} \text{ the quotient. (49).}$$

But $a^{-2m} = \frac{1}{a^{2m}}$, and $b^{-2m} = \frac{1}{b^{2m}}$ (50).

Therefore $\frac{a^{-2m}}{b^{-2m}} = \frac{b^{2m}}{a^{2m}}$, quotient.

To change a fractional quantity into a Series.

79. DIVIDE the numerator by the denominator, and extend the quotient to as many terms as may be thought necessary.

Examples.

1. Let the fraction $\frac{a}{a+x}$ be changed to a series.

$$a+x\Big)\, a \;\Big(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \text{, \&c.}$$
$$a+x$$

Now it is easy to perceive, that the next or 5*th* term of the quotient will be $+\frac{x^4}{a^4}$, and the 6*th* term $-\frac{x^5}{a^5}$, &c., and so on, alternately *plus* and *minus*: this is called *the law of continuation* of the series. And the sum of all the terms when infinitely continued is said to be equal to the fraction $\frac{a}{a+x}$. Thus we say the vulgar fraction $\frac{2}{3}$ when reduced to a decimal, is $= \cdot 6666$, &c. infinitely continued.

N. B. The terms in the quotient are found by dividing the remainders by (a) the first term of the divisor: thus, the first remainder $-x$ divided by a gives $-\frac{x}{a}$ the second term in the quotient; and the second remainder $+\frac{x^2}{a}$ divided by a gives $+\frac{x^2}{a^2}$ the third term, &c.

2. If the fraction is $\frac{a}{a-x}$, the series becomes wholly affirmative:

$$\text{Thus } a-x\,)\,a\,\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\&c.\right.$$

$$\begin{array}{l}
a-x \\
\hline
+x \\
+x-\frac{x^2}{a} \\
\hline
+\frac{x^2}{a} \\
+\frac{x^2}{a}-\frac{x^3}{a^2} \\
\hline
+\frac{x^3}{a^2}-\frac{x^4}{a^3} \\
\hline
+\frac{x^4}{a^3}
\end{array}$$

In this example, if $x < a$, the series is convergent, or the value of the terms continually diminish; but when $x > a$, it is said to diverge:

To explain this by numbers, let $a = 3$, and $x = 2$:

Then.......... $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}$, &c.

corresponding values $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}$, &c. where the fractions or terms of the series grow less and less, and therefore the farther they are extended the more they converge or approximate to 0, which is supposed to be the last term or limit.

But if $a = 2$, and $x = 3$.

$$1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3}, \text{ \&c.}$$

corresponding values $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$, &c. in which the terms become larger and larger. This is called a diverging series.

If $x = 1$, and $a = 1$ in *Examp.* 1.

Then $\frac{a}{a+x} = 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$, &c.

will be $\frac{1}{1+1} = 1 - 1 + 1 - 1$, &c.

Now because $\frac{1}{1+1} = \frac{1}{2}$, it has been said that $1 - 1 + 1 - 1 + 1$, &c. *infinitely continued* is $= \frac{1}{2}$: —— a singular conclusion, when it is perceivable from the terms themselves, that their sum must necessarily be either 0, or $+ 1$, to whatever extent the division is supposed to be continued.

The real quotient however, results from the fractional part, which (by the division) is always $+ \frac{1}{2}$ when the sum of the terms is 0; and $- \frac{1}{2}$ when the sum is $+ 1$: consequently $\frac{1}{2}$ is the true quotient in the former case; and $1 - \frac{1}{2}$ in the other.

3. Let the fraction $\frac{a}{c-x}$ be expanded into a series.

$$c - x)\,a\,\left(\quad \frac{a}{c} + \frac{ax}{c^2} + \frac{ax^2}{c^3} + \frac{ax^3}{c^4} +, \text{ \&c. quotient.}\right)$$

$$\text{or } \frac{a}{c}\left(1 + \frac{x}{c} + \frac{x^2}{c^2} + \frac{x^3}{c^3} +, \text{ \&c.}\right)$$

$$\begin{array}{l}
a - \frac{ax}{c} \\
\hline
\quad + \frac{ax}{c} \\
\quad + \frac{ax}{c} - \frac{ax^2}{c^2} \\
\hline
\qquad + \frac{ax^2}{c^2} \\
\qquad + \frac{ax^2}{c^2} - \frac{ax^3}{c^3} \\
\hline
\qquad\quad + \frac{ax^3}{c^3} \\
\qquad\quad + \frac{ax^3}{c^3} - \frac{ax^4}{c^4} \\
\hline
\qquad\qquad + \frac{ax^4}{c^4}
\end{array}$$

By substituting other quantities for a, c, and x, in the quotient $\frac{a}{c} + \frac{ax}{c^2} + \frac{ax^2}{c^3}$, &c., different series may be produced.

Let $a = 3$, $c = 10$, and $x = 1$:

then $\frac{a}{c-x} = \frac{a}{c} + \frac{ax}{c^2} + \frac{ax^2}{c^3} + \frac{ax^3}{c^4} +$ &c.

will be $\frac{3}{10-1} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} +$ &c. which is the same series as the decimal answering to $\frac{1}{3}$ or $\frac{3}{10-1}$:

for $\frac{1}{3} = .3333$, &c. $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$ &c.

4. To reduce $\frac{a^2 + x^8}{a^4 + x^4}$ to a series.

$$a^4 + x^4)\, a^2 + x^8 \left(\frac{1}{a^2} - \frac{x^4}{a^6} + \frac{x^8}{a^{10}} - \text{\&c.}\right.$$

$$\begin{array}{r}
a^2 + \frac{x^4}{a^2} \\
\hline
-\frac{x^4}{a^2} + x^8 \\
-\frac{x^4}{a^2} - \frac{x^8}{a^6} \\
\hline
+\frac{x^8}{a^6} + x^8 \\
+\frac{x^8}{a^6} + \frac{x^{12}}{a^{10}} \\
\hline
-\frac{x^{12}}{a^{10}} + x^8
\end{array}$$

and the 4th term will be $-\frac{x^{12}}{a^{14}}$, the 5th $+\frac{x^{16}}{a^{18}}$, and so on.

73. It may be worth observing, that the same fraction will give different series if the order of the terms in its denominator is changed.

Thus taking Examp. 1.

$$\frac{a}{a+x} = 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \text{\&c.}$$

$$\text{But } \frac{a}{x+a} = \frac{a}{x} - \frac{a^2}{x^2} + \frac{a^3}{x^3} - \text{\&c.}$$

The two quotients however, will always be equal when the remainders are taken into the account.

OF EQUATIONS.

74. When the symbol $=$ is placed between two quantities that are equal, but differently expressed, it is called an Equation. (5.)

Thus, $7+2$ and $10-1$ are equal:

And $7+2=10-1$ is an equation denoting the equality of $7+2$ and $10-1$.

Also $x=c-d$ is an equation which shews that the quantity x is equal to the difference of the quantities c and d.

Equations take their denominations from the highest power of the unknown quantity which they involve: For example, if that quantity is of one dimension only, it is called a Simple Equation; if of two dimensions, a Quadratic; when of three, a Cubic, &c.

Thus, if x be the unknown quantity;
then $10x-7=4x+a$ is a simple equation.
$ax^2-2x=cd$ a quadratic equation.
$x^3+x^2-cx=a^2b$ a cubic equation.
$x^4+ax^3+dx=abc$ a biquadratic.
$x^m-x^{m-1}+cx^{m-2}=ab$ an equation of m dimensions.

75. When x, y, z, &c. or other symbols, are put to denote unknown quantities, it is from certain given relations they have to each other, and to such as are known, that equations are derived. And to resolve or reduce an equation, is to discover the value of the unknown quantity which it involves; but no rule has yet been found sufficiently general for that purpose in all cases. The resolution of Simple and Quadratic Equations however, principally depend on the following obvious

AXIOMS:

1. *If equal quantities are added to equal quantities, the sums are equal.*

2. If equal quantities are subtracted from equal quantities, the remainders are equal.

3. If equal quantities are multiplied by equal quantities, the products are equal.

4. If equal quantities are divided by equal quantities, the quotients are equal.

5. If two or more quantities are each equal to another quantity, those quantities are equal.

6. Like powers of equal quantities are equal.

7. Like roots of equal quantities are equal.

8. A whole quantity is equal to all its parts taken together.

76. RESOLUTION OF SIMPLE EQUATIONS.

Examples.

1. GIVEN $x-7=22$; to find the value of x.

By adding 7 to each side of the equation

we have $x-7+7=22+7$ (*Ax.* 1.)
or $x=29$ the value required.

Therefore, any quantity may be transposed from one side of an equation to the other, by changing its sign.

For $x-7=22$:
And $x=22+7$, where 7 is transposed from one side of the equation to the other, and its sign changed from $-$ to $+$.

In like manner, if $x-a=b$
then $x=b+a$.

77. Hence also it appears, that if all the signs in an equation are changed, the equality still subsists:

Thus, changing the signs of $x = b + a$
And it becomes.............. $-x = -b - a$.
Now by transposing a... $-x + a = -b$
Next, transpose x......... $a = -b + x$
Lastly, transpose b.... then $a + b = x$, as before.

2. If $x + 8 = 18$; what is the value of x?

Let 8 be subtracted from each side of the equation;

Then $x + 8 - 8 = 18 - 8$ (*Ax.* 2.)
And $x = 10$ the answer.

Or thus. Since $x + 8 = 18$; then $x = 18 - 8 = 10$, the answer.

3. Given $\frac{x + 7}{3} = 13$; to find x.

If each side of the equation be multiplied by 3,

we have $\frac{(x + 7) \times 3}{3} = 39$ (*Ax.* 3.)
or $x + 7 = 39$. (56).
And $x = 39 - 7 = 32$ the answer.

In like manner, if $\frac{x + a}{c} = b$,

then $\frac{(x + a) \times c}{c} = bc$ (*Ax.* 3.)
or $x + a = bc$ (56)
And $x = bc - a$.

4. Suppose $3x + 14 = 50$; what is the value of x?

$3x + 14 = 50$
then ... $3x = 50 - 14$
or ... $3x = 36$
And dividing by 3 gives ... $x = \frac{36}{3}$ (*Ax.* 4.)
or .... $x = 12$ the value required.

And if......... $ax + b = c$
then ... $ax = c - b$
And dividing by the coefficient a... $x = \frac{c - b}{a}$ the value of x.

5. If $\frac{7x-45}{4}=x$; what is the value of x?

By multiplying each side of the equation by 4

we get .... $\frac{(7x-45)\times 4}{4}=4x$

or.......... $7x-45=4x$

by transposing 45...... $7x=4x+45$

Subtracting $4x$ from each side ...... gives $3x=45$

And dividing by the coefficient 3 (*Ax.* 4.)... $x=15$ the answer.

6. Let $\frac{x+5}{2}-2x=\frac{x}{3}$: required x?

In order to clear the equation from fractions, let both sides be first multiplied by 3,

and we have .... $\frac{3x+15}{2}-6x=x$

and multiplying again by 2....gives $3x+15-12x=2x$

But $3x-12x=-9x$, therefore we have $15-9x=2x$

And transposing $9x$......gives $15=11x$

And $x=\frac{15}{11}=1\frac{4}{11}$ — the answer.

Or thus:

The given equation is $\frac{x+5}{2}-2x=\frac{x}{3}$

Which is the same as ...... $\frac{1}{2}x+2\frac{1}{2}-2x=\frac{1}{3}x$

Now transposing $2\frac{1}{2}$ .... gives $\frac{1}{2}x-2x=\frac{1}{3}x-2\frac{1}{2}$

But $\frac{1}{2}x-2x=-1\frac{1}{2}x$ ... therefore $-1\frac{1}{2}x=\frac{1}{3}x-2\frac{1}{2}$

And transposing $\frac{1}{3}x$ .... gives $-1\frac{1}{2}x-\frac{1}{3}x=-2\frac{1}{2}$

or $-1\frac{5}{6}x=-2\frac{1}{2}$

And dividing by the coefficient $-1\frac{5}{6}$ gives $x=1\frac{4}{11}$ as before.

7. Given $\frac{5x+7x-9x}{4}=100$; to find x

Multiplying both sides by 4 gives $5x+7x-9x=400$

or $12x-9x=400$

or $3x=400$

And $x=\frac{400}{3}=133\frac{1}{3}$ Ans.

Or thus.

$$5x+7x-9x=(5+7-9)x$$

therefore $(5+7-9)\ x=400$

then (*Ax.* 4.) $\qquad x=\dfrac{400}{5+7-9}$.

And according to this last method the value of the unknown quantity is denoted when it has literal coefficients:

For let $\dfrac{ax+bx-cx}{4}=m$

Then $ax+bx-cx=4m$

or $(a+b-c)\ x=4m$

Therefore $\qquad x=\dfrac{4m}{a+b-c}$. (*Ax.* 4).

8. Given $\dfrac{a}{x}-\dfrac{b}{x}+\dfrac{c}{x}=d$; to find x.

Both sides of the equation multiplied by x gives $a-b+c=dx$

And dividing both sides by d ...... $\dfrac{a-b+c}{d}=x$.

9. Given $\dfrac{3x}{17-4x}=19$; to find x.

Both sides multiplied by $17-4x$ gives $3x=19\ (17-4x)$

or $3x=323-76x$

And transposing $76x$...... $3x+76x=323$

or........ $79x=323$

And........ $x=\dfrac{323}{79}=4\dfrac{7}{79}$ Ans.

10. Given $\dfrac{ax+bx}{x-c}=d-n$; to find x.

Let both sides of the equation be multiplied by $x-c$;

Then........... $ax+bx=(d-n)\ (x-c)$

or $ax+bx=cn-dc+dx-nx$

And transposing dx and nx... $ax+bx+nx-dx=cn-dc$

or $(a+b+n-d)\ x=cn-dc$

Therefore............ $x=\dfrac{cn-dc}{a+b+n-d}$ Ans.

11. Let $a^2-x^2=bx+ba$; to find x.

When each side of the Equation is resolved into its factors,

we have......... $(a+x)\ (a-x)=b\ (x+a)$

Then dividing by $a+x$ gives $a-x=b$

And transposing x and b, the result is... $a-b=x$.

12. Given $\frac{ax^2+ac^2}{a+x}=ax+b^2$; to find x.

Multiplying by $a+x$ gives... $ax^2+ac^2=(ax+b^2)\ (a+x)$

or $ax^2+ac^2=a^2x+ax^2+b^2x+ab^2$

And subtracting ax^2 from each side,...... $ac^2=a^2x+b^2x+ab^2$

And transposing ab^2............. $ac^2-ab^2=a^2x+b^2x$

Then dividing by the coefficient a^2+b^2 gives $\frac{ac^2-ab^2}{a^2+b^2}=x$.

13. Let $a-x=\frac{x^2}{a-x}$; to find x.

If each side be multiplied by $a-x$, we have $a^2-2ax+x^2=x^2$

And subtracting x^2 from each side gives $a^2-2ax=0$

by transposing $2ax$......... $a^2=2ax$

And dividing by $2a$ gives $\frac{a^2}{2a}=\frac{a}{2}=x$.

Of reducing Simple Equations when the values of two unknown quantities are required.

78. If two independent equations are given, which involve two unknown quantities, find two expressions for one of them, one from each equation, by the foregoing methods; those expressions being put equal, an equation will arise with only one unknown quantity in it, whose value may be found as before.

Examples.

1. Given $3x-4y=1$

$7x+3y=64$. To find x and y.

First,.............. $3x-4y=1$

By transposition $3x=1+4y$

therefore $x=\frac{1+4y}{3}$.

Secondly, ... $7x+3y=64$

By transposition...... $7x=64-3y$

and... $x=\dfrac{64-3y}{7}$

$$\text{Therefore } \frac{1+4y}{3}=\frac{64-3y}{7} \quad (Ax.\ 5).$$

This equation cleared of fractions (Examp. 6.) gives $7+28y=192-9y$.

And by transposition $37y=185$

or $y=\dfrac{185}{37}=5.$

And substituting 5 for y gives $x=\dfrac{1+4y}{3}=\dfrac{1+20}{3}=7$ the value of x.

79. But it will frequently be more expeditious to multiply, or divide the equations by such numbers or quantities as will make the term which contains one of the unknown quantities the same in both equations; then by adding, or subtracting the equations, as the case may require, that term will be exterminated.

Thus, if the first equation in the preceding example be multiplied by 7, and the second by 3,

we have............ $21x-28y=7$

and .... $21x+9y=192$

The upper subtracted from the lower gives $37y=185$ ($Ax.$ 2.).

Therefore ...... $y=\dfrac{185}{37}=5$, as before.

Again, if we would exterminate y, multiply the first equation by 3, and the second by 4:

Then................ $9x-12y=3$

and $28x+12y=256$

The sum is...... $37x \quad =259$ ($Ax.$ 1.)

whence $x=\dfrac{259}{37}=7$, as before.

2. Let $ax+by=c$

$dx+gy=p$. To find x and y.

The first equation multiplied by d, and the second by a.

$$\text{gives} \quad dax + dbx = dc$$
$$\text{and} \quad dax + agx = ap$$

And subtracting the lower from the upper $\quad dbx - agx = dc - ap$ (*Ax.* 2.)

And dividing by the coefficient $db - ag$.... ... $\quad x = \dfrac{dc - ap}{db - ag}$.

But to exterminate x, let the first equation be multiplied by g, and the second by b;

$$\text{Then ... .} \quad gaz + gbx = gc$$
$$\text{and ....} \quad bdz + gbx = bp$$

Subtracting the under from the upper...... $\quad gaz - bdz = gc - bp$

$$\text{whence.........} \quad z = \frac{gc - bp}{ga - bd}$$

Remark. In this example there is nothing to indicate which of the two equations is greatest, and consequently we are at liberty to subtract the upper from the lower; in that case $x = \dfrac{ap - dc}{ag - db}$; and $z = \dfrac{bp - gc}{bd - ga}$. The same expressions however, result from the equations by changing their signs:

For if $dbx - agx = dc - ap$, then (77) $agx - dbx = ap - dc$, whence $x = \dfrac{ap - dc}{ag - db}$,

and $gaz - bdz = gc - bp$, then $bdz - gaz = bp - gc$, and $z = \dfrac{bp - gc}{bd - ga}$.

But such expressions are usually set down thus $x = \dfrac{ap \backsim dc}{ag \backsim db}$, and $z = \dfrac{bp \backsim gc}{bd \backsim ga}$.

3. Given $\dfrac{7x}{2} + z = 52$

$$z - \frac{7x}{16} = 16. \quad \text{To find } x \text{ and } z.$$

From the second equation we have $\quad z = 16 + \dfrac{7x}{16}$

which substituted for z in the first, gives... $\dfrac{7x}{2} + 16 + \dfrac{7x}{16} = 52$

$$\text{or} \quad \frac{7x}{2} + \frac{7x}{16} = 52 - 16 = 36$$

$$\text{Whence} \quad 56x + 7x = 36 \times 16 = 576$$
$$\text{or} \quad 63x = 576$$
$$\text{and} \quad x = 9\tfrac{1}{7}$$

Now put $9\frac{1}{7}$ for x, and we have $z = 16 + \dfrac{7x}{16} = 16 + \dfrac{7 \times 9\frac{1}{7}}{16} = 20$ the value of z.

4. Given $x^2 - z^2 = 3876$

$x + z = 102$. To find x and z.

Divide the first equation by the second... $\dfrac{x^2 - z^2}{x + z} = \dfrac{3876}{102}$ (*Ax.* 4.)

which, by actual division............ gives $x - z = 38$

whence $x = 38 + z$

And putting $38 + z$ for x in the second equation... $38 + z + z = 102$

or $2z = 102 - 38$

Therefore $z = 32$

And $x = 38 + z = 38 + 32 = 70$

When Three Equations are given, involving Three unknown quantities.

80. If the three unknown quantities are found in all the equations, find three expressions for one of them, one from each equation, then compare the first expression with the second, and also with the third, by which means that quantity will be exterminated, and the equations reduced to two; which may be resolved as in the preceding articles. But the method in Art. 79, will generally be found the least tedious.

Examples.

1. Given $x + y = 19$

$x + z = 20$

$z + y = 21$. To find x, y, and z.

Subtracting the first equation from the second gives $z - y = 1$

To this remainder add the third equation, and we have $2z = 22$

whence $z = 11$.

Now putting 11 for z in the 2*d*. and 3*d*. equations,

we have $11 + y = 21$, and $x + 11 = 20$; whence $y = 10$; and $x = 9$.

2. Let $x + y + z = 10$

$x + 2y + 3z = 23$

$2x + 3y + 5z = 38$. To find x, y, and z.

By the first equation $\quad x = 10 - y - z$

From the second.......... $\quad x = 23 - 2y - 3z$

therefore $\quad 10 - y - z = 23 - 2y - 3z$

whence $\quad y = 13 - 2z.$

Next, by the 3*d.* equation $\quad x = \dfrac{38 - 3y - 5z}{2}$

And from the first ...... $\quad x = 10 - y - z$

therefore... $\quad 10 - y - z = \dfrac{38 - 3y - 5z}{2}$

whence $\quad 20 - 2y - 2z = 38 - 3y - 5z$

and......... $\quad y = 18 - 3z$

But......... $\quad y = 13 - 2z$

therefore $\quad 18 - 3z = 13 - 2z$

and.... $\quad z = 5.$

Now substituting 5 for z in the two first equations, we have $x + y = 5$, and $x + 2y = 8$; whence $x = 2$, and $y = 3$.

Or thus:

Subtracting the 1*st.* equation from the 2*d.* gives $y + 2z = 13$

and double the 2*d.* from the 3*d.* gives $-y - z = 38 - 46 = -8$

or $\quad y + z = 8$ (77)

whence. .. $\quad y = 8 - z$

But... $\quad y + 2z = 13$

and... $\quad y = 13 - 2z$

Therefore $\quad 8 - z = 13 - 2z$

which gives $\quad z = 5$, as before.

3. Given $x + \dfrac{y}{2} + \dfrac{z}{3} = 32$

$\dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = 15$

$\dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{6} = 12.$ To find x, y, and z.

The equations cleared of fractions become

$$6x + 3y + 2z = 192$$
$$20x + 15y + 12z = 900$$
$$15x + 12y + 10z = 720.$$

To exterminate y (for example) let the second equation be subtracted from 5 times the first, and the third equation from 4 times the first:

nes the first	$30x + 15y + 10z = 960$	
the second	$20x + 15y + 12z = 900$	
remainder	$10x \quad\quad - 2z = 60$	
nes the first	$24x + 12y + \quad 8z = 768$	
the third	$15x + 12y + 10z = 720$	
remainder	$9x \quad\quad - 2z = 48$	

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iven $ax + by + cz = m$

$dx + gy + hz = n$

$fx + ry + sz = p$. To find x, y, and z in terms of func-
ther quantities.

with exterminating z, let the first equation be multiplied by
s (the coefficients of z in the other equations), the second
s, and the third by hc;

have $hsax + hsby + hscz = hsm$

$csdx + csgy + cshz = csn$

$hcfx + hcry + hcsz = hcp$: where the coefficients of z

acting the first of these equations from the second, and also
l, the results will be

$dx - hsax + csgy - hsby = csn - hsm$

$fx - hsax + hcry - hsby = hcp - hsm$

$dx - hax + cgy - hby = cn - hm$

$fx - sax + cry - sby = cp - sm$ (by dividing the first by s,
by h).

. of these equations... $x = \frac{cn - hm - cgy + hby}{cd - ha}$.

om the second...... $x = \frac{cp - sm - cry + sby}{cf - sa}$.

$$\frac{cn - hm - cgy + hby}{cd - ha} = \frac{cp - sm - cry + sby}{cf - sa},$$

duced gives $y = \frac{fhm - dsm + san - cfn + cdp - hap}{fhb - dsb + sag - cfg + cdr - har}$. (A)

$$\text{or } y = \frac{(fh - ds)m + (sa - cf)n + (cd - ha)p}{(fh - ds)b + (sa - cf)g + (cd - ha)r}. \text{ (B)}$$

From this value of y, the expressions for x and z may be obtained without substitution, or repeating the process:

For it is evident from *Ex.* 2. (Art. 79.) that the expressions for x and z will have the same denominator as this for y. And since the coefficients of m, n, and p, in the expression (B) are respectively the same, and have the same signs as those of b, g, and r in the denominator (these latter being the coefficients of y in the given equations), it is manifest from analogy that m, n, and p will have the same respective coefficients and signs in the required numerators as a, d, and f; and c, h, and s have in the denominator: a, d, and f; and c, h, and s being the coefficients of x and z in the given equations, following the same order as those of y.

Now the given denominator (A) when resolved into factors exhibiting those coefficients

will be $\quad (gs-rh)\,a+(rc-bs)\,d+(bh-cg)\,f$, for x:

and $\quad (dr-gf)\,c+(fb-ra)\,h+(ag-bd)\,s$, for z.

$$\text{Therefore } x=\frac{(gs-rh)\,m+(rc-bs)\,n+(bh-cg)\,p}{(gs-rh)\,a+(rc-bs)\,d+(bh-cg)\,f}.$$

$$-\text{ And } \quad z=\frac{(dr-gf)\,m+(fb-ra)\,n+(ag-bd)\,p}{(dr-gf)\,c+(fb-ra)\,h+(ag-bd)\,s}.$$

Instead of subtracting the first equation from the other two, a contrary order might have been adopted; for that reason, perhaps, the symbol $\backsim$ would be more proper than the negative sign in the final expressions: See Examp. 2. (79.)

The last example is sufficient to direct the process, when four or more unknown quantities are concerned. But methods of reduction different from those we have given, will frequently present themselves in practice.

81. When the number of equations is less than the number of unknown quantities they involve, the problem is said to be indeterminate or unlimited. Thus if $x+y=10$, then x and y may be any two numbers whose sum is 10. Or suppose $x-y=6$, and $z-x=9$, in which case y may be any number whatever; and consequently the values of the three unknown quantities will be indefinite. The like must also take place

when the number of equations and unknown quantities are the same, if one of the equations is deducible from the others:

Thus in *Ex.* 4, let $a=1$, $b=2$, $c=3$, $d=4$, $g=5$, $h=6$, $f=7$, $r=8$, $s=9$, $m=20$, $n=47$, and $p=74$:

Then the three equations become
$$x+2y+3z=20$$
$$4x+5y+6z=47$$
$$7x+8y+9z=74.$$

Now substituting those numbers in the expressions for the values of x, y, and z, the numerators and denominators become $=0$, or the expressions vanish. The reason perhaps is not obvious at first sight; but on examining the equations we find, that double the second is equal to the sum of the other two, and consequently there are only *two* independent equations. Also, with these numeral coefficients, $2n$ must be $=m+p$, otherwise the equations are incongruous.

82. Sometimes equations may involve an absurdity; as when $x-y=8$, and $x+y=7$; for it is impossible that the difference of two quantities should be greater than their sum.

The young Algebraist will now perceive, that the art of resolving Equations consists in bringing each of the unknown quantities on one side of an equation having known quantities on the other.

OF RATIOS AND PROPORTIONS.

83. THE relation or proportion which two quantities of the same kind bear to each other in respect of magnitude, is called the Ratio of those quantities: this is found by considering what part or parts one is of the other, or how often one is contained in the other.

Thus if $12a$ and $4a$ are the two quantities, then by comparing them, we find their magnitudes such, that the former

contains the latter 3 times, and in common language we say it is 3 times as big, because 4 is contained 3 times in 12: the quantities therefore appear to have the same ratio or proportion, the greater to the less, as 3 has to 1. Hence it is, that the equality of two Ratios constitutes Proportion.

The terms of the two equal ratios are sometimes set down thus:

$12a : 4a = 3 : 1$; *viz.* the ratio of $12a$ to $4a$ is equal to that of 3 to 1.

Or thus, $12a : 4a :: 3 : 1$, which may be read thus — $12a$ bears the same proportion to $4a$ as 3 does to 1; or, As $12a$ is to $4a$, so is 3 to 1.

The 4th term 1 is called a *4th* proportional to the other three.

The Antecedents of the two ratios are $12a$ and 3, and their consequents $4a$ and 1.

81. The terms of the ratio $3 : 1$ are like submultiples of $12a : 4a$, the divisor being $4a$. But any other like submultiples or multiples of $12a$ and $4a$ will have the same ratio or proportion; for $\frac{12a}{4a} = \frac{3}{1} = \frac{6a}{2a} = \frac{3a}{a} = \frac{24a}{8a} = \frac{15ab}{5ab}$, &c. where each numerator has the same ratio to its denominator as $12a$ has to its denominator $4a$. This is evident from the nature of fractions.

Hence $12a : 4a :: 3 : 1 :: 6a : 2a :: 3a : a :: 24a : 8a :: 15ab : 5ab$, &c., are a rank of proportionals.

83. The fraction $\frac{12a}{4a}$ or the antecedent divided by the consequent, is by many authors, called the magnitude or quantity of

ratio of $12a$ to $4a$; and sometimes its measure, or exponent. But writers differ on this subject. Some will have the ratios (when considered as magnitudes,) to be the exponents of the powers of their terms. Thus if 1 denotes the magnitude of the ratio of a to b, that of the ratio of a^2 to b^2 will be 2; that of a^3 to b^3 will be 3, &c. these indices or ratios are therefore analogous to the scale of Logarithms; and consequently the ratio of equality will be 0; for if $a = b$, then $\frac{a}{b} = 1$, whose logarithm is 0. Dr. Barrow however, says, "Reason" (Ratio) "is not quantity;" and maintains that the magnitude of a simple Ratio cannot be expressed in numbers: but if its quantity be referred to the fraction formed by the two terms, then he makes the magnitude of the ratio of equality greater than 0. See his *Math. Lectures.*

If the terms of the ratio however, are commensurable, the ratio itself may be expounded by the quotient arising from the division of the antecedent by the consequent, as in Arithmetic. Thus in the progression 128, 64, 32, 16, 8, &c. we call 2 the common ratio of the terms, $\left(\text{for } \frac{128}{64} = 2, \text{ or } \frac{64}{32} = 2, \text{\&c}\right)$. Also in the progression ar^3, ar^2, ar, a, $\frac{a}{r}$, $\frac{a}{r^2}$, &c. r is the common ratio. But when the terms are incommensurable, the ratio cannot be exhibited in this manner: Thus it is impossible to find in numbers the exact ratio, or proportion which the square root of 1 has to the square root of 2. And in comparing geometrical magnitudes, the ratios are not set down fractionwise, except the terms are supposed to be subjected to some common measure.

86. When the terms of two ratios are commensurable, the greatest of the two may be found thus: Let $7a$ and $8a$ be the terms of one ratio, and $8b$ and $9b$ the terms of the other, in *the same order*;

The first equation multiplied by d, and the second by a.

$$\text{gives} \quad dax + dbx = dc$$
$$\text{and} \quad dax + agx = ap$$

And subtracting the lower from the upper $\quad dbx - agx = dc - ap$ (*Ax.* 2.)

And dividing by the coefficient $db - ag$ .... $\quad x = \dfrac{dc - ap}{db - ag}$.

But to exterminate x, let the first equation be multiplied by g, and the second by b;

$$\text{Then} \ldots\ldots\ldots \quad gaz + gbx = gc$$
$$\text{and} \ldots\ldots\ldots \quad bdz + gbx = bp$$

Subtracting the under from the upper...... $\quad gaz - bdz = gc - bp$

$$\text{whence} \ldots\ldots\ldots \quad z = \frac{gc - bp}{ga - bd}$$

Remark. In this example there is nothing to indicate which of the two equations is greatest, and consequently we are at liberty to subtract the upper from the lower; in that case $x = \dfrac{ap - dc}{ag - db}$; and $z = \dfrac{bp - gc}{bd - ga}$. The same expressions however, result from the equations by changing their signs:

For if $dbx - agx = dc - ap$, then (77) $agx - dbx = ap - dc$, whence $x = \dfrac{ap - dc}{ag - db}$.

and $gaz - bdz = gc - bp$, then $bdz - gaz = bp - gc$, and $z = \dfrac{bp - gc}{bd - ga}$.

But such expressions are usually set down thus $x = \dfrac{ap \backsim dc}{ag \backsim db}$, and $z = \dfrac{bp \backsim gc}{bd \backsim ga}$.

3. Given $\dfrac{7x}{2} + z = 52$

$z - \dfrac{7x}{16} = 16$. To find x and z.

From the second equation we have $\quad z = 16 + \dfrac{7x}{16}$

which substituted for z in the first, gives... $\dfrac{7x}{2} + 16 + \dfrac{7x}{16} = 52$

$$\text{or} \quad \frac{7x}{2} + \frac{7x}{16} = 52 - 16 = 36$$

$$\text{Whence} \quad 56x + 7x = 36 \times 16 = 576$$
$$\text{or} \quad 63x = 576$$
$$\text{and} \quad x = 9\tfrac{1}{7}$$

Now put $9\frac{1}{7}$ for x, and we have $z = 16 + \dfrac{7x}{16} = 16 + \dfrac{7 \times 9\frac{1}{7}}{16} = 20$ the value of z.

4. Given $x^2 - z^2 = 3876$
$x + z = 102$. To find x and z.

Divide the first equation by the second... $\frac{x^2 - z^2}{x+z} = \frac{3876}{102}$ (*Ax.* 4.)

which, by actual division............ gives $x - z = 38$

whence $x = 38 + z$

And putting $38 + z$ for x in the second equation... $38 + z + z = 102$

or $2z = 102 - 38$

Therefore $z = 32$

And $x = 38 + z = 38 + 32 = 70$

When Three Equations are given, involving Three unknown quantities.

80. If the three unknown quantities are found in all the equations, find three expressions for one of them, one from each equation, then compare the first expression with the second, and also with the third, by which means that quantity will be exterminated, and the equations reduced to two; which may be resolved as in the preceding articles. But the method in Art. 79, will generally be found the least tedious.

Examples.

1. Given $x + y = 19$
$x + z = 20$
$z + y = 21$. To find x, y, and z.

Subtracting the first equation from the second gives $z - y = 1$
To this remainder add the third equation, and we have $2z = 22$
whence $z = 11$.

Now putting 11 for z in the 2*d*. and 3*d*. equations,
we have $11 + y = 21$, and $x + 11 = 20$; whence $y = 10$; and $x = 9$.

2. Let $x + y + z = 10$
$x + 2y + 3z = 23$
$2x + 3y + 5z = 38$. To find x, y, and z.

By the first equation $\quad x = 10 - y - z$

From the second.......... $\quad x = 23 - 2y - 3z$

therefore $\quad 10 - y - z = 23 - 2y - 3z$

whence $\quad y = 13 - 2z.$

Next, by the 3*d.* equation $\quad x = \dfrac{38 - 3y - 5z}{2}$

And from the first ...... $\quad x = 10 - y - z$

therefore... $\quad 10 - y - z = \dfrac{38 - 3y - 5z}{2}$

whence $\quad 20 - 2y - 2z = 38 - 3y - 5z$

and......... $\quad y = 18 - 3z$

But......... $\quad y = 13 - 2z$

therefore $\quad 18 - 3z = 13 - 2z$

and.... $\quad z = 5.$

Now substituting 5 for z in the two first equations, we have $x + y = 5$, and $x + 2y = 8$; whence $x = 2$, and $y = 3$.

Or thus:

Subtracting the 1*st.* equation from the 2*d.* gives $y + 2z = 13$

and double the 2*d.* from the 3*d.* gives $-y - z = 38 - 46 = -8$

or $\quad y + z = 8$ (77)

whence. .. $\quad y = 8 - z$

But... $\quad y + 2z = 13$

and... $\quad y = 13 - 2z$

Therefore $\quad 8 - z = 13 - 2z$

which gives $\quad z = 5$, as before.

3. Given $x + \dfrac{y}{2} + \dfrac{z}{3} = 32$

$\dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = 15$

$\dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{6} = 12.$ To find x, y, and z.

The equations cleared of fractions become

$$6x + 3y + 2z = 192$$
$$20x + 15y + 12z = 900$$
$$15x + 12y + 10z = 720.$$

To exterminate y (for example) let the second equation be subtracted from 5 times the first, and the third equation from 4 times the first:

$$\begin{array}{rl}
\text{5 times the first} & 30x + 15y + 10z = 960 \\
\text{the second} & 20x + 15y + 12z = 900 \\ \hline
\text{remainder} & 10x \qquad - 2z = 60
\end{array}$$

$$\begin{array}{rl}
\text{4 times the first} & 24x + 12y + 8z = 768 \\
\text{the third} & 15x + 12y + 10z = 720 \\ \hline
\text{remainder} & 9x \qquad - 2z = 48
\end{array}$$

Then the last remainder subtracted from the first gives $x = 60 - 48 = 12$.

Now substituting 12 for x in the first remainder, we have $10 \times 12 - 2z = 60$, whence $z = 30$; then from the first equation y is found $= 20$.

4. Given $ax + by + cz = m$
$dx + gy + hz = n$
$fx + ry + sz = p$. To find x, y, and z in terms or functions of the other quantities.

If we begin with exterminating z, let the first equation be multiplied by the product hs (the coefficients of z in the other equations), the second equation by cs, and the third by hc;

and we have $hsax + hsby + hscz = hsm$
$csdx + csgy + cshz = csn$
$hcfx + hcry + hcsz = hcp$: where the coefficients of z are the same.

Now subtracting the first of these equations from the second, and also from the third, the results will be

$$csdx - hsax + csgy - hsby = csn - hsm$$
$$hcfx - hsax + hcry - hsby = hcp - hsm$$
or $$cdx - hax + cgy - hby = cn - hm$$
and $$cfx - sax + cry - sby = cp - sm$$ (by dividing the first by s, and the latter by h).

From the first of these equations... $x = \dfrac{cn - hm - cgy + hby}{cd - ha}$.

And from the second...... $x = \dfrac{cp - sm - cry + sby}{cf - sa}$.

Therefore $\dfrac{cn - hm - cgy + hby}{cd - ha} = \dfrac{cp - sm - cry + sby}{cf - sa}$.

Which reduced gives $y = \dfrac{fhm - dsm + san - cfn + cdp - hap}{fhb - dsb + sag - cfg + cdr - har}$, (A)

or $y = \dfrac{(fh - ds)m + (sa - cf)n + (cd - ha)p}{(fh - ds)b + (sa - cf)g + (cd - ha)r}$. (B)

From this value of y, the expressions for x and z may be obtained without substitution, or repeating the process:

For it is evident from *Ex.* 2. (Art. 79.) that the expressions for x and z will have the same denominator as this for y. And since the coefficients of m, n, and p, in the expression (B) are respectively the same, and have the same signs as those of b, g, and r in the denominator (these latter being the coefficients of y in the given equations), it is manifest from analogy that m, n, and p will have the same respective coefficients and signs in the required numerators as a, d, and f; and c, h, and s have in the denominator: a, d, and f; and c, h, and s being the coefficients of x and z in the given equations, following the same order as those of y.

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will be $(gs-rh)\,a+(rc-bs)\,d+(bh-cg)\,f$, for x:

and $(dr-gf)\,c+(fb-ra)\,h+(ag-bd)\,s$, for z.

$$\text{Therefore } x=\frac{(gs-rh)\,m+(rc-bs)\,n+(bh-cg)\,p}{(gs-rh)\,a+(rc-bs)\,d+(bh-cg)\,f}.$$

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Instead of subtracting the first equation from the other two, a contrary order might have been adopted; for that reason, perhaps, the symbol $\backsim$ would be more proper than the negative sign in the final expressions: See Examp. 2. (79.)

The last example is sufficient to direct the process, when four or more unknown quantities are concerned. But methods of reduction different from those we have given, will frequently present themselves in practice.

81. When the number of equations is less than the number of unknown quantities they involve, the problem is said to be indeterminate or unlimited. Thus if $x+y=10$, then x and y may be any two numbers whose sum is 10. Or suppose $x-y=6$, and $z-x=9$, in which case y may be any number whatever; and consequently the values of the three unknown quantities will be indefinite. The like must also take place

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{f}{g} = \frac{h}{k}$;

And $\frac{a}{b} \times \frac{f}{g} = \frac{c}{d} \times \frac{h}{k}$, or $\frac{af}{bg} = \frac{ch}{dk}$ (75. *Ax.* 3.)

Whence $af : bg :: ch : dk$. And so for any number of ranks.

This is called *compounding* the proportions.

97. **In a rank of continued proportionals, the ratio of the first term to the last is compounded of the ratios of all the antecedents to their consequents.**

Let $a : b :: b : c :: c : d :: d : f$.

Then $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{f}$, and the continued product

is $\frac{abcd}{bcdf}$, which in its lowest terms is $\frac{a}{f}$ the ratio of a to f.

Cor. Hence the ratio of the first term to the third will be equal to the ratio of the squares of the first and second.

For $b^2 = ca$
And $ab^2 = ca^2$
Whence $a : c :: a^2 : b^2$;

And the ratio of the first to the fourth is equal to the ratio of the cubes of the first and second.

Because $a : c :: a^2 : b^2$,

Therefore $\frac{a}{c} = \frac{a^2}{b^2}$; But $\frac{c}{d} = \frac{a}{b}$;

Whence $\frac{a}{c} \times \frac{c}{d} = \frac{a^2}{b^2} \times \frac{a}{b}$, or $\frac{a}{d} = \frac{a^3}{b^3}$; (75. *Ax.* 3.)

Therefore $a : d :: a^3 : b^3$. And in like manner it is proved that $a : f :: a^4 : b^4$. And so of others.

And the first term is said to have to the third, the duplicate ratio of that which it has to the second; and to the fourth, the triplicate ratio of that which it has to the second; and so on.

N. B. This compounding of ratios by multiplication is called addition of ratios by those who consider ratios to be the exponents of the powers of their terms.

98. If there be four proportional qua
e sum of the least and greatest is gre
her two.

Suppose $a : b :: c : d$; and let a be the leas
he quantities are commensurable, c and d v
iples, of a and b, respectively:

herefore if c and d are expounded by $(n +$

we have $a : b :: na + a : nb + b$.

And $nb + b + a$ is the sum of the leas

$na + a + b$ is the sum of the oth
the other sum, because b is greater tha
efore nb greater than na.

OF INVOLUTI

9. If a quantity be continually m
d to be involved or raised to a powe
es it has been employed in the mul

Thus $a \times a = a^2$, the 2*d*. power,
$a \times a \times a = a^3$, the 3*d*. power,
$a \times a \times a \times a = a^4$, the 4*th*. or biq

$a^n =$ the *n*th. power. (

Here $+ a$ is the root; and all the powers a
gative, then its odd powers, or the 3*d*, 5*th*

For $-a \times -a = a^2$, the square (
$a^2 \times -a = -a^3$, the cube o
$-a^3 \times -a = a^4$, the 4*th*. pow
$a^4 \times -a = -a^5$, the 5*th*.

100. But simple quantities are rai
ying the index of every factor in the
the power.

Thus the square of a^3 is $a^{3\cdot 2}=a^6$

The cube of a^2 is $a^{2\cdot 3}=a^6$

The cube of ab^2 is $a^{1\cdot 3}\,b^{2\cdot 3}=a^3b^6$. For $ab^2\times ab^2\times ab^2=a^3b^6$.

Also $-a^m$ raised to the nth. power is $+a^{mn}$ or $-a^{mn}$, according as n is even, or odd.

And $3a^{-4}$ raised to the nth. power is $3^n a^{-4n}$.

101. Fractions are raised to given powers by involving their terms:

Thus the square of $\dfrac{3ab}{4cd}$ is $\dfrac{9a^2b^2}{16c^2d^2}$.

And the cube of $\dfrac{2ab^2}{xy}$ is $\dfrac{8a^3b^6}{x^3y^3}$.

102. Compound quantities are involved by actual multiplication, as in Art. 44:

Thus if the root be $a+b$:

$$\begin{array}{rl}
 & a+b \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \underline{\quad\;\; +ab+b^2} \\
(a+b)^2= & a^2+2ab+b^2 \text{ the square or 2d. power.} \\
 & \underline{\qquad\qquad a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \underline{\quad +\;a^2b+2ab^2+b^3} \\
(a+b)^3= & a^3+3a^2b+3ab^2+b^3 \text{ the cube or 3d. power.} \\
 & \underline{\qquad\qquad\qquad a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \underline{\quad +\;a^3b+3a^2b^2+3ab^3+b^4} \\
(a+b)^4= & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \text{ the 4th. power.}
\end{array}$$

&c.

If the root be $a-b$ then the terms which involve the odd powers of b will be negative, viz. the signs are alternately *plus* and *minus*:

Thus $(a-b)^3=a^3-3a^2b+3ab^2-b^3$, the exponent of b being an odd *number* in the *2d.* and *4th.* terms.

Sometimes it is most convenient to represent the powers by the indices: Thus $(ax-x)^3$ denotes the third power of $ax-x$.

OF EVOLUTION OR THE EXTRACTION OF ROOTS.

103. THIS is the reverse of Involution, and consists in discovering the roots of given powers, or quantities.

The rule for Simple Quantities is,

Divide the exponent, or exponents, of the given quantity by the index of that power whose root is required. This follows from Involution, Art. 100.

Thus, to find the 3*d*. or cube root of $64a^6$?

The exponent of 64 being 1, we have $64^{\frac{1}{3}}a^{\frac{6}{3}}$, or $64^{\frac{1}{3}}a^2$, or $4a^2$ the root required: for $64^{\frac{1}{3}}=4$. And $4a^2\times4a^2\times4a^2=64a^6$.

And the cube root of $-64a^6$ is $-4a^2$.

Therefore the root of the product of two or more powers is the product of their roots; for $4a^2=4\times a^2$; and the square root $=2\times a$.

Also the cube root of $10x^3$ is $10^{\frac{1}{3}}x^{\frac{3}{3}}$ or $10^{\frac{1}{3}}x$. The coefficient $10^{\frac{1}{3}}$ is called a *surd*, because 10 will not admit of an exact root.

104. An even root of an affirmative quantity may be either $+$ or $-$;

Thus the square root of a^2 is $+a$, or $-a$; for $+a\times+a=a^2$; and $-a\times-a=a^2$.

But any even root of a negative quantity is impossible:

Thus the 2*d*. or square root of $-a^2$ is impossible, for $+a\times+a$, nor $-a\times-a$ will produce $-a^2$. In this case the root is represented thus, $\sqrt{(-a^2)}$, or thus $(-a^2)^{\frac{1}{2}}$. Also $(b+x^2)^{\frac{1}{3}}$ denotes the 3*d*. or cube root of $(b+x^2)$. And $(a^2-xy)^{\frac{1}{n}}$ the *n*th. root of a^2-xy.

105. An odd root of any quantity will have the same sign as the quantity itself: thus the 3d. root of $+64a^6$ is $+4a^2$; and the 3d. root of $-64a^6$ is $-4a^2$, as above.

106. The root of a fraction is found by taking the roots of the numerator and denominator:

Thus the cube root of $\frac{8x^6}{27x^3}$ is $\frac{2x^2}{3x}$. And the nth. root of $\frac{a^2}{b^3}$ is $\frac{a^{\frac{2}{n}}}{b^{\frac{3}{n}}}$. Also the square root of $\frac{a^{-2}}{b^{-4}}$ is $\pm\frac{a^{-1}}{b^{-2}}$.

If the numerator of the fraction be 1, the root may be denoted by the root of the denominator with a contrary sign to the index;

Thus, the cube root of $\frac{1}{x^3}$ is $\frac{1}{x}$ or x^{-1}: For $\frac{1}{x} = \frac{x}{x^2} = x^{1-2} = x^{-1}$. (71. *Examp.* 5.)

To extract the Square Root of a Compound Quantity.

107. In order to discover the rule for this purpose, it may be necessary to consider the formation of a square.

The square of a binomial $x+a$ is $x^2+2ax+a^2$, or x^2+a^2+ twice the product of the roots x and a:

of a trinomial $x+a+c$, is $(x+a)^2+c^2+$ twice the product of the roots (considering $x+a$ as making one term):

of a quadrinomial $x+a+c+d$, is $(x+a+c)^2+d^2+$ twice the product of the roots, &c. &c.

Therefore $(x+a+c)^2 = (x+a)^2 + (x+a)\,c \times 2 + c^2$

or $x^2+2ax+a^2+2cx+2ca+c^2$, which consists of the three following products;

viz. $x \times x$
$(2x+a)\,a$
$(2x+2a+c)\,c$:

Consequently if x^2 the first term in the square, be divided by x, the quotient is x the first term in the root:

And $2ax+a^2$ divided by $2x+a$ gives a the second term in the root.

Also the remainder $2cx+2ca+c^2$ divided by $2x+2a+c$ gives c the third term.

Again,

$$(x+a+c+d)^2 = x^2+2ax+a^2+2cx+2ca+c^2+(x+a+c)d\times 2+d^2,$$

$$\text{or } x^2+2ax+a^2+2cx+2ca+c^2+2dx+2da+2dc+d^2,$$

which is made up of the 4 products

$$x\times x$$
$$(2x+a)\,a$$
$$(2x+2a+c)\,c$$
$$(2x+2a+2c+d)\,d.$$

Now the three first terms of the root are found as in the preceding trinomial; and $2x+2a+2c+d$ is the divisor which gives the 4*th* term d.

And here we may observe that the terms in the root are constantly doubled, and the next term added, to form the divisors: Hence the following rule,

Range the quantities according to the dimensions of some letter, and set the root of the first term in the quotient:

Subtract the square of the root thus found from the first term; and bring down the two next terms for a dividend:

Divide the first term of the dividend by double the root, and set the result in the quotient, and also in the divisor:

Multiply the divisor thus augmented by the last result, and subtract the product from the dividend: then proceed as before, till all the terms are brought down, or the root extracted as far as may be thought necessary.

Examples.

1. To find the square root of $x^2+2ax+a^2+2cx+2ca+c^2+2dx+2da+2dc+d^2$.

$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 + 2ad + 2bd + 2cd + d^2$ ($a + b + c + d$, the root.

$$
\begin{array}{llll}
 & a^2 & & \\
a)\ & 0 \quad 2ab + b^2 & & \\
 & \quad\ \ 2ab + b^2 & & \\
2a + b) & \quad\ 0 & 2ac + 2bc + c^2 & \\
 & & 2ac + 2bc + c^2 & \\
+ 2a + 2b + c) & & \quad 0 & 2ad + 2bd + 2cd + d^2 \\
 & & & 2ad + 2bd + 2cd + d^2 \\
 & & & \quad\quad 0
\end{array}
$$

ence the rule for extracting the square root of a number is immediately ed: and if the root consists of 4 figures, a will stand for thousands, b undreds, c for tens, and d for units.

To extract the square root of $4a^8 - 16a^6 - 16a^5 + 12a^4 + 32a^3 + 24a^2$ $a + 1$.

$$
\begin{array}{lll}
 & 4a^8 - 16a^6 - 16a^5 + 12a^4 + 32a^3 + 24a^2 + 8a + 1 & (2a^4 - 4a^3 - \\
 & 4a^8 & \quad 4a - 1, \text{root.} \\
-4a^3) & 0 - 16a^6 - 16a^5 & \\
 & \ - 16a^6 + 16a^4 & \\
 & \qquad - 16a^5 - 16a^4 + 12a^4 & \\
-8a^3 - 4a) & \text{or} \ - 16a^5 - \ 4a^4 + 32a^3 + 24a^2 & \\
 & \qquad - 16a^5 + 32a^3 + 16a^2 & \\
{}^4 - 8a^3 - 8a - 1) & \qquad\quad - \ 4a^4 + \ 8a^2 + \ 8a + 1 & \\
 & \qquad\quad - \ 4a^4 + 8a^2 \ + 8a \ + 1 & \\
 & \qquad\qquad\qquad 0 &
\end{array}
$$

To find the square root of $a^2x^2 + a^2$.

$a^2x^2 + a^2 = a^2(x^2 + 1)$, therefore we have to approximate the root of $+ 1$:

$$
\begin{array}{ll}
 & x^2 + 1 \quad \left(x + \frac{1}{2x} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{5}{128x^7}, \text{ \&c. root.}\right. \\
 & x^2 \\
+ \frac{1}{2x}) & \ + 1 \\
 & \ + 1 + \frac{1}{4x^2} \\
\frac{1}{x} - \frac{1}{8x^3}) & \quad - \frac{1}{4x^2} \\
 & \quad - \frac{1}{4x^2} - \frac{1}{8x^4} + \frac{1}{64x^6} \\
\frac{1}{x} - \frac{1}{4x^3} + \frac{1}{16x^5}) & \qquad + \frac{1}{8x^4} - \frac{1}{64x^6} \\
 & \qquad + \frac{1}{8x^4} + \frac{1}{16x^6} - \frac{1}{64x^8} + \frac{1}{16^2x^{10}} \\
 & \qquad\qquad 5 \quad . \quad 1 \quad\quad 1
\end{array}
$$

To extract the Cube Root of a Compound Quantity.

108. We shall derive the rule from the formation of a Cube.

$$(a+c)^3 = a^3 + 3a^2c + 3ac^2 + c^3:$$
$$\text{or } a^3 + \overline{3(a)^2 + 3(a)c + c^2} \times c.$$

And $(a+c+d)^3 = (a+c)^3 + \overline{3(a+c)^2 + 3(a+c)d + d^2} \times d$, (by considering $a+c$ as making one term of a binomial);

And so on, for the cube of any multinomial.

Hence

$$(a+c)^3 = \begin{cases} a^3 \\ +(3a^2+3ac+c^2)c \end{cases}$$

$$(a+c+d)^3 = \begin{cases} a^3 \\ +(3a^2+3ac+c^2)c \\ +(3a^2+6ac+3c^2+3ad+3cd+d^2)\,d, \text{ \&c.} \end{cases}$$

(From the preceding formulæ, the rule, *art.* 117. *Arith.* is readily derived)

Or

$$(a+c)^3 = \begin{cases} a^3 \\ +3a^2c+3ac^2+c^3 \end{cases}$$

$$(a+c+d)^3 = \begin{cases} a^3 \\ +3a^2c+3ac^2+c^3 \\ +3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3 \end{cases}$$

Hence, in the cube of the trinomial $a+c+d$ it appears, that the 2*d.* term $3a^2c$ divided by $3a^2$ (three times the square of the root a) gives c the 2*d.* term in the root: and when the cube of the two first terms $a+c$ is subtracted, the next term $(3a^2d)$ in the remainder divided by the same divisor $(3a^2)$ gives d the 3*d.* term of the root, &c. whence the following Rule,

Arrange the terms according to the dimensions of some letter, as in extracting the square root, and set the root of the first term in the quotient, and subtract its cube from the quantity whose root is required:

Divide the first term of the remainder by 3 times the square of the root, and the quotient is the second term in the root:

Subtract the cube of the root already found from the given quantity, and divide the first term of the remainder by 3 times the square of the first term, as before, and the quotient is the third term in the root:

Subtract the cube of this augmented root from the proposed quantity; and proceed as before, should there be any remainder.

Examples.

1. To find the cube root of $a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3$.

$$a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3\ (a+c+d,\ \textit{root}.$$

$$\begin{array}{rl} & a^3 \\ \hline 3a^2) & 0 \quad +3a^2c \ldots \text{first term of the remainder.} \\ \hline & a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3, \\ (a+c)^3 = & a^3+3a^2c+3ac^2+c^3 \\ \hline & 0 \qquad 3a^2) \quad \underline{+3a^2d} \ldots\ldots \text{first term of remainder.} \end{array}$$

Now $(a+c+d)^3$ will be the quantity proposed, and consequently $a+c+d$ is the root required.

2. To extract the cube root of x^3+a^3

$$x^3+a^3\ \left(x+\frac{a^3}{3x^2}-\frac{a^6}{9x^5},\ \text{\&c. } \textit{root},\right.$$

$$\begin{array}{rl} & x^3 \\ \hline 3x^2) & +a^3 \ldots \text{first term of the remainder,} \end{array}$$

$$\begin{array}{rl} & x^3+a^3 \\ \left(x+\frac{a^3}{3x^2}\right)^3 = & x^3+a^3+\frac{a^6}{3a^3}+\frac{a^9}{27a^6} \\ \hline 3x^2) & \quad -\frac{a^6}{3a^3}-\frac{a^9}{27a^6} \\ & \qquad \text{\&c.} \end{array}$$

109. This rule adapted to the extraction of a cube root in numbers, will be as follows;

Point the number into periods of three figures each (beginning at the units) and find the greatest cube in the first period on the

left hand, and set its root in the quotient for the first figure of the required root:

Subtract the cube from the period above it, and bring down the next period to the remainder for a dividend:

Divide the dividend by 3 times the square of the root, and the first quotient figure will be the second figure in the root:

Subtract the cube of the root from the two first periods on the left hand, and to the remainder bring down the next period for a new dividend:

Divide this dividend by 3 times the square of the root, and the first quotient figure is the third figure in the root:

Subtract the cube of the root from the three left hand periods; then proceed as before till all the periods are brought down.

N. B. In dividing by 3 times the square, it is only the first quotient figure that is wanting, and therefore it will not be necessary to continue the division.

3. To extract the cube root of 269210725993.

```
                      269210725993  (6457 root.
                      216
      6² × 3 = 108)    53210 (4
                       269210
         64³... =      262144
   64² × 3 = 12288)    7066725 (5
                       269210725
           645³ =      268336125
 645² × 3 = 1248075)   874600993 (7
                       269210725993
          6457³ =      269210725993
                                  0
```

This rule is easily remembered; and may be made general by changing the indices: Thus if the mth. root is required; then, instead of the 3*d*. root, and 3*d*. power, we must take the mth. root, and mth. power; and for 3 times the square, make use of m times the $m-1$th power: &c.

But in extracting the higher roots of numbers, the divisors frequently give the quotient figures too great; the true figure, however, is found by a trial or two: Thus to extract the 5*th* root of 5559251349024.

$$\begin{array}{rl}
 & \dot{5}5\dot{5}9251\dot{3}49\dot{0}24 \text{ (354 } root. \\
3^5 = & \underline{243} \\
3^4 \times 5 = 405) & 31292513 \text{ (5} \\
 & \overline{55592513} \text{ ... two first periods} \\
35^5 = & \underline{52521875} \\
35^4 \times 5 = 7503125) & 307063849024 \text{ (4} \\
354^5 = & \overline{5559251349024}\text{, therefore 354 is the } root.
\end{array}$$

The second divisor 405 will give 7 for the second quotient figure, but 6 is too great; and 5 the true figure is found by raising 35 to the 5*th*. power.

In Algebra, when the proposed quantity is an exact power, its root may frequently be found by inspection: Thus in *Ex*. 1. we find the three cubes a^3, c^3, and d^3 in the given quantity; and the three roots connected with their proper signs, is the root required.

Of Dr. Halley's rational Formulæ for the Roots of pure Powers.

110. Let a^2+b be a quantity whose *square root* is required: where b is supposed to be small when compared with a^2.

Put $a+x=(a^2+b)^{\frac{1}{2}}$. Then squaring both sides (75. *Ax*. 6.) we have $a^2+2ax+x^2=a^2+b$; and $2ax+x^2=b$; whence $x=\dfrac{b}{2a+x}$.

But if b is small when compared with a^2, then x^2 will be much less than $2ax$, and consequently $2ax$ may be taken for the value of b, nearly: therefore if we reject x^2, in the equation $2ax + x^2 = b$ *, we have $2ax = b$, and $x = \frac{b}{2a}$ for the first approximate value of x; which being substituted for x in the fraction $\frac{b}{2a+x}$ gives $x = \frac{b}{2a + \frac{b}{2a}} = \frac{2ab}{4a^2+b}$ the second approximation; therefore the root $a + x = a + \frac{ab}{2a^2 + \frac{1}{2}b} = (a^2+b)^{\frac{1}{2}}$.

Again. Let the *cube root* of $a^3 + b$ be required; (b being supposed small, as before): and put $a + x = (a^3 + b)^{\frac{1}{3}}$:

Then $a^3 + 3a^2x + 3ax^2 + x^3 = a^3 + b$;

whence $3a^2x + 3ax^2 + x^3 = b$; and rejecting x^3 on account of its smallness, we get $3a^2x + 3ax^2 = b$, whence $x = \frac{b}{3a^2 + 3ax}$.

Now in the equation $3a^2x + 3ax^2 = b$, the term $3ax^2$ is supposed to be small when compared with $3a^2x$, therefore *that* being rejected, we have $3a^2x = b$, and $x = \frac{b}{3a^2}$ the first approximate value of x in this case, which put for x in the fraction $\frac{b}{3a^2 + 3ax}$, gives $x = \frac{b}{3a^2 + \frac{3ab}{3a^2}}$ the second approximation;

Whence the root $a + x = a + \frac{ab}{3a^3+b} = (a^3+b)^{\frac{1}{3}}$.

And if $a + x = (a^4 + b)^{\frac{1}{4}}$, then (omitting all the terms in which x is above the 2d. power) $a + x$ will be $= a + \frac{ab}{4a^4 + \frac{3}{2}b} = (a^4 + b)^{\frac{1}{4}}$.

$$\text{Hence } (a^2+b)^{\frac{1}{2}} = a + \frac{ab}{2a^2 + \frac{1}{2}b}.$$
$$(a^3+b)^{\frac{1}{3}} = a + \frac{ab}{3a^3 + b}.$$
$$(a^4+b)^{\frac{1}{4}} = a + \frac{ab}{4a^4 + \frac{3}{2}b}.$$
$$(a^5+b)^{\frac{1}{5}} = a + \frac{ab}{5a^5 + 2b}.$$
$$(a^6+b)^{\frac{1}{6}} = a + \frac{ab}{6a^6 + \frac{5}{2}b}.$$

&c. &c.

* Let $a = 50$, and $b = 1$ (a small number) then $2ax + x^2 = b$ becomes $100x + x^2 = 1$, in which equation x must be less than $\frac{1}{100}$, and consequently x^2 less than $\frac{1}{10000}$.

When the given quantity is a *residual*, the latter factors in the root will be negative: Thus $(a^6 - b)^{\frac{1}{6}} = a - \frac{ab}{6a^6 - \frac{5}{2}b}$. And so of others.

These are the rational formulæ of Dr. Halley (*Philos. Trans.* 1694.) who has informed us however, that M. de Lagney first gave the rule for the cube root. The irrational formulæ are surds derived nearly in the same manner.

111. But the foregoing expressions may be rendered more commodious for practice as follows:

Let N represent the quantity whose root is required, and r its root:

$$\text{Then because } a + \frac{ab}{2a^2 + \frac{1}{2}b} = \frac{4a^3 + 3ab}{4a^2 + b} = \frac{a^2 + 3\ (a^2 + b)}{3a^2 + a^2 + b} \times a$$

$$a + \frac{ab}{3a^3 + b} = \frac{6a^4 + 4ab}{6a^3 + 2b} = \frac{2a^3 + 4\ (a^3 + b)}{4a^3 + 2\ (a^3 + b)} \times a,$$

$$a + \frac{ab}{4a^4 + \frac{3}{2}b} = \frac{8a^5 + 5ab}{8a^4 + 3b} = \frac{3a^4 + 5\ (a^4 + b)}{5a^4 + 3\ (a^4 + b)} \times a,$$

&c. &c. &c.

If N be substituted for $a^2 + b$, $a^3 + b$, &c.

$$\text{we have } \frac{a^2 + 3\ (a^2 + b)}{3a^2 + a^2 + b} \times a = \frac{a^2 + 3N}{3a^2 + N} \times a = r.$$

$$\frac{2a^3 + 4\ (a^3 + b)}{4a^3 + 2\ (a^3 + b)} \times a = \frac{2a^3 + 4N}{4a^3 + 2N} \times a = r.$$

$$\frac{3a^4 + 5\ (a^4 + b)}{5a^4 + 3\ (a^4 + b)} \times a = \frac{3a^4 + 5N}{5a^4 + 3N} \times a = r.$$

&c. &c.

Which converted into analogies (88. cor. 3.) will be

$3a^2 + N : a^2 + 3N :: a : r.$

$4a^3 + 2N : 2a^3 + 4N :: a : r.$

$5a^4 + 3N : 3a^4 + 5N :: a : r.$

&c.

And since it appears that the numeral coefficients are constantly 1 greater, and 1 less than the *index* of the power whose root is to be extracted, the law of continuation is manifest: Therefore, putting n for that index, we have the following general Rule:

$(n+1)\ a^n + (n-1)\ N : (n-1)\ a^n + (n+1)\ N :: a : r$ (110. Arith.)

For, suppose the 4th. root is required;

Then, in the proportion $5a^4 + 3N : 3a^4 + 5N :: a : r$,

n is $= 4$

the cofficients 5 and 3...... are $n + 1$ and $n - 1$;

and $N = a^4 + b$, where b is small when compared with a^4.

If n is an odd number, $\frac{n+1}{2}$ and $\frac{n-1}{2}$ will be whole numbers which may be used instead of $n + 1$ and $n - 1$, as in extracting the cube root. (118. Arith. *Ex.* 1.)

This method, when applied to the extraction of the higher roots of numbers, is the most expeditious of any, if we except that by Logarithms. See Arith. *Art.* 118, &c.

OF SURDS.

112. SURDS or Radical Quantities are such as have no exact root. The roots however, are designated by means of the radical sign $\sqrt{\ }$, or by fractional indices.

Thus, the square root of 5 is expressed by $\sqrt{5}$, or $5^{\frac{1}{2}}$.

Also $\sqrt[2]{5^3}$, or $5^{\frac{3}{2}}$ denotes the square root of the cube of 5.

And $\sqrt[5]{(a + b)}$, or $(a + b)^{\frac{1}{5}}$ the 5*th*. root of $a + b$: &c. (104.)

113. A rational quantity may be exhibited under various surd forms;

Thus, taking the number 6, for example;

Then $6 = \sqrt{36} = \sqrt{(6 \times 6)} = \sqrt{(4 \times 9)} = \sqrt{(3 \times 12)} = \sqrt{(2 \times 18)} = \sqrt{(1 \times 36)}$

$= \sqrt{4} \times \sqrt{9} = 2\sqrt{9} = 3\sqrt{4} = \sqrt{3} \times \sqrt{12} = \sqrt{6} \times \sqrt{6} = 216^{\frac{1}{3}}$, &c.

And if the quantity be a,

We have $a = \sqrt{a^2} = \sqrt{(a \times a)} = \sqrt{(\frac{1}{4}a \times 4a)} = \sqrt{a} \times \sqrt{a} = \sqrt{a} \times \sqrt{\frac{a^2}{a}}$

$= a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{2}{2}} = a^{\frac{1}{n}} \times a^{\frac{n-1}{n}} = \left(a^{\frac{1}{n}}\right)^n = \left(a^n\right)^{\frac{1}{n}}$, and innumerable other expressions, which will be evident if we consider that the square, or cube, &c. root of any quantity when squared, or cubed, &c. *must give the quantity itself.*

114. Hence, to bring a rational quantity to the form of a square, or a cube, &c. root, raise it to the 2d. or 3d. &c. power, and set this quantity under the index denoting the root.

Thus 4 under the forms of the 2d, 3d, 4th, and $\frac{2}{3}$ roots, will be $16^{\frac{1}{2}}$, $64^{\frac{1}{3}}$, $256^{\frac{1}{4}}$, $8^{\frac{2}{3}}$.

Also $a^{\frac{1}{3}}$ reduced to the form of the square root is $(a^{\frac{2}{3}})^{\frac{1}{2}}$.

And a^{r} reduced to the mth. root is $(a^{rm})^{\frac{1}{m}}$.

Generally: $a^{\frac{r}{s}}$ reduced to the form of the $\frac{n}{m}$th. root is $(a^{\frac{r}{s}\times\frac{m}{n}})^{\frac{n}{m}}$.

or $(a^{\frac{mr}{ns}})^{\frac{n}{m}}$.

For if the multiplication by $\frac{m}{n}$ involves to the power $\frac{m}{n}$, the multiplication by its reciprocal $\frac{n}{m}$ must reduce it again to the root.

115. To reduce quantities with different indices, to other equivalent ones having a common index.

Reduce the indices to a common denominator; then involve each quantity to the power denoted by its numerator.

Examples.

1. Reduce $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$ to equivalent quantities having a common index.

$\frac{1}{3}=\frac{2}{6}$; and $\frac{1}{2}=\frac{3}{6}$.

Therefore $8^{\frac{1}{3}}=8^{\frac{2}{6}}=64^{\frac{1}{6}}$

$9^{\frac{1}{2}}=9^{\frac{3}{6}}=729^{\frac{1}{6}}$.

And the quantities are $64^{\frac{1}{6}}$ and $729^{\frac{1}{6}}$, having the common index $\frac{1}{6}$.

For $8^{\frac{1}{3}}=2$, and $64^{\frac{1}{6}}=2$. Also $9^{\frac{1}{2}}=3$, and $729^{\frac{1}{6}}=3$.

116. When the quantities are to be reduced to equivalent ones having a *given index*, it may be done by the general form in the preceding article: thus,

2. Let $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$ be reduced to equivalent quantities having the common index $\frac{1}{4}$.

$$\text{Then } \left(8^{\frac{1}{3}\times\frac{4}{1}}\right)^{\frac{1}{4}} = \left(8^{\frac{4}{3}}\right)^{\frac{1}{4}} = \left(4096^{\frac{1}{3}}\right)^{\frac{1}{4}} = 16^{\frac{1}{4}}.$$

$$\text{And } \left(9^{\frac{1}{2}\times\frac{4}{1}}\right)^{\frac{1}{4}} = \left(9^{\frac{4}{2}}\right)^{\frac{1}{4}} = \left(9^{2}\right)^{\frac{1}{4}} = 81^{\frac{1}{4}}.$$

Ans. $16^{\frac{1}{4}}$ and $81^{\frac{1}{4}}$.

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{2}{3}}$ to the common index $\frac{1}{6}$.

$$\left(3^{\frac{1}{2}\times\frac{6}{1}}\right)^{\frac{1}{6}} = \left(3^{3}\right)^{\frac{1}{6}} = 27^{\frac{1}{6}}.$$

$$\left(2^{\frac{2}{3}\times\frac{6}{1}}\right)^{\frac{1}{6}} = \left(2^{4}\right)^{\frac{1}{6}} = 16^{\frac{1}{6}}.$$

Ans. $27^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$.

4. Let $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, and $c^{\frac{1}{4}}$ be reduced to the common index $\frac{1}{3}$.

$$\left(a^{\frac{1}{2}\times\frac{3}{1}}\right)^{\frac{1}{3}} = \left(a^{\frac{3}{2}}\right)^{\frac{1}{3}};\ \left(b^{\frac{1}{3}\times\frac{3}{1}}\right)^{\frac{1}{3}} = \left(b^{\frac{1}{1}}\right)^{\frac{1}{3}};\ \left(c^{\frac{1}{4}\times\frac{3}{1}}\right)^{\frac{1}{3}} = \left(c^{\frac{3}{4}}\right)^{\frac{1}{3}}.$$

Ans. $\left(a^{\frac{3}{2}}\right)^{\frac{1}{3}}$, $\left(b^{\frac{1}{1}}\right)^{\frac{1}{3}}$, $\left(c^{\frac{3}{4}}\right)^{\frac{1}{3}}$.

5. Reduce $a^{\frac{r}{n}}$ and $b^{\frac{r}{m}}$ to a common index.

$$\frac{r}{n} = \frac{mr}{nm};\ \text{and } \frac{r}{m} = \frac{nr}{nm}.$$

$$\text{Therefore } a^{\frac{r}{n}} = a^{\frac{mr}{nm}} = \left(a^{m}\right)^{\frac{r}{nm}}.$$

$$\text{and } b^{\frac{r}{m}} = b^{\frac{nr}{nm}} = \left(b^{n}\right)^{\frac{r}{nm}}.$$

Ans. $\left(a^{m}\right)^{\frac{r}{nm}}$, and $\left(b^{n}\right)^{\frac{r}{nm}}$.

Of Multiplying Surd quantities together.

117. It appears from Art. 113. that the product of surds having a common index, is the product of the quantities themselves with the same common index.

$$\text{Thus } \sqrt{3}\times\sqrt{12} = \sqrt{36}:\ \text{or } 3^{\frac{1}{2}}\times 12^{\frac{1}{2}} = 36^{\frac{1}{2}}.$$

$$\text{And } \sqrt{a}\times\sqrt{a} = \sqrt{a^2};\ \text{or } a^{\frac{1}{2}}\times a^{\frac{1}{2}} = (a^2)^{\frac{1}{2}}.$$

$$\text{Also } \sqrt{a}\times\sqrt{b} = \sqrt{ab};\ \text{or } a^{\frac{1}{2}}\times b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}.$$

And since $2\sqrt{9} \times 3\sqrt{4} = 2 \times 3\sqrt{(9 \times 4)} = 6\sqrt{36}$;

Therefore if surds have coefficients their product must be prefixed:

Thus $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.

And $a\,(b^{\frac{1}{n}}) \times c\,(d^{\frac{1}{n}}) = ac\,(bd)^{\frac{1}{n}}$, or $ac\,b^{\frac{1}{n}}d^{\frac{1}{n}}$.

118. The product of like quantities in the form of surds with the same, or different indices, is found by adding those indices together (45):

Thus $64^{\frac{1}{3}} \times 64^{\frac{1}{3}} = 64^{\frac{1}{3}+\frac{1}{3}} = 64^{\frac{2}{3}}$.

And $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{1} = a$.

Also $(a+b)^{\frac{1}{n}} \times (a+b)^{\frac{1}{m}} = (a+b)^{\frac{1}{n}+\frac{1}{m}} = (a+b)^{\frac{n+m}{nm}}$

119. If surds have different indices, reduce them to the same index (115); then find the product as in the preceding articles.

Thus, to find the product of $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$;

$\frac{1}{3} = \frac{2}{6}$; and $\frac{1}{2} = \frac{3}{6}$;

Then $8^{\frac{1}{3}} = 8^{\frac{2}{6}} = 64^{\frac{1}{6}}$,

and $9^{\frac{1}{2}} = 9^{\frac{3}{6}} = 729^{\frac{1}{6}}$

Therefore $64^{\frac{1}{6}} \times 729^{\frac{1}{6}} = (64 \times 729)^{\frac{1}{6}} = 46656^{\frac{1}{6}}$, the answer.

Other Examples.

1. Required the product of $6\sqrt{10}$ and $10\sqrt{6}$.

$6 \times 10\sqrt{(10 \times 6)} = 60\sqrt{60}$. *Ans.*

2. What is the product of $5\,(9)^{\frac{1}{3}}$ and $2\,(6)^{\frac{1}{3}}$?

$5 \times 2\,(9 \times 6)^{\frac{1}{3}} = 10\,(54)^{\frac{1}{3}}$. *Ans.*

3. Required the product of $\frac{1}{2}\left(\frac{4}{5}\right)^{\frac{1}{3}}$ and $\frac{1}{3}\left(\frac{5}{6}\right)^{\frac{1}{3}}$?

$\frac{1}{2} \times \frac{1}{3}\left(\frac{4}{5} \times \frac{5}{6}\right)^{\frac{1}{3}} = \frac{1}{6}\left(\frac{4}{6}\right)^{\frac{1}{3}} = \left(\frac{4}{162}\right)^{\frac{1}{3}} = \left(\frac{2}{81}\right)^{\frac{1}{3}}$. *Ans.*

4. Required the product of $a\,(a+\sqrt{c})^{\frac{1}{2}}$ and $b\,(a\sqrt{c})^{\frac{1}{2}}$?

$$ab \times \overline{(a+\sqrt{c})\ (a-\sqrt{c})}^{\frac{1}{2}} = ab\ (a^2-c)^{\frac{1}{2}}. \quad \textit{Ans.}$$

5. Required the product of $(x-y)^{-2}$ and $(x-y)^{3}$?

$$(x-y)^{-2} \times (x-y)^{3} = (x-y)^{-2+3} = x-y. \quad \textit{Ans.}$$

6. What is the product of $(x-y)^{\frac{1}{2}}$ and $\frac{x}{(x^2-y^2)^{\frac{1}{2}}}$?

$$(x-y)^{\frac{1}{2}} \times \frac{x}{(x^2-y^2)^{\frac{1}{2}}} = x\left(\frac{x-y}{x^2-y^2}\right)^{\frac{1}{2}} = \frac{x}{(x+y)^{\frac{1}{2}}}. \quad \textit{Ans.}$$

7. What is the product of $(a-x)^{\frac{1}{3}}$ and $x^{\frac{1}{2}}$?

$$(a-x)^{\frac{1}{3}} = (a-x)^{\frac{2}{6}} = (a^2-2ax+x^2)^{\frac{1}{6}}$$

$$x^{\frac{1}{2}} = x^{\frac{3}{6}} = (x^3)^{\frac{1}{6}}$$

$$(a^2-2ax+x^2)^{\frac{1}{6}} \times (x^3)^{\frac{1}{6}} = (a^2x^3-2ax^4+x^5)^{\frac{1}{6}}. \quad \textit{Ans.}$$

8. Required the product of $\sqrt{(-a)}$ and $\sqrt{(-a)}$?

$\sqrt{(-a)} \times \sqrt{(-a)} = \sqrt{(-a \times -a)} = \sqrt{a^2}$: but the square root of a^2 is $+a$, or $-a$.

Or thus: $(-a)^{\frac{1}{2}} \times (-a)^{\frac{1}{2}} = (-a)^{\frac{1}{2}+\frac{1}{2}} = (-a)^1 = -a$.

This last method shews that the square of $\sqrt{(-a)}$ is $-a$; but it does not prove that it is not $+a$. Now the preceding operation gives both $+a$, and $-a$, conformable to the rules of multiplication, and the extraction of roots: for if $\sqrt{(-a)}$ be a negative root, its square by actual multiplication will be positive, and this positive square will have a positive, and a negative root. But it may be said that $\sqrt{(-a)}$ denotes the root of a negative, not a negative root; this objection however, is obviated by the process; for $\sqrt{(-a)} \times \sqrt{(-a)}$ or $\sqrt{(-a \times -a)}$, and $-\sqrt{a} \times -\sqrt{a}$, both give $+\sqrt{a^2}$. It seems therefore not more repugnant to algebraic method, in making the square of $\sqrt{(-a)}$ equal to $+a$ or $-a$, than in admitting $+a$, or $-a$ to be the square root of $+a^2$.

Hence it appears that the square of $\sqrt{(-a)}$ is $\pm a$.

of $\sqrt{(-a^2)}$ is $\pm a^2$.

of $1+\sqrt{(-2)}$ is $1+\sqrt{(-8)}\pm 2$.

And that the product of $\sqrt{(-a)}$ and $\sqrt{(-b)}$ is $\pm\sqrt{ab}$.

Division of Surds.

160. REDUCE the surds to the same index; then the quotient of the surd quantities, with that index, will be the answer.

Examples.

1. Divide $\sqrt{18}$ by $\sqrt{2}$.

$$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3. \quad \textit{Ans.}$$

2. Divide $46656^{\frac{1}{6}}$ by $8^{\frac{1}{3}}$?

$$8^{\frac{1}{3}} = 8^{\frac{2}{6}} = 64^{\frac{1}{6}};$$

$$\frac{46656^{\frac{1}{6}}}{64^{\frac{1}{6}}} = \left(\frac{46656}{64}\right)^{\frac{1}{6}} = 729^{\frac{1}{6}} = 3. \quad \textit{Ans.}$$

3. Divide $\sqrt{a}$ by $\sqrt{b}$.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}. \quad \textit{Ans.}$$

4. Divide $a^{\frac{1}{m}}$ by $b^{\frac{1}{n}}$.

$$a^{\frac{1}{m}} = a^{\frac{n}{mn}} = (a^n)^{\frac{1}{mn}}; \text{ and } b^{\frac{1}{n}} = b^{\frac{m}{mn}} = (b^m)^{\frac{1}{mn}}.$$

$$\frac{(a^n)^{\frac{1}{mn}}}{(b^m)^{\frac{1}{mn}}} = \left(\frac{a^n}{b^m}\right)^{\frac{1}{mn}}. \quad \textit{Ans.}$$

161. If the surds have coefficients their quotient must be prefixed. And the quotient of like surds is found by subtracting the index of the divisor from that of the dividend. (49).

5. Divide $8\sqrt{10}$ by $4\sqrt{2}$.

$$\frac{8\sqrt{10}}{4\sqrt{2}} = 2\sqrt{\tfrac{10}{2}} = 2\sqrt{5}. \quad \textit{Ans.}$$

6. Divide $\frac{3}{4}\sqrt{\frac{5}{8}}$ by $\frac{3}{5}\sqrt{\frac{5}{6}}$?

$$\frac{\frac{3}{4}\sqrt{\frac{5}{8}}}{\frac{3}{5}\sqrt{\frac{5}{6}}} = \frac{5}{4}\sqrt{\frac{3}{4}} = \frac{4\times5}{4\times4}\sqrt{\frac{3}{4}} = \frac{5}{16}\sqrt{\frac{3\times16}{4}} = \frac{5}{16}\sqrt{12}. \quad \textit{Ans.}$$

7. Divide $a\sqrt{b}$ by $c\sqrt{d}$?

$$\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}. \quad \textit{Ans.}$$

8. Divide $ax^{\frac{1}{2}}$ by $bx^{\frac{1}{3}}$?

$$\frac{ax^{\frac{1}{2}}}{bx^{\frac{1}{3}}} = \frac{a}{b}x^{\frac{1}{2}-\frac{1}{3}} = \frac{a}{b}x^{\frac{1}{6}}. \text{ Ans.}$$

9. Divide $(a-)^{\frac{1}{2}}$ by $(a-x)^{\frac{2}{3}}$?

$$\frac{(a-x)^{\frac{1}{2}}}{(a-x)^{\frac{2}{3}}} = (a-x)^{\frac{1}{2}-\frac{2}{3}} = (a-x)^{-\frac{1}{6}} = \frac{1}{(a-x)^{\frac{1}{6}}}. \text{ Ans.}$$

10. Divide $1+\sqrt{(-8)}\pm 2$ by $1+\sqrt{(-2)}$?

$$\sqrt{(-8)} = 2\sqrt{(-2)}:$$

$$\begin{array}{r|l}
1+\sqrt{(-2)} & 1+2\sqrt{(-2)}\pm 2 \;\left(1+\sqrt{(-2)}\right. \text{ Quotient.} \\
 & \underline{1+\sqrt{(-2)}\qquad\quad} \\
 & \qquad\sqrt{(-2)}\pm 2 \\
 & \qquad\underline{\sqrt{(-2)}\pm 2} \\
 & \qquad\qquad 0
\end{array}$$

11. Divide ax^{-n} by bx^{m}.

$$\frac{ax^{-n}}{bx^{m}} = \frac{a}{b}\left(\frac{x^{-n}}{x^{m}}\right) = \frac{a}{b}\left(x^{-n-m}\right) = \frac{a}{b}\left(x^{-(n+m)}\right). \text{ Ans.}$$

Of reducing Surds to their simplest terms.

122. This principally consists in resolving the quantity into its component factors, or separating the rational and irrational parts; which being done, the root of the greatest rational power it contains, will be the coefficient of the irrational part.

These reductions are frequently necessary when surds are to be added, or subtracted.

Examples.

1. Reduce $\sqrt[3]{(a^3b^3x)}$ to its simplest terms.

$$\sqrt[3]{(a^3b^3x)} \text{ or } (a^3b^3x)^{\frac{1}{3}} = (a^3b^3)^{\frac{1}{3}} \times x^{\frac{1}{3}} = abx^{\frac{1}{3}}. \text{ Ans.}$$

2. Reduce $\sqrt{363}$ to its simplest terms.

$$\sqrt{363} = \sqrt{(11\times 11\times 3)} = 11\sqrt{3}. \text{ Ans.}$$

3. Reduce $\left(\frac{2}{81}\right)^{\frac{1}{3}}$ to its simplest terms.

$$\left(\frac{2}{81}\right)^{\frac{1}{3}} = \left(\frac{2 \times 9}{81 \times 9}\right)^{\frac{1}{3}} = \frac{18^{\frac{1}{3}}}{729^{\frac{1}{3}}} = \frac{18^{\frac{1}{3}}}{9} = \frac{1}{9}\,(18)^{\frac{1}{3}}. \quad \textit{Ans.}$$

4. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms.

$$\sqrt{\frac{50}{147}} = \sqrt{\frac{25 \times 2}{49 \times 3}} = \frac{5}{7}\sqrt{\frac{2}{3}} = \frac{5 \times 3}{7 \times 3}\sqrt{\frac{2}{3}} = \frac{5}{21}\sqrt{\frac{2 \times 9}{3}} = \frac{5}{21}\sqrt{6}. \quad \textit{Ans.}$$

5. Reduce $(54ax^3 - 54ax^4)^{\frac{1}{3}}$ to its simplest terms.

$$(54ax^3 - 54ax^4)^{\frac{1}{3}} = (27x^3 \times 2a - 27x^3 \times 2ax)^{\frac{1}{3}} = 3x\,(2a - 2ax)^{\frac{1}{3}}. \quad \textit{Ans.}$$

Of the Addition and Subtraction of Surds.

123. Reduce the proposed quantities, by the preceding article, so that the surd parts are the same (if they admit of such reduction); then denote the sum, or difference, by means of their coefficients.

Examples.

1. Required the sum, and difference of $\sqrt{a^2b}$ and $\sqrt{c^2b}$.

$\sqrt{a^2b} = a\sqrt{b}$: and $\sqrt{c^2b} = c\sqrt{b}$.

Therefore $(a + c)\sqrt{b}$, is the *sum*:

And $(a \backsim c)\sqrt{b}$, the *difference*.

2. Required the sum, and difference of $\sqrt{245}$ and $\sqrt{605}$.

$\sqrt{245} = \sqrt{(49 \times 5)} = 7\sqrt{5}$. And $\sqrt{605} = \sqrt{(121 \times 5)} = 11\sqrt{5}$.

Hence $18\sqrt{5}$, the *sum*. And $4\sqrt{5}$ the *difference*.

3. What is the sum, and also the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$?

$$\sqrt{\frac{2}{3}} = \frac{3}{3}\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{\frac{2 \times 9}{3}} = \frac{1}{3}\sqrt{6}.$$

And $\sqrt{\frac{27}{50}} = \sqrt{\frac{9 \times 3}{25 \times 2}} = \frac{3}{5}\sqrt{\frac{3}{2}} = \frac{3 \times 2}{5 \times 2}\sqrt{\frac{3}{2}} = \frac{3}{10}\sqrt{\frac{3 \times 4}{2}} = \frac{3}{10}\sqrt{6}.$

$\frac{1}{3}\sqrt{6} = \frac{10}{30}\sqrt{6}$. And $\frac{3}{10}\sqrt{6} = \frac{9}{30}\sqrt{6}$.

$$\frac{10}{30}\sqrt{6}+\frac{9}{30}\sqrt{6}=\frac{19}{30}\sqrt{6}, \text{ the } sum.$$

$$\frac{10}{30}\sqrt{6}-\frac{9}{30}\sqrt{6}=\frac{1}{30}\sqrt{6}, \text{ the } difference.$$

4. Required the sum, and difference of $(24a^4x)^{\frac{1}{3}}$ and $(40b^3x)^{\frac{1}{3}}$.

$$(24a^4x)^{\frac{1}{3}}=24^{\frac{1}{3}}a^{\frac{4}{3}}x^{\frac{1}{3}}. \text{ And } (40b^3x)^{\frac{1}{3}}=40^{\frac{1}{3}}bx^{\frac{1}{3}}:$$

Therefore $(24^{\frac{1}{3}}a^{\frac{4}{3}}+40^{\frac{1}{3}}b)x^{\frac{1}{3}}$ is the *sum*.

And $(24^{\frac{1}{3}}a^{\frac{4}{3}} \backsim 40^{\frac{1}{3}}b)x^{\frac{1}{3}}$ the *difference*.

Of involving Surds: And extracting their Roots.

124. SURDS are involved by multiplying the index by the exponent of the power to which it is to be raised. (100).

Thus, the cube of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3}\times 3}=a^2$.

And the square of $(a^3-x^3)^{\frac{1}{2}}$ is $(a^3-x^3)^{\frac{1}{2}\times 2}=a^3-x^3$.

Also the 4*th* power of $(a-x)^{\frac{1}{2}}$ is $(a-x)^{\frac{1}{2}\times 4}=(a-x)^2=a^2-2ax+x^2$.

If the Surds have coefficients, their powers must be prefixed.

Thus, the cube of $3x^{\frac{1}{3}}$ is $27x$.

And the $\frac{2}{3}$ power of $8x^{\frac{1}{2}}$ is $8^{\frac{2}{3}}x^{\frac{1}{2}\times\frac{2}{3}}=4x^{\frac{1}{3}}$.

Other Examples.

1. What is the *m*th. power of $ax^{\frac{1}{n}}$? *Ans.* $a^m x^{\frac{m}{n}}$

2. Required the square of $7(9^{\frac{1}{3}})$? *Ans.* $49(81^{\frac{1}{3}})$.

3. Required the cube of $2^{\frac{1}{2}}$?

$$2^{\frac{1}{2}\times 3}=2^{\frac{3}{2}}=8^{\frac{1}{2}}=2(2^{\frac{1}{2}}). \text{ } Ans.$$

4. Required the 4th. power of $\frac{1}{7}\sqrt{7}$?

$$\frac{1}{7}\sqrt{7}=\sqrt{\frac{7}{49}}=\left(\frac{1}{7}\right)^{\frac{1}{2}}; \text{ and } \left(\frac{1}{7}\right)^{\frac{1}{2}\times 4}=\left(\frac{1}{7}\right)^2=\frac{1}{49}. \text{ } Ans.$$

5. What is the square of $\sqrt{a}+\sqrt{b}$? *Ans.* $a+2\sqrt{ab}+b$.

6. Required the square, and also the cube of $\sqrt{3}-\sqrt{2}$?

$$\begin{array}{l} \sqrt{3}-\sqrt{2} \\ \underline{\sqrt{3}-\sqrt{2}} \\ \quad 3-\sqrt{6} \\ \quad\quad \underline{-\sqrt{6}+2} \\ \quad 5-\sqrt{24} \text{.........} \text{ the } \textit{square.} \\ \underline{\sqrt{3}-\sqrt{2}} \\ 5\sqrt{3}-\sqrt{72} \\ \quad\quad \underline{-5\sqrt{2}+\sqrt{48}} \\ 5\sqrt{3}-\sqrt{72}-5\sqrt{2}+\sqrt{48}=9\sqrt{3}-11\sqrt{2}, \text{ the } \textit{cube.} \end{array}$$

For $\sqrt{72}=6\sqrt{2}$; and $\sqrt{48}=4\sqrt{3}$; whence $4\sqrt{3}+5\sqrt{3}=9\sqrt{3}$; and $6\sqrt{2}+5\sqrt{2}=11\sqrt{2}$.

125. *To find the root of a surd,* divide its index by the index of the root to be extracted. And when the surds have rational coefficients, their roots must be prefixed.

Examples.

1. What is the square root of $a^2\,(b^{\frac{2}{3}})$?

The square root of a^2 is a; and the square root of $b^{\frac{2}{3}}$ is $b^{\frac{2}{3\times2}}=b^{\frac{1}{3}}$; therefore $ab^{\frac{1}{3}}$ is the root required.

2. What is the square root of $49c^2\,(81^{\frac{1}{3}})$?

The square root of $49c^2$ is $7c$.

And $81^{\frac{1}{3\times2}}=81^{\frac{1}{6}}$ or $9^{\frac{1}{3}}$ is the square root of $81^{\frac{1}{3}}$.

Ans. $7c\,(9^{\frac{1}{3}})$.

3. Required the square root of 12^3?

$12^{\frac{3}{2}}=1728^{\frac{1}{2}}$ or $12\,(12^{\frac{1}{2}})$. *Ans.*

4. Required the $4th.$ root of $5x^2y^2$? *Ans.* $5^{\frac{1}{4}}\,(xy)^{\frac{1}{2}}$.

5. What is the $mth.$ root of $ax^{\frac{2}{n}}$? *Ans.* $a^{\frac{1}{m}}x^{\frac{2}{mn}}$.

6. What is the cube root of $(a^3c^3x)^{\frac{3}{5}}$?

$(a^3c^3x)^{\frac{3}{5}}=(ac)^{\frac{9}{5}}x^{\frac{3}{5}}$; and $(ac)^{\frac{9}{5\times3}}x^{\frac{3}{5\times3}}=a^{\frac{3}{5}}c^{\frac{3}{5}}x^{\frac{1}{5}}$. *Ans.*

7. What is the $-m$th. root of ax^{-n}?

$$a^{\frac{1}{-m}} x^{\frac{-n}{-m}} = a^{-\frac{1}{m}} x^{\frac{n}{m}}. \quad \textit{Ans.}$$

8. Required the square root of $a - 2\sqrt{ab} + b$?

Ans. $\sqrt{a} - \sqrt{b}$, or $\sqrt{b} - \sqrt{a}$.

And the square root of $a + 2\sqrt{ab} + b$, is $\sqrt{a} + \sqrt{b}$.

126. The roots in the two last examples are obvious by inspection only; but if numbers are substituted for the letters, the square becomes a binomial: thus, let $5 = a$, and $3 = b$, then $(\sqrt{5} + \sqrt{3})^2 = 8 + \sqrt{60}$, where the rational part 8 is the sum of the squares of the two surds, and the irrational part $\sqrt{60}$, twice their product. To discover the root in such cases, suppose $a + \sqrt{b}$ is the binomial, and let x and y denote the two members of its root, viz. put $(x + y)^2 = a + \sqrt{b}$.

Then $x^2 + y^2 + 2xy$ being $= a + \sqrt{b}$,

we shall have $x^2 + y^2 = a$, and $2xy = \sqrt{b}$.

$\therefore$ And $y^2 = a - x^2$. Also $4x^2y^2 = b$; whence $y^2 = \dfrac{b}{4x^2}$.

Therefore $a - x^2 = \dfrac{b}{4x^2}$.

And $ax^2 - x^4 = \dfrac{b}{4}$, or $x^4 - ax^2 = -\dfrac{b}{4}$ (77).

Now if $\dfrac{a^2}{4}$ be added to each side of this equation, we get

$$x^4 - ax^2 + \frac{a^2}{4} = \frac{a^2}{4} - \frac{b}{4} = \frac{a^2 - b}{4}; \text{ (75. } \textit{Ax.} \text{ 1)}$$

But $x^4 - ax^2 + \dfrac{a^2}{4}$ is a square, its root being $x^2 - \dfrac{a}{2}$;

$$\text{Therefore } x^2 - \frac{a}{2} = \sqrt{\frac{a^2 - b}{4}} = \frac{\sqrt{(a^2 - b)}}{2}. \text{ (75. } \textit{Ax.} \text{ 7).}$$

$$\text{whence } x^2 = \frac{a}{2} + \frac{\sqrt{(a^2 - b)}}{2} = \frac{a + \sqrt{(a^2 - b)}}{2}; \text{ and } x = \sqrt{\left(\frac{a + \sqrt{(a^2 - b)}}{2}\right)}.$$

Now y being $= \sqrt{(a - x^2)}$,

$$\text{we have } y = \sqrt{\left(a - \frac{a + \sqrt{(a^2 - b)}}{2}\right)} = \sqrt{\left(\frac{a - \sqrt{(a^2 - b)}}{2}\right)}.$$

Therefore the square root of $a+\sqrt{b}$ is $\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$.

And if the surd be a residual, $a-\sqrt{b}$;

Then $\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$ is the root.

Examp. Let $12-\sqrt{140}$ be the binomial;

Then the expression becomes $\sqrt{\frac{12+\sqrt{4}}{2}}-\sqrt{\frac{12-\sqrt{4}}{2}}=\sqrt{7}-\sqrt{5}$, the square root of $12-\sqrt{140}$.

The above expressions are general for a binomial: but when applied to surds, it is necessary that $\sqrt{(a^2-b)}$ be *rational* to bring out one or both members of the root in simple quadratic surds.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

127. According to the natural arrangement of parts into which Algebra may be divided, this should immediately have followed the Resolution of Simple Equations, *Art.* 74, &c. But we found some knowledge of equations requisite in the articles on Proportion, the extraction of roots, &c. And an acquaintance with these latter branches frequently becomes necessary in the resolution of Problems, whether they produce simple, or quadratic equations.

128. When a Question is proposed, the student should represent the unknown or required quantity, or quantities, by a letter, or letters, as x, y, z, &c. then let him operate with both the given and unknown quantities, by addition, subtraction, multiplication, &c. according to the conditions and tenor of the question; by which means he will obtain one or more Equations involving the unknown letter or letters.—But on this head, a few examples are preferable to many precepts.

Examples.

1. What number is that to which 7 being added, and the sum divided by 3, gives the quotient 13?

Let x denote the required number:

Then the sum of x and 7 is .... $x+7$

which divided by 3 gives .... $\frac{x+7}{3}$

And this must be $=13$, viz.......... $\frac{x+7}{3}=13$

Whence $x=32$, the number sought. See Art. 77. *Ex.* 3.

2. There is a number to which if we add 5, and subtract its double from ½ the sum, the remainder will be equal to the said number divided by 3. Required the number?

Let x represent the number sought:

To which adding 5 gives .... $x+5$

Half of this is .... $\frac{x+5}{2}$

And subtracting double the required number, leaves...... $\frac{x+5}{2}-2x$

Which, by the question is $=\frac{1}{3}$ of that number, viz. $\frac{x+5}{2}-2x=\frac{x}{3}$

Whence $x=1\frac{4}{11}$. Art. 77. *Ex.* 6.

3. What two numbers are those whose sum is 20, and difference 6.

Suppose the less number to be x

Then the greater must be .... $x+6$

Their sum is.. $2x+6$

This sum is $=20$, viz. .... $2x+6=20$

Whence $2x=20-6=14$

And $x=\frac{14}{2}=7$ the *less.*

And $7+6=13$ the *greater.*

Or thus. To find two numbers whose sum is s, and difference d.

Let the less be x; then the greater is $x+d$

And their sum is $2x+d$

Therefore $2x+d=s$

Whence $2x=s-d$, and $x=\frac{1}{2}s-\frac{1}{2}d$ the *less.*

And $\frac{1}{2}s-\frac{1}{2}d+d=\frac{1}{2}s+\frac{1}{2}d$, the *greater.*

Therefore when the sum, and difference, of two numbers are given, half the difference added to, and subtracted from half the sum, will be the greater and less, respectively.

4. The sum of two numbers is 19, and the difference of their squares 95: What are the numbers?

Put x for the greater number, and y for the less.

Then by the question.................. $x+y=19$

And.............................. $x^2-y^2=95$

Now dividing the second equation by the first, we have $\frac{x^2-y^2}{x+y}=\frac{95}{19}$ (73. Art.)

Or by actual division.................. $x-y=5$

Therefore we have the sum of the two numbers, $x+y=19$

and their difference $x-y=5$

Whence, by the preceding example, $\frac{19}{2}+\frac{5}{2}=12$ the *greater*; and $\frac{19}{2}-\frac{5}{2}=7$ the *less*.

5. What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 10?

For the less put.............................. x

Then the greater will be .... $1-x$

Whence by the question .... $\frac{1-x}{x}=10$

or............ $1-x=10x$

therefore... $1=11x$

And... $\frac{1}{11}=x$, the *less*

And $1-\frac{1}{11}=\frac{10}{11}$ the *greater*.

6. A General having detached 620 men to take possession of a strong post, and $\frac{3}{7}$ of the remainder of his troops to watch the motions of the enemy, finds that he has only $\frac{3}{13}$ of his army left; what was his whole force?

Let the whole number of men be.............................. x

Then after 620 were detached, the remainder was............ $x-620$

$\frac{3}{7}$ of these are.. $\frac{3x-1860}{7}$

And $\frac{3}{13}$ of the whole is.. $\frac{3x}{13}$

Now by the question, the two last parts with

620 must make the whole; viz......... $620+\frac{3x-1860}{7}+\frac{3x}{13}=x$

which cleared of fractions gives $56420+39x-24180+21x=91x$

whence... $56420-24180=31x$

or....... $32240=31x$

and $x=1040$, the *Answer*.

7. Three battalions of unequal force are in column of march; the extent of the first battalion is 216 paces, the extent of the second is equal to that of the first and third together, and the extent of the third is equal to that of the first and half the second: what is the length of the column?

Let the length of the third be x

Then that of the second will be .... $216+x$

The first and half the second together is...... $216+\frac{216+x}{2}$

Which, by the quest. is equal to the third, viz. $216+\frac{216+x}{2}=x$

whence......... $432+216+x=2x$

And......... $x=648$ the third

$648+216=864$ the second

216 the first

The whole $=$ 1728 *paces*.

8. The Double Rule of False is founded on the supposition, That the differences between the true and supposed numbers are directly proportional to the respective differences between the true and erroneous results: now it is required to show if the Arithmetical process is conformable to that supposition. (Arith. *Art.* 109.)

Let S and s be the two suppositions, D and d the corresponding errors or differences between the results and the number with which they are compared; also, let x denote the number required.

Then $x-S$, and $x-s$ will be the differences between the true and supposed numbers when the latter are both too little:

And $S-x$, and $s-x$, when they are both too great.

Now by the supposition, $x - S : x - s :: D : d$; whence $Dx - Ds = dx - dS$,

and $Dx - dx = Ds - dS$

therefore $x = \dfrac{Ds - dS}{D - d}$.

And $S - x : s - x :: D : d$ gives .... $x = \dfrac{dS - Ds}{d - D}$.

Now in both these cases, the errors are alike. And each expression is the *difference* of the products divided by the *difference* of the errors. (*First rule.*)

But when the errors are unlike, we shall have

either $S - x : x - s :: D : d$, whence $x = \dfrac{dS + Ds}{d + D}$;

or $x - S : s - x :: D : d$, and $x = \dfrac{Ds + dS}{D + d}$;

Where the expressions give the *sum* of the products divided by the *sum* of the errors: which is the *second rule.*

9. Divide 10 into three such parts, that when the first is multiplied by 2, the second by 3, and the third by 4, the three products may be equal?

Let x, y, and z, denote the three parts:

Then, by the question.......... $x + y + z = 10$

and.... $2x = 3y = 4z$

Now because $2x = 4z$, therefore $x = 2z$

Also, since $3y = 4z$, we have $y = \dfrac{4z}{3}$

Now putting $2z$, and $\dfrac{4z}{3}$ for x and y in the first equation

and we have $2z + \dfrac{4z}{3} + z = 10$

whence $6z + 4z + 3z = 30$

and.......... $z = \dfrac{30}{13} = 2\dfrac{4}{13}$

Or thus. Assume three quantities which being multiplied by 2, 3, and 4, respectively, shall give the same quotient; thus,

Suppose $\dfrac{x}{2}$, $\dfrac{x}{3}$, and $\dfrac{x}{4}$; then $\dfrac{x}{2} + \dfrac{x}{3} + \dfrac{x}{4} = 10$

And $9\frac{3}{13}$ divided by 2, 3, and 4, respectively, give $4\frac{8}{13}$, $3\frac{1}{13}$, $2\frac{4}{13}$, the three parts required.

10. Let 10 be divided into 4 parts such, that when they are respectively divided by 2, 3, 4, and 5, the quotients shall be in the same proportion as 6, 7, 8, and 9?

Assume $2\times 6x$, $3\times 7x$, $4\times 8x$, $5\times 9x$ for the 4 parts; (these being divided by 2, 3, 4, and 5, produce quotients in the given proportions).

Then .... $12x+21x+32x+45x=10$

or.......................... $110x=10$

and.......................... $x=\frac{1}{11}$

Therefore

$$\left.\begin{array}{l} \frac{1}{11}\times 12=1\frac{1}{11} \\ \frac{1}{11}\times 21=1\frac{10}{11} \\ \frac{1}{11}\times 32=2\frac{10}{11} \\ \frac{1}{11}\times 45=4\frac{1}{11} \end{array}\right\} \text{the 4 parts required.}$$

11. There is a number consisting of two digits, and if 72 be subtracted from it, the digits will be inverted. What is the number?

The answer is found from the following property, namely; The difference of a number consisting of two digits, and the number when those digits are inverted, is 9 *times the difference of the digits:* Thus, if 35 be the number, then the difference of 35, $(3\times 10+5)$ and 53, $(5\times 10+3)$ is 18, or 2 (the difference of 3 and 5) multiplied by 9.

Generally. If a and b are the digits, and $10a+b$ the number, then $10b+a$ is the number when the digits a and b change places: now subtracting the latter from the former, we have

$$10a+b-10b-a=9a-9b=(a-b)\times 9.$$

To apply this to the question, we have only to divide 72 by 9, and the quotient 8 is the difference of the digits; therefore 1 and 9 must be the digits; and 91 the number. For $91-72=19$.

Corol. Hence the difference between any number, and the number made by its digits in a contrary order, is always divisible by 9.

6. Required the square, and also the cube of $\sqrt{3}-\sqrt{2}$?

$$\begin{array}{l}
\sqrt{3}-\sqrt{2} \\
\underline{\sqrt{3}-\sqrt{2}} \\
\quad 3-\sqrt{6} \\
\quad\quad \underline{-\sqrt{6}+2} \\
\quad 5-\sqrt{24} \text{......... the square.} \\
\underline{\sqrt{3}-\sqrt{2}} \\
5\sqrt{3}-\sqrt{72} \\
\quad\quad \underline{-5\sqrt{2}+\sqrt{48}} \\
5\sqrt{3}-\sqrt{72}-5\sqrt{2}+\sqrt{48}=9\sqrt{3}-11\sqrt{2}, \text{ the cube.}
\end{array}$$

For $\sqrt{72}=6\sqrt{2}$; and $\sqrt{48}=4\sqrt{3}$; whence $4\sqrt{3}+5\sqrt{3}=9\sqrt{3}$; and $6\sqrt{2}+5\sqrt{2}=11\sqrt{2}$.

195. *To find the root of a surd*, divide its index by the index of the root to be extracted. And when the surds have rational coefficients, their roots must be prefixed.

Examples.

1. What is the square root of $a^2\ (b^{\frac{2}{3}})$?

The square root of a^2 is a; and the square root of $b^{\frac{2}{3}}$ is $b^{\frac{2}{3\times2}}=b^{\frac{1}{3}}$; therefore $ab^{\frac{1}{3}}$ is the root required.

2. What is the square root of $49c^2\ (81^{\frac{1}{3}})$?

The square root of $49c^2$ is $7c$.

And $81^{\frac{1}{3\times2}}=81^{\frac{1}{6}}$ or $9^{\frac{1}{3}}$ is the square root of $81^{\frac{1}{3}}$.

Ans. $7c\,(9^{\frac{1}{3}})$.

3. Required the square root of 12^3?

$12^{\frac{3}{2}}=1728^{\frac{1}{2}}$ or $12\ (12^{\frac{1}{2}})$. *Ans.*

4. Required the *4th.* root of $5x^2y^2$? *Ans.* $5^{\frac{1}{4}}\,(xy)^{\frac{1}{2}}$.

5. What is the *m*th. root of $ax^{\frac{2}{n}}$? *Ans.* $a^{\frac{1}{m}}x^{\frac{2}{mn}}$.

6. What is the cube root of $(a^3c^3x)^{\frac{1}{2}}$?

$(a^3c^3x)^{\frac{1}{2}}=(ac)^{\frac{3}{2}}x^{\frac{1}{2}}$; and $(ac)^{\frac{3}{2\times3}}x^{\frac{1}{2\times3}}=a^{\frac{1}{2}}c^{\frac{1}{2}}x^{\frac{1}{6}}$. *Ans.*

7. What is the $-m$th. root of ax^{-n}?

$$a^{\frac{1}{-m}}x^{\frac{-n}{-m}} = a^{-\frac{1}{m}}x^{\frac{n}{m}}. \quad \textit{Ans.}$$

8. Required the square root of $a - 2\sqrt{ab} + b$?

Ans. $\sqrt{a} - \sqrt{b}$, or $\sqrt{b} - \sqrt{a}$.

And the square root of $a + 2\sqrt{ab} + b$, is $\sqrt{a} + \sqrt{b}$.

126. The roots in the two last examples are obvious by inspection only; but if numbers are substituted for the letters, the square becomes a binomial: thus, let $5 = a$, and $3 = b$, then $(\sqrt{5} + \sqrt{3})^2 = 8 + \sqrt{60}$, where the rational part 8 is the sum of the squares of the two surds, and the irrational part $\sqrt{60}$, twice their product. To discover the root in such cases, suppose $a + \sqrt{b}$ is the binomial, and let x and y denote the two members of its root, viz. put $(x + y)^2 = a + \sqrt{b}$.

Then $x^2 + y^2 + 2xy$ being $= a + \sqrt{b}$,

we shall have $x^2 + y^2 = a$, and $2xy = \sqrt{b}$.

And $y^2 = a - x^2$. Also $4x^2y^2 = b$; whence $y^2 = \frac{b}{4x^2}$.

Therefore $a - x^2 = \frac{b}{4x^2}$.

And $ax^2 - x^4 = \frac{b}{4}$, or $x^4 - ax^2 = -\frac{b}{4}$ (77).

Now if $\frac{a^2}{4}$ be added to each side of this equation, we get

$$x^4 - ax^2 + \frac{a^2}{4} = \frac{a^2}{4} - \frac{b}{4} = \frac{a^2 - b}{4}; \quad (75.\ \textit{Ax.}\ 1)$$

But $x^4 - ax^2 + \frac{a^2}{4}$ is a square, its root being $x^2 - \frac{a}{2}$;

Therefore $x^2 - \frac{a}{2} = \sqrt{\frac{a^2 - b}{4}} = \frac{\sqrt{(a^2 - b)}}{2}$. (75. *Ax.* 7).

whence $x^2 = \frac{a}{2} + \frac{\sqrt{(a^2 - b)}}{2} = \frac{a + \sqrt{(a^2 - b)}}{2}$; and $x = \sqrt{\left(\frac{a + \sqrt{(a^2 - b)}}{2}\right)}$.

Now y being $= \sqrt{(a - x^2)}$,

we have $y = \sqrt{\left(a - \frac{a + \sqrt{(a^2 - b)}}{2}\right)} = \sqrt{\left(\frac{a - \sqrt{(a^2 - b)}}{2}\right)}$.

Again, by the third equation, $cxz + czv - zvx = -cxv$

$$\text{whence } z = \frac{-cxv}{cx + cv - vx}.$$

$$\text{And } \frac{-byv}{bv + by - yv} = \frac{-cxv}{cx + cv - xv} \text{ (two values of } z \text{ made equal)}$$

This reduced gives $byx - bcy = cxy - bcx$

We now have three equations involving three unknown quantities,

viz. $axv - abx = bvx - abv$

$byx - bcy = cxy - bcx$

$dyv + dxv + dxy = xyv.$

By the first, $axv + abv - bvx = abx,$

$$\text{whence } v = \frac{abx}{ax + ab - bx}.$$

From the third equation, $dyv + dxv - xyv = -dxy,$

$$\text{whence } v = \frac{-dxy}{dy + dx - xy}:$$

$$\text{Therefore } \frac{abx}{ax + ab - bx} = \frac{-dxy}{dy + dx - xy}$$

which reduced becomes $2abdy + abdx - abxy = bdxy - adxy.$

Now v being exterminated, the equations are reduced to the two following, involving only x and y,

viz. $2abdy + abdx - abxy = bdxy - adxy.$

$bxy - bcy = cxy - bcx.$

By the first, $2abdy - abxy - bdxy + adxy = -abdx$

$$\text{whence } y = \frac{-abdx}{2abd - abx - bdx + adx}.$$

From the second, $bxy - bcy - cxy = -bcx,$

$$\text{which gives } y = \frac{-bcx}{bx - bc - cx}:$$

$$\text{Therefore, } \frac{-bcx}{bx - bc - cx} = \frac{-abdx}{2abd - abx - bdx + adx} \text{ (the two values of } y.)$$

$$\text{This reduced gives } x = \frac{3abcd}{abc + bcd + bad - 2adc}.$$

Now the values of y, z, and v, are easily discovered: For $3abcd$ will evidently be a common numerator: And since b is not found in the negative part of the denominator in the expression for x, it follows that $-2abd$, $-2abc$, and $-2bcd$, are the other negative quantities, where c, d, and a, are respectively excluded.

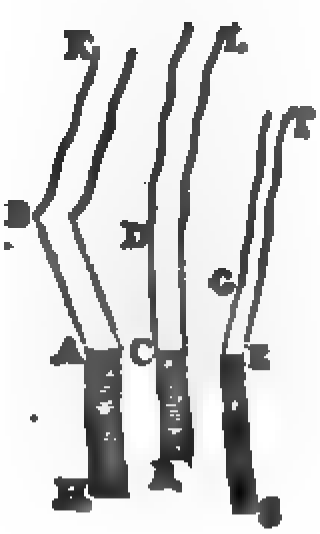
$$\text{Therefore } y = \frac{3abcd}{abc + bcd + acd - 2abd}$$

$$z = \frac{3abcd}{bcd + acd + abd - 2abc}$$

$$u = \frac{3abcd}{acd + abd + abc - 2bcd}$$

A single example proposed in numbers and wrought arithmetically, is less tedious than the preceding operation. But the algebraic method has the advantage of giving general formulæ or expressions for all questions of the kind.

14. Suppose B battalions of troops, of equal strength, are in three columns HA, RC, SE, and that they have to pass through the roads or defiles BK, DL, GT, whose breadths admit of different fronts. Let the times of marching from A to K, from C to L, and from E to T, be denoted by a, b, and c, respectively; also put r, s, and t for the respective times in which a battalion can march its own length in BK, DL, and GT. Now it is required to determine the number of battalions of which each column should be composed in order to enable their rears to quit the defiles at K, L, and T, at one and the same time, or that the whole march through the defiles may be made in the least time possible?



Let x, y, and z denote the number of battalions in the columns HA, RC, and SE, respectively.

Then rx, sy, and tz will be the times in which they can march their own lengths in the respective defiles.

$$\text{And } \left.\begin{array}{r} a + rx \\ b + sy \\ c + tz \end{array}\right\} \begin{array}{c}\text{times of} \\ \text{marching from}\end{array} \left\{\begin{array}{l} \text{H to K} \\ \text{R to L} \\ \text{S to T} \end{array}\right.$$

Now, by the question, those times must be equal;

$$\text{or, } a + rx = b + sy$$
$$a + rx = c + tz$$

From the first equation $\dfrac{a - b + rx}{s} = y$.

But $z = B - x - y$ (because $x + y + z = B$) which put for z in the second equation, gives $a + rx = c + Bt - tx - ty$,

$$\text{whence } \frac{a - c - Bt + rx + tx}{-t} = y:$$

$$\text{Therefore } \frac{a - b + rx}{s} = \frac{a - c - Bt + rx + tx}{-t},$$

$$\text{which reduced gives } x = \frac{Bts + sc + bt - ta - sa}{tr + sr + st}.$$

And repeating the operation for y and z, we have

$$y = \frac{Btr + ta + rc - br - bt}{tr + sr + st}.$$

$$z = \frac{Bsr + sa + rb - cr - cs}{tr + sr + st}.$$

Examp. Suppose the number of battalions to be $20 = B$.

And let BK = 2 miles = 4224 paces of $2\frac{1}{2}$ feet each, and the rate of marching 70 paces *per minute*.

DL = $2\frac{1}{4}$ miles = 4752 paces, ——— rate 65 p. *per min.*
GT = 1 mile = 2112 paces, ——— rate 50 p. *per min.*
AB = 1 mile = 2112 p.
CD = $\frac{3}{4}$ mile = 1584 p. $\Big\}$ rate of marching 80 p. *per min.*
EG = $\frac{1}{2}$ mile = 1056 p.

Paces
205 depth or extent of a battal. in the defile BK
270 .... in DL
350 .... in GT.

Then

$\frac{4224}{70} = 60.34$ *min.* time of marching from B to K.

$\frac{2112}{80} = 26.4$ *min.* time of marching from A to B.

86.74 *min.* $= a$, time of marching from A to K.

$\frac{4752}{65} = 73.11$ *min.* time of marching from D to L

$\frac{1584}{80} = 19.8$ *min.* time of marching from C to D

92.91 *min.* $= b$, time of marching from C to L.

$\frac{2112}{50} = 42.24$ *min.* time of marching from G to T

$\frac{1056}{80} = 13.2$ *min.* time of marching from E to G

55.44 min. $= c$, time of marching from E to T.

$$\frac{205}{70} = 2.93 \text{ min.} = r.$$

$$\frac{270}{65} = 4.154 \text{ min.} = s$$

$$\frac{350}{50} = 7.0 \text{ min.} = t.$$

Those values being substituted in the foregoing expressions, give $x = 8$, $y = 4$, and $z = 8$, the nearest integers. Therefore the columns HA, RC, must each consist of 8 battalions, and RC of 4.

In this example, the three columns are supposed to begin their march at the same time: but should it be found necessary to delay the movement of either column, the numeral value of the corresponding letter must be varied accordingly, and a new division of the battalions take place. Thus suppose the troops at A and C are to begin their march 25 minutes before those which pass the defile GT,

Then c will be $55.44 + 25 = 80.44$ *min.* and the resulting values of x, y, and z, are 10, 5, and 5 (the nearest integers) for the number of battalions in the columns AH, RC, and SE, so that the whole body may clear the defiles in the least time possible, *in that case*.

Should the value of either expression be less than $\frac{1}{2}$, the whole body will pass in two columns only: Thus suppose the rate of marching in DL, the middle defile, is only 45 paces *per minute*:

Then $\frac{4752}{45} = 105.6$, and $105.6 + 19.8 = 125.4$ *min.* $= b$.

And $\frac{270}{45} = 6$ *min* $= s$. Whence $y = \frac{1}{10}$ nearly;

And retaining a, c, r, and t, as in the first example, x and z will be found 12 and 8, (the nearest whole numbers) respectively, for the number of battalions in the columns HA, and SE.

But when it is proposed to make the division for two roads or defiles only, the expressions become much more simple; for in that case we have but two equations,

namely, $a + rx = b + sy$,

and $x + y = B$.

Whence $x = \dfrac{Bs + b - a}{r + s}$, and $y = \dfrac{Br + a - b}{r + s}$.

Now suppose the 20 battalions are to march through BK, and DL only; and let $a = 86.74$, $b = 92.91$, $r = 2.93$, and $s = 4.154$, as before.

$$\text{Then } x = \frac{89.05}{7.084} = 12.6, \text{ and } y = \frac{52.43}{7.084} = 7.4$$

or 13, and 7 battalions in the columns HA, and RC, respectively.

15. In drawing up a certain number of men into a square column, it was found that 21 men were left; but when the side of the square was increased by 1 man, then 34 men were wanting to complete the square. Required the number of men?

Let x be the number of men in the side of the first square;

Then x^2+21 is the whole number of men:

And $(x+1)^2-34$, or $x^2+2x+1-34$, is also the number:

Therefore $x^2+21=x^2+2x-33$

or $21=2x-33$

Whence $x=27$; therefore $27^2+21=750$, the *answer*.

16. To find 3 numbers in Harmonical Proportion, when the difference of the first and second is denoted by a, and that of the second and third by b.

If 3 numbers are in musical proportion, the first will be to the third, as the difference between the first and second, is to the difference of the second and third.

Let the first number be x;

Then the second will be $x+a$;

And the third............ $x+a+b$.

Whence, as $x : x+a+b :: a : b$

therefore $ax+a^2+ab=bx$

$$\text{and } x = \frac{a^2+ab}{b-a}.$$

Let $a=2$, $b=3$. Then $\frac{a^2+ab}{b-a}=10$, the first number.

$10+2=12$ the second.

$10+2+3=15$ the third.

This Harmonic Proportion relates to the lengths of Musical Cords. *Thus*, if 6 strings of equal thickness and tension, are made to sound or *vibrate* together, the greatest harmony they can produce, will be when

their lengths are in the proportion of $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$: Whence those fractions are said to be in Musical Proportion.

Since the denominators 1, 2, 3, 4, 5, 6, are in Arithmetical proportion, it follows, that numbers in harmonic proportion are the reciprocals of numbers in arithmetical proportion, (and *vice versa*). For $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, are the reciprocals of 1, 2, 3, 4, 5, 6.

If the fractions are reduced to a common denominator, the numerators will be 60, 30, 20, 15, 12, 10; which are 6 numbers in harmonic proportion.

17. What is the least number of weights, and the weight of each, that will weigh any number of pounds from 1lb. to an Hundred weight?

It is evident that one of the weights must be 1lb:.

Now let x denote the next greater weight: then in order to weigh 2lb. $x-1$ must be equal to 2,

viz. $x-1=2$, or $x=3$:

And since $3+1=4$, it follows that 1, 2, 3, and 4lb. may be weighed with 1lb. and a 3lb.

Again, put x for the *greatest weight next greater* than 3: Then, to weigh 5lb. with the weights 1, 3, and x, the value of $x-3-1$ must not exceed 5;

therefore making $x-3-1=5$, gives $x=9$.

And $9+3+1=13$; consequently any number of pounds up to 13 may be weighed with the weights 1, 3, and 9.

For $9+3+1$	$=13$	9	$=8+1$
$9+3$	$=12$	$9+1$	$=7+3$
$9+3$	$=11+1$	9	$=6+3$
$9+1$	$=10$	9	$=5+3+1$
9	$=9$		

And if x denote the weight next greater than 9; then $x-9-3-1=14$; and $x=27$: Hence it appears, that the least number of weights, in all cases, will be the geometric series 1, 3, 9, 27, 81, 243, &c. The first 5 however, are sufficient in the present example: for $1+3+9+27+81=121$lb. that may be weighed with those 5 weights.

OF QUADRATIC EQUATIONS.

129. It has already been observed (74) that when the highest power of the unknown quantity in an equation is of two dimensions, the equation is called a *Quadratic.*

If the equation involves the square only, it is a simple quadratic; as $x^2 = bc$, where the value of x is $= \sqrt{bc}$.

But when one term contains the square, and another its root, the equation is an adfected or affected one: These are all reducible to the three following forms:

$$x^2 + ax = b.$$
$$x^2 - ax = c.$$
$$x^2 - ax = -d.$$

The method of resolving these equations is easily deduced from the square of a binomial, thus:

Let $x + r$ be the binomial, then its square is $x^2 + 2rx + r^2$, in which we are to remark, that half $2r$ the coefficient of x in the middle term, is r the root of r^2 the third term. Therefore the third term of the square of which $x^2 + ax$ are the two first terms, will be $\frac{1}{4}a^2$ the square of half the coefficient a; the whole square being $x^2 + ax + \frac{1}{4}a^2$, and its root $x + \frac{1}{2}a$. This is called *completing the square.*

To find the value of x in the equation $x^2 + ax = b$: add $\frac{1}{4}a^2$ to each side of the equation, and we have

$$x^2 + ax + \tfrac{1}{4}a^2 = b + \tfrac{1}{4}a^2 \text{ (75. Ax. 1.)}$$

And taking the root of each side, $x + \frac{1}{2}a = \sqrt{(b + \frac{1}{4}a^2)}$ (Ax. 7.)

whence $x = \sqrt{(b + \frac{1}{4}a^2)} - \frac{1}{2}a$.

But (104) the square root of $b + \frac{1}{4}a^2$ is also denoted by $-\sqrt{(b + \frac{1}{4}a^2)}$, therefore $x = -\sqrt{(b + \frac{1}{4}a^2)} - \frac{1}{2}a$, which is the negative value of x; the former being the affirmative one.

The value of x in the second equation is found in the same manner: for by adding $\frac{1}{4}a^2$ (the square of half the coefficient a) to each side of the equation, we get

$$x^2 - ax + \tfrac{1}{4}a^2 = c + \tfrac{1}{4}a^2$$

and extracting the roots, ... $x - \frac{1}{2}a = \sqrt{(c + \frac{1}{4}a^2)}$

whence $x = \frac{1}{2}a + \sqrt{(c + \frac{1}{4}a^2)}$

which is the affirmative root.

But in this case, $\frac{1}{2}a - x$ is also the square root of $x^2 - ax + \frac{1}{4}a^2$, for $(\frac{1}{2}a - x)^2 = x^2 - ax + \frac{1}{4}a^2$,

therefore $\frac{1}{2}a - x = \sqrt{(c + \frac{1}{4}a^2)}$

which gives $x = \frac{1}{2}a - \sqrt{(c + \frac{1}{4}a^2)}$, the negative root.

This ambiguity is usually denoted by means of the double or uncertain sign, $\pm$, thus $x = \frac{1}{2}a \pm \sqrt{(c + \frac{1}{4}a^2)}$.

By completing the square, and extracting the roots in the third form $x^2 - ax = -d$,

we get $x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - d)}$, where both roots or expressions will be affirmative. For since $\sqrt{\frac{1}{4}a^2}$ is $= \frac{1}{2}a$, $\sqrt{(\frac{1}{4}a^2 - d)}$ must be less than $\frac{1}{2}a$; therefore $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - d)}$ will be affirmative.

If d be greater than $\frac{1}{4}a^2$, then $\sqrt{(\frac{1}{4}a^2 - d)}$ is an imaginary quantity whose root cannot be assigned; in which case the roots, or values of x are both impossible.

Respecting the two affirmative roots $\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - d)}$ that result from this third form, the nature of the problem will determine which is to be taken: both values however, frequently answer the conditions of the question.

130. *Other Examples.*

1. Given $x^2 + 8x = 209$. To find x.

The square of $\frac{1}{2}$ the coefficient 8 is 16,

Therefore $x^2 + 8x + 16 = 209 + 16$,

or $x^2 + 8x + 16 = 225$.

And taking the square roots, $x + 4 = \sqrt{225} = 15$,
whence $x = 15 - 4 = 11$, the *Answer*.

2. Given $x^2 - 6x = 72$. To find x.

The square of half 6 is 9;

Whence $x^2 - 6x + 9 = 72 + 9 = 81$,

And by evolution, $x - 3 = \sqrt{81} = 9$.

Therefore $x = 3 \pm 9 = 12$ and -6, the positive, and negative roots, or values of x.

3. Given $x^2 - 12x = -35$. To find x.

Completing the square gives $x^2 - 12x + 36 = 36 - 35 = 1$.

And by evolution we get $x - 6 = 1$,

whence $x = 6 \pm 1 = 7$ and 5 the two roots:

And both answer the conditions of the question:

For $7^2 - 12 \times 7 = -35$
And $5^2 - 12 \times 5 = -35$.

4. Given $3x^2 + 21x = 180$. To find x.

In this, and all other examples in which the square of the unknown quantity is affected with a coefficient, it is evident the whole equation must be divided by that coefficient before the square can be completed.

Now dividing by 3 gives $x^2 + 7x = 60$

And completing the square, $x^2 + 7x + 12\frac{1}{4} = 60 + 12\frac{1}{4} = 72\frac{1}{4}$.

Whence, by evolution, $x + 3\frac{1}{2} = \sqrt{72\frac{1}{4}} = 8\frac{1}{2}$
therefore $x = 8\frac{1}{2} - 3\frac{1}{2} = 5$, the *Answer*.

5. Given $x^2 - x = -\frac{2}{9}$. To find the value of x.

Here 1 is the coefficient of the second term x. Therefore the square being compleated, we have $x^2 - x + \frac{1}{4} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$;

And taking the roots,......$x - \frac{1}{2} = \sqrt{\frac{1}{36}} = \frac{1}{6}$,
whence..........$x = \frac{1}{2} \pm \frac{1}{6} = \frac{2}{3}$ and $\frac{1}{3}$ the two values of x; both of which answer the question.

6. Given $ax^2 - x = b$. To find x.

The whole divided by a gives.............. $x^2 - \frac{1}{a}x = \frac{b}{a}$.

And completing the square...... $x^2 - \frac{1}{a}x + \frac{1}{4a^2} = \frac{b}{a} + \frac{1}{4a^2}$;

Whence, by evolution................ $x - \frac{1}{2a} = \sqrt{\left(\frac{b}{a} + \frac{1}{4a^2}\right)}$

Therefore $x = \frac{1}{2a} \pm \sqrt{\left(\frac{b}{a} + \frac{1}{4a^2}\right)}$.

7. Given $ax^2 + bx^2 + dx - cx = m$. To find x.

Dividing by $a + b$ the coefficient of x^2, we have

$$x^2 + \frac{d-c}{a+b}x = \frac{m}{a+b}$$

And completing the square, $x^2 + \frac{d-c}{a+b}x + \left(\frac{d-c}{2a+2b}\right)^2 = \frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2$

By evolution,......................... $x + \frac{d-c}{2a+2b} = \left(\frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2\right)^{\frac{1}{2}}$

whence $x = \left(\frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2\right)^{\frac{1}{2}} - \frac{d-c}{2a+2b}$.

8. Given $x + \sqrt{x} = a$. To find x.

By transposition......................... $a - x = \sqrt{x}$

And squaring both sides....... $a^2 - 2ax + x^2 = x$

Whence, by transposition,... $x^2 - 2ax - x = -a^2$

or $x^2 - (2a + 1)x = -a^2$

And completing the square $x^2 - (2a + 1)x + (a + \frac{1}{2})^2 = (a + \frac{1}{2})^2 - a^2$.

or $x^2 - (2a + 1)x + (a + \frac{1}{2})^2 = a + \frac{1}{4}$

And taking the roots......................... $x - (a + \frac{1}{2}) = \sqrt{(a + \frac{1}{4})}$

whence $x = a + \frac{1}{2} \pm \sqrt{(a + \frac{1}{4})}$.

Let $a = 15\frac{3}{4}$. Then $x = 16\frac{1}{4} \pm 4 = 12\frac{1}{4}$ and $20\frac{1}{4}$, the two values of x; but it is only the first which answers the conditions of the equation. The other value $20\frac{1}{4}$ is what would result, supposing $x - \sqrt{x} = 15\frac{3}{4}$ (a); for in that case, $-\sqrt{x} = a - x$, whence, by squaring both sides, we get

$x = a^2 - 2as + s^2$ as before, because the square of $-\sqrt{x}$ is the same as that of $\sqrt{x}$.

Sometimes the process of resolving an equation may be abridged by making use of a substitution, as in the two next examples.

9. Given $\sqrt{(a+x)} - b\,(a+x)^{\frac{1}{4}} = m$. To find x.

Let $z^4 = a + x$.

Then.................................$z = (a+x)^{\frac{1}{4}}$

And.................................$z^2 = \sqrt{(a+x)}$.

Therefore.........................$z^2 - bz = m$;

Whence, by completing the square, and extracting the roots,

we get $z = \frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}$.

Consequently $z^4 = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4$;

or $a + x = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4$;

Therefore $x = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4 - a$.

10. Given $x^2 + xy = 918$.

$xy - 3y^2 = 42$. To find x and y.

Put $zy = x$:

Then $z^2y^2 + zy^2 = 918$: whence $y^2 = \frac{918}{z^2 + z}$.

And $zy^2 - 3y^2 = 42$: whence $y^2 = \frac{42}{z - 3}$:

Therefore $\frac{42}{z - 3} = \frac{918}{z^2 + z}$:

And, by reduction, $42z^2 + 42z = 918z - 2754$;

whence $z^2 - 20\frac{6}{7}z = -65\frac{4}{7}$:

Now by completing the square, and extracting the roots,

we get $z = 10\frac{3}{7} \pm 6\frac{4}{7} = 17$, and $3\frac{6}{7}$. These values being substituted for z in the equation $y^2 = \frac{42}{z - 3}$, give $y = \sqrt{3}$, and $y = 7$;

And the corresponding values of x will be $17\sqrt{3}$, and 21.

Therefore $\left.\begin{array}{l}x=27\\ y=\ \ 7\end{array}\right\}$ the rational values: And $\left.\begin{array}{l}x=17\sqrt{3}\\ y=\ \ \sqrt{3}\end{array}\right\}$ the irrational.

11. Given $x^6-4x^3=621$. To find x.

Since x^3 is the square root of x^6, this equation is solved after the manner of a quadratic: thus,

Add 4, the square of half the coefficient 4, to each side of the equation, and we have

$$x^6-4x^3+4=625$$

And extracting the square roots, $x^3-2=25$

whence $x^3=27$; and $x=3$.

In general; any equation of this form $x^{2n}-ax^n=b$, (where x is the unknown quantity, and the indices $2n$, and n, are one double the other,) is resolved in the same manner:

For by adding $\frac{1}{4}a^2$ to each side of the equation,

we have $x^{2n}-ax^n+\frac{1}{4}a^2=b+\frac{1}{4}a^2$

Now $x^n-\frac{1}{2}a$ is the square root of $x^{2n}-ax^n+\frac{1}{4}a^2$,

Therefore $x^n-\frac{1}{2}a=\sqrt{(b+\frac{1}{4}a^2)}$

whence $x^n=\frac{1}{2}a\pm\sqrt{(b+\frac{1}{4}a^2)}$

And $x=[\frac{1}{2}a\pm\sqrt{(b+\frac{1}{4}a^2)}]^{\frac{1}{n}}$.

131. QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To divide 40 into two such parts, that the sum of their squares shall be 818?

Let one of the parts be.................... x

Then the other will be .... $40-x$

By the question.............. $(40-x)^2+x^2=818$

or $\quad 1600-80x+x^2+x^2=818$

that is $\quad 2x^2-80x=818-1600=-782$

or $\quad x^2-40x=-391$

And completing the square $x^2 - 40x + 400 = 9$

By evolution $x - 20 = 3$

whence $x = 20 \pm 3 = 23$ and 17, the two parts required.

2. To divide 11 into two such parts, that the sum of their cubes may be 407 ?

Let one part be.................................x

Then the other must be.............$11 - x$

And, by the question.......$(11 - x)^3 + x^3 = 407$

or....$1331 - 363x + 33x^2 - x^3 + x^3 = 407$

that is.... ...$1331 - 363x + 33x^2 = 407$

or.....................$33x^2 - 363x = -924$

And dividing by 33.................................$x^2 - 11x = -28$

And completing the square.............$x^2 - 11x + 30\frac{1}{4} = 2\frac{1}{4}$

whence, by evolution,.................................$x - 5\frac{1}{2} = 1\frac{1}{2}$

therefore $x = 5\frac{1}{2} \pm 1\frac{1}{2}$ $= 7$ and 4, the parts required.

3. To divide 146 into two such parts, that the difference of their square roots may be 6.

Suppose the least root to be..........x

Then the other must be..............$x + 6$

Whence, by the question, $(x + 6)^2 + x^2 = 146$

that is $2x^2 + 12x + 36 = 146$

or $x^2 + 6x + 18 = 73$

Whence $x^2 + 6x = 55$

And completing the square, $x^2 + 6x + 9 = 55 + 9 = 64$;

By evolution, $x + 3 = 8$

and $x = 5$, one of the roots; whence $5 + 6 = 11$ the other. And the two parts are 5^2 and 11^2.

4. The sum of two numbers being $2s$, and the sum of their 4*th.* powers p, to find the numbers.

Let ½ the difference of the numbers be denoted by x; then ½ their sum being s,

we have $s + x$ for the greater,
and $s - x$ the less. (128. *Ex.* 3.)

Then, by the question, $(s+x)^4 + (s-x)^4 = p$.

$$\text{Now } (s+x)^4 = s^4 + 4s^3x + 6s^2x^2 + 4x^3s + x^4$$
$$(s-x)^4 = s^4 - 4s^3x + 6s^2x^2 - 4sx^3 + x^4$$
$$\text{sum } \quad 2s^4 \quad + 12s^2x^2 \quad + 2x^4$$

or .... $2s^4 + 12s^2x^2 + 2x^4 = p$;

And dividing by 2 .... $x^4 + 6s^2x^2 + s^4 = \frac{1}{2}p$,

or .... $x^4 + 6s^2x^2 = \frac{1}{2}p - s^4$

And completing the square, $x^4 + 6s^2x^2 + 9s^4 = \frac{1}{2}p + 8s^4$.

Whence, by evolution, ... $x^2 + 3s^2 = \sqrt{(\frac{1}{2}p + 8s^4)}$

Therefore .... $x^2 = \sqrt{(\frac{1}{2}p + 8s^4)} - 3s^2$

and .... $x = (\sqrt{(\frac{1}{2}p + 8s^4)} - 3s^2)^{\frac{1}{2}}$.

Suppose the sum $= 12 = 2s$ (or $s = 6$) and the sum of the biquadrates $= 3026 = p$.

Then, those values substituted for s and p in the expression for x, and we have $x = \sqrt{(109 - 108)} = 1$.

Therefore $6 \pm 1 = 7$ and 5, the two numbers.

5. The sum (s), and sum of the squares (p) of four numbers in arithmetical progression, being given; to find the numbers.

Numbers in arithmetical progression have a common difference (121. Arith.) Therefore, if $2x$ be that difference, and $4a = s$.

Then $a - 3x$, $a - x$, $a + x$, $a + 3x$, will denote the four numbers; for their sum is $4a$ (or s), and common difference $2x$.

Now by the question, $(a-3x)^2+(a-x)^2+(a+x)^2+(a+3x)^2=p$.

$$\begin{aligned}
(a-3x)^2 &= a^2-6ax+9x^2\\
(a-x)^2 &= a^2-2ax+x^2\\
(a+x)^2 &= a^2+2ax+x^2\\
(a+3x)^2 &= a^2+6ax+9x^2\\
\hline
\text{sum}\quad 4a^2 &\qquad +20x^2=p:\\
\hline
\text{or}\quad 20x^2 &= p-4a^2=p-\tfrac{1}{4}s^2\\
\text{whence}\quad x &= \sqrt{\left(\frac{4p-s^2}{80}\right)}
\end{aligned}$$

Or the whole difference $2x=\sqrt{\dfrac{4p-s^2}{20}}$.

And in the same manner, if the number of terms be three, their common difference will be found $=\sqrt{\dfrac{3p-s^2}{6}}$. Also if 5 be the number, the common difference is $\sqrt{\dfrac{5p-s^2}{50}}$. Hence it appears, that the coefficient of p is the number of terms; and that the denominators 6, 20, 50, &c. resulting from the coefficients of s^2, are each equal to half the number of terms drawn into the sum of the series $1+3+6+10+$ &c. continued to $n-1$ terms, n being the number of terms whose sum is given.

Now the sum of the series $1+3+6+10$, &c. continued to $n-1$ terms, is $n\times\dfrac{n^2-1}{6}$ (144), which drawn into $\dfrac{n}{2}$ gives $\dfrac{n^2}{2}\times\dfrac{n^2-1}{6}$:

Therefore, if the sum (s), and sum of their squares (p) of any number (n) of terms in arithmetical progression, are given, then $\sqrt{\left(\dfrac{np-s^2}{\frac{n^2}{2}\times\frac{n^2-1}{6}}\right)}$ or its equal $\dfrac{2}{n}\sqrt{\dfrac{3np-3s^2}{n^2-1}}$ is the common difference of the terms.

Examp. Let the number of terms be 7, their sum $=49$, and the sum of their squares $=455$; then, putting those numbers for n, s, and p, respectively, we get 2 for the difference; whence the numbers are 1, 3, 5, 7, 9, 11, and 13.

6. The sum (s) and continued product (p) of 5 numbers in arithmetical progression, being given; to find those numbers.

Let $5a = s$; and put $2x$ for the common difference.

Then $a - 4x$, $a - 2x$, a, $a + 2x$, $a + 4x$, will denote the 5 numbers.

Therefore, by the Quest. $(a - 4x)(a - 2x)(a)(a + 2x)(a + 4x) = p$;

or $64\,ax^4 - 20a^3x^2 + a^5 = p$:

Whence, by division, $x^4 - \frac{5}{16}a^2x^2 + \frac{a^4}{64} = \frac{p}{64a}$

or $x^4 - \frac{5}{16}a^2x^2 = \frac{p - a^5}{64a}$.

Now, by completing the square and extracting the roots,

we get $x^2 = \frac{5}{32}a^2 \pm \sqrt{\frac{16p + 9a^5}{32^2\,a}}$.

If $s = 25$, and $p = 945$; then a being $= 5$; we shall have $x^2 = 1$; and $x = 1$, therefore $2x = 2$ the common difference of the numbers or terms; whence the 5 numbers are found to be 1, 3, 5, 7, and 9.

7. The sum (s) and product (p) of any two numbers being given; to find the sum of their squares, cubes, biquadrates, &c.

Let the two numbers be .... x and y

Then, by the question, ... $x + y = s$

and .... $xy = p$

The first equation squared gives .... $x^2 + 2xy + y^2 = s^2$

whence .... $x^2 + y^2 = s^2 - 2xy$

But $2xy = 2p$; therefore $x^2 + y^2 = s^2 - 2p$, the sum of the squares.

Now multiply the last equation by the first,

and we have .... $(x^2 + y^2)(x + y) = (s^2 - 2p)s$

or .... $x^3 + xy\,(x + y) + y^3 = s^3 - 2sp$.

But $xy\,(x + y) = sp$, which substituted in the last equation

gives $x^3 + sp + y^3 = s^3 - 2sp$

or $x^3 + y^3 = s^3 - 3sp$ the sum of the cubes.

Again, let this last equation be multiplied by the first,

Then $(x^3 + y^3)(x + y) = (s^3 - 3sp)\,s$

or $x^4 + xy\,(x^2 + y^2) + y^4 = s^4 - 3s^2p$

But $xy(x^2+y^2) = p(s^2-2p)$, therefore, by substitution,

$$x^4 + p(s^2-2p) + y^4 = s^4 - 3s^2p$$

or $x^4 + y^4 = s^4 - 3s^2p - p(s^2-2p) = s^4 - 4s^2p + 2p^2$ the sum of the *4th.* powers.

And the sum of the 5th. powers will be

$$(s^4 - 4s^2p + 2p^2)\,s - (s^3 - 3sp)\,p\,;$$

That is, the sum of the next superior powers is constantly obtained by multiplying the sum of the powers last found by s, and subtracting from that product the sum of the next preceding ones multiplied by p.

And the sum of the nth. powers will be

$$s^n - ns^{n-2}p + n\times\frac{n-3}{2}\times s^{n-4}p^2 - n\times\frac{n-4}{2}\times\frac{n-5}{3}\times s^{n-6}p^3$$

$$+\,n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}\times s^{n-8}p^4, \text{ \&c.}$$

Where it is evident the series, or expression for the sum of the powers, will terminate when the least index of s becomes $=0$.

8. Given the sum (s), and sum of the squares (p) of four numbers in geometrical progression; to find those numbers.

Let the first number be denoted by $\frac{y^2}{x}$, and the common ratio by $\frac{x}{y}$;

Then the 4 numbers will be $\frac{y^2}{x}$, y, x, $\frac{x^2}{y}$, (142. Arith.)

By the question................. $\frac{y^2}{x} + y + x + \frac{x^2}{y} = s$

and..................... $\left(\frac{y^2}{x}\right)^2 + y^2 + x^2 + \left(\frac{x^2}{y}\right)^2 = p$.

Put $x+y=z$, and $xy=v$. Then $\frac{y^2}{x} + \frac{x^2}{y} = s - z$,

And........ .... $\frac{y^2}{x}\times\frac{x^2}{y} = xy = v$ (144 Arith.)

But, by the preceding question, if the sum $x+y=z$, and the product $xy=v$; then $x^2+y^2=z^2-2v$.

And for the like reason, if the sum $\frac{y^2}{x} + \frac{x^2}{y} = s - z$, and the product $\frac{y^2}{x}\times\frac{x^2}{y} = v$,

then............. .... $\left(\frac{y^2}{x}\right)^2 + \left(\frac{x^2}{y}\right)^2 = (s-z)^2 - 2v$.

Consequently the sum of the four squares will be

$$x^2+y^2+\left(\frac{y^2}{x}\right)^2+\left(\frac{x^2}{y}\right)^2=z^2-2v+(s-z)^2-2v=z^2+(s-z)^2-4v=p.$$

And since............................ $\frac{y^2}{x}+\frac{x^2}{y}=s-z,$

we have, by reduction,........ $y^3+x^3=xy\,(s-z)$

Or, putting v for its equal xy, gives.. . $y^3+x^3=v\,(s-z)=vs-vz$

But, by the preceding quest.......... $y^3+x^3=z^3-3vz$

Therefore, by equality, ... $z^3-3vz=vs-vz$

whence.................. $v=\frac{z^3}{2z+s}$

this value of v substituted in the equation $z^2+(s-z)^2-4v=p$

gives............ $z^2+(s-z)^2-\frac{4z^3}{2z+s}=p$

which reduced becomes....... $z^2+\frac{p}{s}\,z=\frac{s^2-p}{2}$

whence........ $z=\sqrt{\left(\frac{s^2-p}{2}+\frac{p^2}{4s^2}\right)}-\frac{p}{2s}.$

Examp. Let $s=80$, and $p=3280$. Then $z=24$, which put for z in the equation $v=\frac{z^3}{2z+s}$, gives $v=108$. Therefore $x+y=24$, and $xy=108$; whence $x=6$, and $y=18$: and the 4 numbers are 2, 6, 18, 54.

9. To find two numbers such, that their sum, product, and sum of their squares shall, if possible, be equal to each other.

Let the two numbers be represented by x and y.

Then, by the question,... .... $x+y=xy$

and.......... $x^2+y^2=xy$

Now, if $2xy$ be added to each side of this last equation,

we have............ $x^2+2xy+y^2=3xy$

And extracting the square root... $x+y=\sqrt{3xy}$

But $x+y=xy$, therefore by equality, $\sqrt{3xy}=xy$

whence......... $3xy=x^2y^2$

Consequently, by division.... .... $3=xy$

But $x+y=xy$, therefore............... $x+y=3$

And $x+y=x^2+y^2$, whence.......... $x^2+y^2=3$

Now $x+y$ being $=3$, we have $x=3-y$

Therefore......... .... $(3-y)^2+y^2=3$

From this equation...$y-1\frac{1}{2}=\sqrt{-\frac{3}{4}}$, which being an impossible quantity, it follows that no two numbers can be found to answer the conditions of the question.

10. A regiment of Foot was ordered to send 216 men on Garrison duty, each Company to furnish a like number; but before the detachment marched, three of the Companies were sent on another service, when it was found that each Company which remained was obliged to furnish 12 additional men in order to make up the complement 216. Hence the number of Companies composing the regiment is required?

Let the number of companies in the regiment be... x

Then the number of men which each would have sent, will be $\frac{216}{x}$

But the number of Companies left when 3 were sent away, is $x-3$

And the number of men which each sent on Garrison duty, in that case, is .... $\frac{216}{x-3}$

Therefore the difference must be 12,...... viz. $\frac{216}{x-3}-\frac{216}{x}=12$

which reduced gives......... $x^2-3x=54$

whence $x=9$, the *Answer*.

11. The number of men in both fronts of two columns of troops A and B when each consisted of as many ranks as it had men in front, was 84: but when the columns changed ground, or A was drawn up with the front that B had, and B with the front that A had, then the number of ranks in both columns was 91. Required the number of men in each column?

Let x^2 and y^2 denote the men in the two columns, respectively:

Then, by the question, $x+y=84$.

And $\frac{x^2}{y}$ was the number of ranks in the column x^2 when drawn up with the front y;

And $\frac{y^2}{x}$ the number of ranks in the column y^2 with the front x. Therefore by the quest. $\frac{x^2}{y}+\frac{y^2}{x}=91$.

Put $z=\frac{x-y}{2}$, and $a=\frac{84}{2}=42$

Then $x=a+z$, and $y=a-z$ (128 *Ex.* 3.)

And $\frac{(a+z)^2}{a-z}+\frac{(a-z)^2}{a+z}=91$

Or $(a+z)^3 + (a-z)^3 = 91\,(a^2 - z^2)$

Which reduced gives $z^2 = \dfrac{91a^2 - 2a^3}{91 + 6a} = 36$; whence $z = 6$.

Therefore $42 \pm 6 = 48$, and 36, the values of x and y.

And $\left.\begin{array}{l} 48^2 = 2304 \\ 36^2 = 1296 \end{array}\right\}$ The No. of Troops.

Of UNLIMITED or INDETERMINATE PROBLEMS.

132. If the independent equations expressing the conditions of a Question are fewer in number than the unknown quantities they involve, the Problem is said to be indeterminate or unlimited (81) because it frequently admits of innumerable answers. But the number of results are generally limited by restricting the values of the unknown quantities to integers. Thus, if $x + y = 5$, then x and y may be any two numbers whatever, whose sum is 5; but 1, 4, 2 and 3 are all their integral values.

133. When the relation of two unknown quantities only are expressed in a simple equation, let the whole equation be divided by the least of the two coefficients, then put the fractional part of the quotient equal to some letter denoting a whole number, positive, or negative, according to the value of the fraction, and a new equation will be obtained; then proceed with the least coefficient as before, and so on, till the last assumed letter has 1 for its coefficient resulting from division; and the expression will come under one of these forms, $\dfrac{z \pm n}{m}$, or $\dfrac{z}{m}$, where the unknown quantity or letter (z) may, in general, be assumed so as to give two extreme integral values in the required answer, whence the others are readily found.

Examples.

1. Given $9x + 13y = 200$: required the values of x and y in whole positive numbers.

By transposition................ $9x = 200 - 13y$

And dividing the whole equation by 9... $x = \frac{200-13y}{9} = 22 + \frac{2-4y}{9} - y$.

Now x and y are to be whole numbers, therefore $\frac{2-4y}{9}$ must also be a whole number, because the sums or differences of whole numbers are integers.

Also, since y is a whole number, $4y$ must be greater than 2, and consequently $\frac{2-4y}{9}$ the fractional part of the quotient, will be negative;

Therefore, put $\frac{2-4y}{9} = -a$ (a negative whole number)

$$\text{Then } 2 - 4y = -9a$$

$$\text{Whence } 4y = 9a + 2$$

$$\text{Therefore } y = \frac{9a+2}{4} = 2a + \frac{a+2}{4}$$

Now to make $\frac{a+2}{4}$ a positive integer, a must be expounded by some term in the series 2, 6, 10, 14, &c. but its least possible value is when $a = 2$, the expression in that case being 1; which gives 5 for the least value of y; and the corresponding or greatest value of x is 19.

If a be taken $= 6$; then $y = 14$, and $x = 2$

$$\text{Therefore } \begin{array}{r|l} y = 5 & 14. \\ x = 19 & 2. \end{array}$$

Which are all the values in positive integers.

The foregoing process is evidently analogous to that of finding the greatest common measure of two numbers in Arithmetic, (40. Arith.) and founded on the same principle, namely, *if a number measures another number, and also a part of that number, it will measure the remaining part.*

2. Given $256x - 87y = 1$; to find the least possible values of x and y in whole positive numbers.

By transposition ... . $87y = 256x - 1$

And dividing by the least coefficient 87, gives $y = 2x + \frac{82x-1}{87}$

Let $\frac{82x-1}{87} = a$; then $82x - 1 = 87a$

$$\text{whence } x = \frac{87a+1}{82} = a + \frac{5a+1}{82}$$

Next, put $\frac{5a+1}{32} = b$; then $5a+1 = 32b$

$$\text{whence } a = \frac{32b-1}{5} = 16b + \frac{2b-1}{5};$$

Now assume $\frac{2b-1}{5} = c$; then $2b = 5c+1$

$$\text{whence } b = \frac{5c+1}{2} = 2c + \frac{c+1}{2};$$

If $c = 1$, then $\frac{c+1}{2} = 1$ the least possible integer. Whence $b = 3$, and $a = 49$; therefore $x = 32$, and the corresponding value of y is 133, the required values of x and y.

The other values of x and y are unlimited in number, because c may be any integer so that $c+1$ is divisible by 2.

3. Given $19x - 14y = 11$; to find the least possible values of x and y in whole numbers.

By transposition.................. $14y = 19x - 11$

And dividing by 14 gives....... $y = x + \frac{5x-11}{14}$

Put $\frac{5x-11}{14} = a$; then $5x - 11 = 14a$,

$$\text{whence } x = \frac{14a+11}{5} = 2a + \frac{4a+1}{5} + 2.$$

Next, let $\frac{4a+1}{5} = b$; then $a = b + \frac{b-1}{4}$: now to make this expression the least possible integer (1) the value of b must be 5, in that case $x = 19$, and $y = 25$, which are two values of the unknown quantities, but not the least; these however, are discovered from the fraction $\frac{4a+1}{5}$ which, when $a = 1$, becomes 1, and $x\left(= 2a + \frac{4a+1}{5} + 2\right) = 5$, and the corresponding value of y is 6; the numbers answering the conditions of the question.

4. Given $5x + 7y = 29$; to find x and y in positive integers.

From the given equation, $x = \frac{29-7y}{5} = 5 + \frac{4-2y}{5} - y$;

Let $\frac{4-2y}{5} = a$; then $4 - 2y = 5a$

$$\text{whence } y = \frac{4-5a}{2} = 2 - \frac{a}{2} - 2a;$$

Now if $2-\frac{a}{2}-2a$ (or y) is a positive integer, a cannot be any affirmative whole number whatever; therefore making $a=0$, y becomes $=2$, and thence $x=3$; which are all the integral values of x and y in the proposed equation.

5. Let $11x+16y=100$; required the values of x and y in positive integers.

By transposition we get............ $11x=100-16y$

whence...... $x=\frac{100-16y}{11}=9+\frac{1-5y}{11}-y$:

Let $\frac{1-5y}{11}=-a$ $\left(\frac{1-5y}{11}\right.$ being evidently negative$\left.\right)$

Then $1-5y=-11a$, whence $5y=11a+1$,

and $y=\frac{11a+1}{5}=2a+\frac{a+1}{5}$, where the least value of a to make this a whole number, must be 4, which gives $y=8+1=9$:

But from the given equation $11x+16y=100$, it follows that y must be less than 6: And therefore no whole numbers can be found to answer the question.

6. Given $17x+19y=2000$; to find all the values of x and y in affirmative whole numbers.

By transposition we get $17x=2000-19y$

And dividing by 17 gives $x=117+\frac{11-2y}{17}-y$:

Now it is evident that $\frac{11-2y}{17}$, the fractional part of the quotient, cannot be made a positive integer if y is a positive integer, whatever be its value,

therefore put $\frac{11-2y}{17}=-a$

Then, by reduction, $2y=17a+11$, and $y=8a+\frac{a+1}{2}+5$; where, if $a=1$, then $y=14$ the least affirmative value; and the corresponding or greatest value of x is 102.

The next value for a which gives $\frac{a+1}{2}$ an integer is 3, this being substituted, and we get $y=31$, whence $x=83$: now the difference between

102 and 83, the two values of x, is 19 the coefficient of y; and the difference of the two values of y is 17, the coefficient of x; therefore by constantly adding 17 to the last value of y, and subtracting 19 from that of x, we get all the other integral values.

$x =$ 102	83	64	45	26	7
$y =$ 14	31	48	65	82	99.

7. Suppose it is required to find integral values of x and y in the equation $9x + 15y = 100$.

Then since the coefficients 9 and 15 are divisible by 3, the sum $9x + 15y$ and also its equal 100 must be divisible by 3, whatever be the values of x and y; but 100 is not divisible by 3 without a remainder; therefore in this, and all similar cases, the unknown quantities x and y cannot be found in whole numbers.

8. How many different ways it is possible to pay 100£. with 7 shilling pieces, and dollars at 4*s*. 3*d*. each?

$$\left.\begin{array}{rcr} 100£. & = & 24000 \\ 7s. & = & 84 \\ 4s.\ 3d. & = & 51 \end{array}\right\} \text{pence.}$$

Let x denote the number of dollars, and y that of the 7*s*. pieces:

Then.............................. $51x + 84y = 24000$

Or, dividing by 3............. $17x + 28y = 8000$

$$\text{whence.........}\ x = \frac{8000 - 28y}{17} = 470 + \frac{10 - 11y}{17} - y.$$

Put $\dfrac{10 - 11y}{17} = -a$; then $10 - 11y = -17a$; whence $y = a + \dfrac{6a + 10}{11}$;

Now, let $\dfrac{6a + 10}{11} = b$; then $a = b + \dfrac{5b - 4}{6}$.

And making $\dfrac{5b - 4}{6} = c$, we have $b = c + \dfrac{c + 4}{5}$; Now if $c = 1$, then $b = 2$, and $a = 2$, whence $y = 4$ the least affirmative value of y; and the corresponding or greatest value of x is 464.

Hence, by adding 51 to the value of y, and subtracting 84 from that of x, we get the following answers, being 6 in number:

$x =$ 464	380	296	212	128	44
$y =$ 4	55	106	157	208	259.

9. To find a whole number which being divided by 15 shall leave 7, but when divided by 19 the remainder shall be 9.

If x be the required number; then $\frac{x-7}{15}$, and $\frac{x-9}{19}$ must be whole numbers, by the nature of the question.

Let $\frac{x-7}{15}=a$ (an integer); then $x-7=15a$, and $x=15a+7$, which being put for x in the second expression $\frac{x-9}{19}$, and we have $\frac{15a-2}{19}$ a whole number:

Now put $\frac{15a-2}{19}=b$; then $15a-2=19b$, and $a=b+\frac{4b+2}{15}$:

Again, make $\frac{4b+2}{15}=c$; then $4b+2=15c$, and $b=3c+\frac{3c-2}{4}$:

Next, let $\frac{3c-2}{4}=d$; and we get $c=d+\frac{d+2}{3}$:

Lastly, make $\frac{d+2}{3}=h$; then $d=3h-2$.

Therefore......... $d=3h-2$

$c=d+h=4h-2$

$b=3c+d=12h-6+3h-2=15h-8$

$a=b+c=15h-8+4h-2=19h-10$

$x=15(19h-10)+7=285h-143$: Where h may be any affirmative integer whatever; consequently if it be 1, the value of x will be the least possible (viz. 142).

10. To find in what year of Christ the cycle of Indiction was 10, the Golden number or Lunar cycle 10, and the cycle of the Sun 8.

These Periods are found thus; Add 3, 1, and 9 to the year, and divide the sums by 15, 19, and 28, respectively; then the remainders will be the cycles.

Let x denote the year:

$$\text{Then}\quad \left.\begin{array}{lcl} \frac{x+3-10}{15} & \text{or} & \frac{x-7}{15} \\ \frac{x+1-10}{19} & \text{or} & \frac{x-9}{19} \\ \frac{x+9-8}{28} & \text{or} & \frac{x+1}{28} \end{array}\right\}\ \text{must each be a whole number.}$$

By the preceding example, if $x = 285h - 143$, then $\frac{x-7}{15}$, and $\frac{x-9}{19}$ will be whole numbers; therefore substituting that value in the other expression gives $\frac{285h - 143 + 1}{28}$ or $\frac{285h - 142}{28}$, a whole number; put this $= a$,

Then $285h - 142 = 28a$, and $a = \frac{285h - 142}{28} = 10h + \frac{5h - 2}{28} - 5$.

Let $\frac{5h-2}{28} = b$ (an integer); then $h = 5b + \frac{3b+2}{5}$:

Again, putting $\frac{3b+2}{5} = c$, gives $b = c + \frac{2c-2}{3}$:

And if $\frac{2c-2}{3} = d$, then $c = d + \frac{d+2}{2}$, where if $d = 0$, then $c = 1$, and $b = 1$, which gives $h = 6$, and $x = 1567$ the required year: which is also the least possible integral value of x.

11. Suppose the qualities of three ingredients are denoted by 10, 15, and 16; how many pounds of each must be taken to make a mixture of 80*lb.* with the quality 12?

Or, if 10, 15, and 16 pence are the prices per pound; what quantity of each will make a mixture of 80*lb.* at 12 pence per *lb.*?

Let x, y, and z denote the respective numbers of pounds:

Then.......... $x + y + z = 80$

And $10x + 15y + 16z = 80 \times 12 = 960$

From the last equation subtract 10 times the first,

$$\begin{array}{r} 10x + 15y + 16z = 960 \\ 10x + 10y + 10z = 800 \\ \hline \text{there remains......} \quad 5y + 6z = 160 \end{array}$$

whence $y = \frac{160 - 6z}{5} = 32 - z - \frac{z}{5}$:

Let $\frac{z}{5} = a$; then $z = 5a$.

Therefore $y = 32 - 6a$

And $x = 48 + a$ (or $80 - (32 - 6a) - 5a$).

Here a may be any positive number whatever less than $\frac{32}{6}$, and therefore the problem is unlimited. But if the values of the unknown quantities are restricted to integers, then, expounding a by 1, 2, 3, 4, and 5, we get the 5 following answers, which are all the question admits of in whole numbers.

$x=5$	10	15	20	25
$y=26$	20	14	8	2
$z=49$	50	51	52	53.

12. How much gold of 15, of 17, and of 22 carats fine, must be mixed with 5 oz. of 18 carats fine, so that the composition may be 20 carats fine?

If x, y, and z, are the respective quantities, and $a=5$ oz. Then from the same principles upon which the preceding solution is founded, we shall have $15x+17y+22z+18a=20\ (x+y+z+a)$

or $15x+17y+22z+18a=20x+20y+20z+20a$;

whence $2z=5x+3y+2a$.

Now it is manifest without farther process, that the number of answers will be indefinite, for x and y may have any positive values; but if they are whole numbers, both must be even, or both odd, to give z an integer also: Thus, let $x=8$ oz. and $y=2$ oz. then $z=\frac{40+6+10}{2}=28$ oz. And if $x=5$, and $y=5$; then $z=\frac{25+15+10}{2}=25$ oz. &c.

13. Suppose a mass of Gold, another of Copper, and a third which is a mixture of those metals, when separately immersed in the same vessel filled with water, expell 8.2, 17.9 and 11.5 ounces of the fluid, respectively: now if each mass weighs 160 *oz.* what quantity of copper is in the compound mass?

The specific gravities of the metals will be reciprocally as the numbers 8.2, 17.9, and 11.5, or as 82, 179, and 115, which, therefore will denote the *rates* of the two simples, and the compound.

Let x be the copper, and y the gold in the compound mass,

Then $x+y=160$

And $179x+82y=115x+115y$

or............. $64x=33y$.

But $y=160-x$

Therefore $64x=33\ (160-x)=5280-33x$

And $97x=5280$, or $x=54.4$ oz. the quantity required.

In this manner it is said Archimedes discovered the quantity of alloy in a Crown that Hiero King of Sicily had ordered to be made with gold.

The Learner will perceive that the three last Examples belong to the Rule of *Alligation* in Arithmetic.

14. Suppose 960 troops are to be drawn up in three columns of march with 11, 14, and 19 men in front, respectively; now, how many different ways can this be done without any broken rank; and what number of ranks will each column consist of when their depths are the nearest possible equal to each other?

If x, y, and z denote the number of ranks in the respective columns,

then $11x+14y+19z=960$

And dividing the whole equation by 11 gives $x+y+\frac{3y}{11}+z+\frac{8z}{11}=87+\frac{3}{11}$

whence $x+y+z=87+\frac{3-3y-8z}{11}$;

Now, if y and z are any positive whole numbers, the expression $\frac{3-3y-8z}{11}$ must be negative;

therefore let.......... $\frac{3-3y-8z}{11}=-a$

Then.................... $3-3y-8z=-11a$

and.............. $3y=11a+3-8z$

or...... $y=3a+\frac{2a}{3}+1-2z-\frac{2z}{3}$

that is $y=3a+1-2z+\frac{2a-2z}{3}$,

Let....... $\frac{2a-2z}{3}=b$

then $2a-2z=3b$

and $z=a-b-\frac{b}{2}$;

But it appears from the equation $y=3a+\frac{2a}{3}+1-2z-\frac{2z}{3}$, that a must be 3, or some multiple of 3, to make $\frac{2a}{3}$ a whole number; therefore to obtain the least integral value of z, make $a=3$, and $b=0$,

Then $z=a$

whence $y=a+1$; and substituting these values in the original equation, we get $x=86-3a$

Consequently, if a or $z=3$

then $y=4$

And $x=77$;

And the other values of y and x are found by constantly adding 11 (the coefficient of x) last value of y, and subtracting 14 (the coefficient of y) from that of x. In this manner, by making z equal to 6, 9, 12, 15, &c. the multiples of 3, we get the following 28 answers;

$z=3$						6					9				
$y=4$	15	26	37	48	59	7	18	29	40	51	10	21	32	43	54
$x=77$	63	49	35	21	7	68	44	30	[illegible]	2	59	45	31	17	3

$z=12$				15			18			21	
$y=13$	24	35	46	16	27	38	19	30	41	22	33
$x=50$	36	22	8	41	27	13	32	18	4	23	9

Hence it appears that 23, 22, and 21 are the ranks in the respective columns when their depths are nearest alike.

And a similar method of solution may be followed when more than three unknown quantities are concerned. But different expedients will present themselves in the course of practice.

Of DIOPHANTINE PROBLEMS.

134. These are another kind of indeterminate Problems, called Diophantine, from *Diophantus* of Alexandria, an ancient Greek Mathematician who left a work on the subject, which chiefly relates to square and cube numbers. The Problems, for the most part, are of an abstruse nature, and do not seem to admit of any general method of solution. We shall subjoin a few easy examples in order to give the learner an idea of this part of Analysis.

Examples.

1. To divide a given number n into two such parts that the difference of their squares may be a given square a^2.

If x be the least part, then $n-x$ is the other;

And $(n-x)^2-x^2=n^2-2nx$ is the difference of their squares;

Whence, by the question, $n^2-2nx=a^2$

or $2nx=n^2-a^2$; and $x=\frac{n^2-a^2}{2n}$ the *least* part;

$$\text{and } n - \frac{n^2 - a^2}{2n} = \frac{n^2 + a^2}{2n} \text{ the } greater.$$

$$\text{For } \left(\frac{n^2 + a^2}{2n}\right)^2 - \left(\frac{n^2 - a^2}{2n}\right)^2 = a^2.$$

Here a^2 must be less than n^2, otherwise the problem is impossible.

Suppose $n = 3$, and $a = 2$; then $\frac{n^2 - a^2}{2n} = \frac{5}{6}$; and $\frac{n^2 + a^2}{2n} = \frac{13}{6}$: and the two parts are $\frac{5}{6}$ and $2\frac{1}{6}$.

2. To divide a given number n into two such parts that the sum of their squares shall be a square.

If x denotes one part, then $n - x$ will be the other:

And $(n - x)^2 + x^2$ or $n^2 - 2nx + 2x^2$ must be a square number: let its root be $n - ax$;

Then $n^2 - 2nx + 2x^2 = (n - ax)^2 = n^2 - 2anx + a^2x^2$;

whence, by reduction, $x = \frac{2n - 2an}{2 - a^2}$ one part:

And $n - \frac{2n - 2an}{2 - a^2} = \frac{2an - a^2n}{2 - a^2}$ the other, where a may be any assumed fraction less than unity.

Suppose the given number $(n) = 8$; and let $a = \frac{1}{2}$:

Then $\frac{16 - 8}{2 - \frac{1}{4}} = 4\frac{4}{7}$ one part; and $\frac{8 - 2}{2 - \frac{1}{4}} = 3\frac{3}{7}$ the other.

3. To find two square numbers whose difference shall be a given number d.

Let $\frac{1}{2}x + \frac{1}{2}y$, and $\frac{1}{2}x - \frac{1}{2}y$, denote the roots of the required squares;

$$\begin{aligned} \text{Then } (\tfrac{1}{2}x + \tfrac{1}{2}y)^2 &= \tfrac{1}{4}x^2 + \tfrac{1}{2}xy + \tfrac{1}{4}y^2 \\ (\tfrac{1}{2}x - \tfrac{1}{2}y)^2 &= \tfrac{1}{4}x^2 - \tfrac{1}{2}xy + \tfrac{1}{4}y^2 \\ \text{difference} &\quad xy \end{aligned}$$

Therefore $xy = d$ (by the question). Hence it appears that x and y may be any two unequal numbers whose product is the difference d: for should they be equal, then $\frac{1}{2}x - \frac{1}{2}y = 0$.

Let $d = 5$, $x = 5$, and $y = 1$; then $5 \times 1 = 5$:

$$\left.\begin{aligned} \text{And } \left(\frac{5}{2} + \frac{1}{2}\right)^2 &= 9 \\ \left(\frac{5}{2} - \frac{1}{2}\right)^2 &= 4 \end{aligned}\right\} \text{ are two squares whose difference is 5.}$$

4. **To divide a given square number s^2 into two other square numbers.**

If x^2 be one of the squares, the other must be s^2-x^2.

Assume $(nx-s)^2=s^2-x^2$: then $n^2x^2-2nsx+s^2=s^2-x^2$:

And, by reduction, we get $x=\left(\frac{2ns}{n^2+1}\right)$; therefore $x^2=\left(\frac{2ns}{n^2+1}\right)^2$ one of the required squares; and $s^2-\left(\frac{2ns}{n^2+1}\right)^2=\left(\frac{sn^2-s}{n^2+1}\right)^2$ the other; where n may be any number whatever, unity excepted.

Suppose 4 is the given square $(=s^2)$; and let $n=3$:

Then $\left(\frac{2ns}{n^2+1}\right)^2=\frac{36}{25}$ one square; and $\left(\frac{sn^2-s}{n^2+1}\right)^2=\frac{64}{25}$ the other, their sum being $\frac{36}{25}+\frac{64}{25}=4$.

5. **To divide a given number consisting of two square numbers, into two other square numbers.**

Suppose the given number is a^2+b^2; and let $x+a$, and $nx-b$ denote the roots of the required squares.

Then $(x+a)^2+(nx-b)^2=a^2+b^2$

or $x^2+2ax+a^2+n^2x^2-2nbx+b^2=a^2+b^2$

whence, by reduction, $x=\dfrac{2nb-2a}{n^2+1}$:

Therefore $x+a=\dfrac{2nb+n^2a-a}{n^2+1}$; and $nx-b=\dfrac{bn^2-2na-b}{n^2+1}$:

And the two required squares, $\left(\frac{2nb+n^2a-a}{n^2+1}\right)^2$, and $\left(\frac{bn^2-2na-b}{n^2+1}\right)^2$, where n may be any assumed number except 1.

If the given number be 13 $(4+9)$; then $a=2$, and $b=3$: and let $n=2$:

Then the two squares are $\left(\frac{12+8-2}{4+1}\right)^2=\frac{324}{25}$, and $\left(\frac{12-8-3}{4+1}\right)^2=\frac{1}{25}$: for $\frac{324}{25}+\frac{1}{25}=13$.

Remark. Every number cannot be divided into two rational squares: For let the number 3 be proposed: then since it is not the sum of two in-

tegral squares, assume it equal to two fractional ones, or suppose $3 = \frac{a^2}{z^2} + \frac{b^2}{z^2} = \frac{a^2 + b^2}{z^2}$; now multiplying each side by 3 gives $9 = \frac{3a^2 + 3b^2}{z^2}$, therefore $3a^2 + 3b^2$ must be a square, but no two integral numbers can be found, such, that 3 times the sum of their squares is a square number; whence it appears that 3 is not resolvable into two squares. And Euler's conclusion in his Algebra is, that when a whole number is not the sum of two integral squares, it cannot be the sum of two fractional ones.

6. To find two numbers whose sum shall be equal to the sum of their cubes.

This admits of one solution in integers, viz. when each of the numbers is 1; for $1 + 1 = 1^3 + 1^3$. But other answers may be found in fractions thus,

Let x and y denote the two numbers; then $x + y = x^3 + y^3$; and dividing the whole equation by $x + y$, gives $1 = \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2$.

Put $nx = y$; then $1 = x^2 - nx^2 + n^2x^2$, whence $x^2 = \frac{1}{n^2 - n + 1}$ which must be a square number because x^2 is a square; and consequently $n^2 - n + 1$ must also be a square.

Let $n^2 - n + 1 = a^2$; then $n^2 - n = a^2 - 1$;
and completing the square, $n^2 - n + \frac{1}{4} = a^2 - 1 + \frac{1}{4} = a^2 - \frac{3}{4}$,

whence $n - \frac{1}{2} = \sqrt{(a^2 - \frac{3}{4})}$:

The problem is now reduced to that of making $a^2 - \frac{3}{4}$ a rational square number, which is done by *Examp.* 3: for if we assume two numbers whose product is $= \frac{3}{4}$, half their sum will be the value of a:

Thus, let 1 and $\frac{3}{4}$ be assumed; then $1 \times \frac{3}{4} = \frac{3}{4}$ and half of $1 + \frac{3}{4}$ is $\frac{7}{8} = a$; therefore $a^2 = \frac{49}{64}$, and $\frac{49}{64} - \frac{3}{4} = \frac{1}{64}$:

Therefore $n - \frac{1}{2} = \sqrt{\frac{1}{64}} = \frac{1}{8}$, whence $n = \frac{5}{8}$:

Now, $x^2 = \frac{1}{n^2 - n + 1} = \frac{64}{49}$, and $x = \frac{8}{7}$; therefore y or $nx = \frac{5}{8} \times \frac{8}{7} = \frac{5}{7}$. Hence $\frac{8}{7}$ and $\frac{5}{7}$ are two fractions answering the conditions of the problem.

If 3 and $\frac{1}{4}$ are assumed; then $3 \times \frac{1}{4} = \frac{3}{4}$; and we get $\frac{15}{13}$ and $\frac{8}{13}$, which are two other fractions whose sum is equal to the sum of their cubes.

7. Should it be required to find two numbers whose *difference* is equal to the *difference* of their cubes:

Then $x - y = x^3 - y^3$; and $1 = \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$; and putting $nx = y$, as before, we get $x^2 = \frac{1}{n^2 + n + 1}$: now assuming $n^2 + n + 1 = a^2$ (a square), gives $n = \sqrt{\left(a^2 - \frac{3}{4}\right)} - \frac{1}{2}$, where the root $\sqrt{\left(a^2 - \frac{3}{4}\right)}$ must be *less* than $1\frac{1}{2}$, but *greater* than $\frac{1}{2}$.

Let 2 and $\frac{3}{8}$ be assumed, their product being $2 \times \frac{3}{8} = \frac{6}{8}$ or $\frac{3}{4}$: then $\frac{2 + \frac{3}{8}}{2} = \frac{19}{16} = a$, and $a^2 = \frac{361}{256}$; therefore $n = \sqrt{\left(\frac{361}{256} - \frac{3}{4}\right)} - \frac{1}{2} = \frac{5}{16}$; now $\frac{1}{n^2 + n + 1} = \frac{256}{361} = x^2$, whence $x = \frac{16}{19}$; and nx or $y = \frac{5}{16} \times \frac{16}{19} = \frac{5}{19}$. Therefore $\frac{16}{19}$ and $\frac{5}{19}$ are two fractions whose difference is equal to the difference of their cubes. And in like manner other answers may be found.

The three following Theorems will frequently be found useful in problems which relate to square numbers.

1. If $2ab$, $a^2 - b^2$, and $a^2 + b^2$, denote the roots of three square numbers; then $(2ab)^2 + (a^2 - b^2)^2 = (a^2 + b^2)^2$. By this Theorem two square numbers may be found whose sum or difference shall be square numbers.

2. If d be any number; then $(d^2 + (d + 1)^2) \times ((d + 1)^2 + (d + 2)^2) \pm (d + 1)^2 \times 4$ will be two squares, whose roots are $2d^2 + 4d + 3$, and $2d^2 + 4d + 1$.

3. If a and b be any two numbers; then $(a^2 + b^2)^2 \pm (a^2 - b^2) \times 4ab$ will be two squares, the roots being $a^2 \pm 2ab - b^2$.

Of ARITHMETICAL PROGRESSIONS.

135. The nature of Arithmetical proportion and progression has already been explained, (Arith. *Art.* 121). It is by the help of analysis however, that we must discover the different relations which the several terms have to one another: besides, algebraic formulæ are better adapted to practice, and more concise than verbal enunciations.

136. Let f be the first term of an arithmetical progression.
d the common difference of the terms.
l the last term.
n the number of terms.
s the sum of all the terms.

Then f, $f+d$, $f+2d$, $f+3d$, $f+4d$, &c. will be an ascending series or progression, (122 Arith.)

And f, $f-d$, $f-2d$, $f-3d$, $f-4d$, &c. a descending one.

Hence it appears that the last term is always $=f+(n-1)d$ in an ascending progression, and $f-(n-1)d$ in a descending one.

137. The sum of all the terms in an arithmetical progression is equal to the sum of the first and last terms multiplied by half the number of terms; *viz.* $s=(f+l)\frac{n}{2}$. (132 Arith.)

Let $f+f+d+f+2d+f+3d+f+4d=s$;

then $5f+10d=s$, *viz.* $(f+f+4d)\times\frac{5}{2}=s$, or $(f+l)\frac{n}{2}=s$.

And if the number of terms be 6, the last term will be $f+5d$;

And the sum $=6f+15d=s$, or $(f+f+5d)\frac{6}{2}=s$, that is $(f+f+(n-1)d)\frac{n}{2}$, or $(f+l)\frac{n}{2}=s$.

Now, from the equations $f + (n-1)d = l$, $(f + f + (n-1)d)\frac{n}{2} = s$, and $(f + l)\frac{n}{2} = s$, the following theorems or formulæ are readily obtained, where it is to be noted, that when the progression is descending, the signs of the terms affected with d must be changed, or f taken for l, and *vice versâ*; these forms being adapted to an ascending series.

138. $f = l - nd + d = \sqrt{(-2sd + l^2 + dl + \frac{1}{4}d^2)} + \frac{1}{2}d = \frac{s}{n} + \frac{1}{2}d - \frac{1}{2}nd = \frac{2s}{n} - l.$

$$d = \frac{l-f}{n-1} = \frac{l^2 - f^2}{2s - l - f} = \frac{2ln - 2s}{n^2 - n} = \frac{2s - 2nf}{n^2 - n}.$$

$$l = f + nd - d = \frac{s}{n} + \frac{1}{2}nd - \frac{1}{2}d = \sqrt{(2sd + f^2 - df + \frac{1}{4}d^2)} - \frac{1}{2}d = \frac{2s - nf}{n}.$$

$$s = \frac{nf + nl}{2} = (f + \frac{1}{2}nd - \frac{1}{2}d)n = (l - \frac{1}{2}nd + \frac{1}{2}d)n = \frac{l^2 - f^2 + dl + df}{2d}.$$

$$n = \frac{2s}{f+l} = \frac{l-f}{d} - 1.$$

Let $\frac{2f \backsim d}{2d} = r$, then

$n = \sqrt{\left(\frac{2s}{d} + r^2\right)} - r$ when $2f$ is *greater* than d.

$n = \sqrt{\frac{2s}{d}}$ .... when $2f = d$.

$n = \sqrt{\left(\frac{2s}{d} + r^2\right)} + r$ when $2f$ is *less* than d.

139. A few examples will show the use of these expressions.

1. Required the sum of the series $1 + 3 + 5 + 7 +$ &c. continued to 20 terms?

Here $f = 1$, $d = 2$, and $n = 20$, which substituted in the form $s = (f + \frac{1}{2}nd - \frac{1}{2}d)n$ gives $s = (1 + 20 - 1)\ 20$, or $20 \times 20 = 400$ the sum required.

Hence it appears, that the sum of the odd numbers $1 + 3 + 5 + 7$ &c. continued to n terms, is always $= n^2$.

2. What is the 17*th.* term of the series 10, $9\frac{2}{3}$, $9\frac{1}{3}$, 9, &c.

In this progression $f = 10$, $d = \frac{1}{3}$, $n = 17$; and the corresponding expression is $l = f + nd - d$ which, when the signs of the terms $+ nd - d$ are changed (the series being a descending one) becomes $l = f - nd + d$, or $l = 10 - 17 \times \frac{1}{3} + \frac{1}{3} = 4\frac{2}{3}$ the required term.

3. Two detachments, distant from each other 39 leagues, and both designing to occupy an advantageous post equidistant from each other's camp, set out at different times; the first detachment increasing every day's march one league and a half, and the second detachment decreasing each day's march 2 leagues: both detachments arrive at the same time; the first after 5 days march, and the second after 4 days march: What is the number of leagues marched by each detachment each day?

The whole distance marched by each detachment is $\frac{39}{2} = 19\frac{1}{2}$ leagues. Therefore, for the first detachment, we have $d = 1\frac{1}{2}$, $n = 5$, and $s = 19\frac{1}{2}$; and to find the first term or distance marched the first day, the expression is $f = \frac{s}{n} + \frac{1}{2}d - \frac{1}{2}nd$, or $\frac{19\frac{1}{2}}{5} + \frac{3}{4} - 3\frac{3}{4} = \frac{9}{10}$ of a league; whence $\frac{9}{10}$, $2\frac{4}{10}$, $3\frac{9}{10}$, $5\frac{4}{10}$, $6\frac{9}{10}$ are the respective distances marched each day.

And the same theorem or expression answers for the distance marched by the other detachment on the first day when the signs of the two last terms are changed; for $d = 2$, $n = 4$, and $s = 19\frac{1}{2}$, whence $f = \frac{s}{n} - \frac{1}{2}d + \frac{1}{2}nd = \frac{19\frac{1}{2}}{4} - 1 + 4 = 7\frac{7}{8}$ leagues the first day's march; therefore the distances are $7\frac{7}{8}$, $5\frac{7}{8}$, $3\frac{7}{8}$, $1\frac{7}{8}$.

4. A detachment of dragoons being sent after a deserter, marched the first day 9 miles, the second 19, the third 29, and so on, increasing the distance 10 miles each day; in what time did they overtake him, supposing he travelled at the rate of 34 miles a day?

This is readily answered by means of the expression $s = \frac{nf + nl}{2}$; for the number of days will be the number of terms, and $34n$ is equal to s the whole distance travelled; therefore substituting 9 the first term, for f, we

have $\frac{9n+nl}{2} = 34n$, or $\frac{9+l}{2} = 34$, whence $l = 59$ the number of miles which the detachment marched on the last day; consequently they overtook the deserter in 6 days.

5. A company of foot leave London for Plymouth, and at the same time a party of horse are ordered from Plymouth to London; the foot march 14 miles the first day, 13 the second, 12 the third, &c. constantly lessening each day's march 1 mile; but the horse travel 8 miles the first day, and increase their march 4 miles every day; what distance will each party have travelled when they meet, if Plymouth is 217 miles from London?

In this example we have the sum of a descending and an ascending progression	$= 217$
The first term of one progression........................	$= 14$
The common difference of the terms........................	$= 1$
The first term of the other series........................	$= 8$
And common difference	$= 4$

And the number of terms (or days) in each progression is the same.

The formula adapted to this case is $n = \sqrt{\left(\frac{2s}{d} + r^2\right)} - r$; therefore if 8 and 4 are substituted for f and d we get $\frac{2f-d}{2d} = \frac{16-4}{8} = 1\frac{1}{2} = r$; and putting x for the miles travelled by the foot, the distance travelled by the horse will be $217 - x = s$; whence $n = \sqrt{\left(\frac{2s}{d} + r^2\right)} - r = \sqrt{\left(\frac{434-2x}{4} + \frac{9}{4}\right)} - 1\frac{1}{2}$ the number of terms, or days travelled by the party of horse.

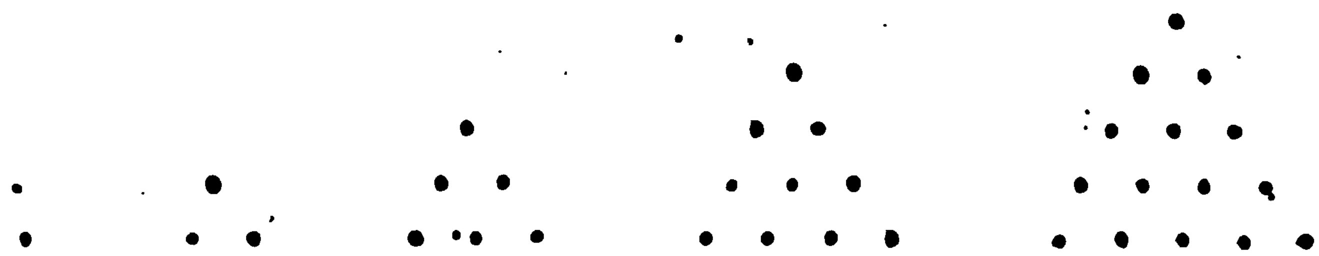
But the expression $n = \sqrt{\left(\frac{2s}{d} + r^2\right)} - r$ is derived from $f = \frac{s}{n} - \frac{1}{2}nd + \frac{1}{2}d$, which becomes $f = \frac{s}{n} + \frac{1}{2}nd - \frac{1}{2}$ when f is the first term of a descending progression, and in this case $r = \frac{2f+d}{2d}$, and $n = r - \sqrt{\left(r^2 - \frac{2s}{d}\right)}$.

Now, in the descending series, $f = 14$, and $d = 1$; therefore $\frac{2f+d}{2d} = \frac{28+1}{2} = 14\frac{1}{2} = r$, and $n = r - \sqrt{\left(r^2 - \frac{2s}{d}\right)} = 14\frac{1}{2} - \sqrt{(210\frac{1}{4} - 2s)}$ the number of terms, or days travelled by the foot:

Consequently we have this equation $\sqrt{\left(\frac{434-2x}{4}+\frac{9}{4}\right)} - 1\frac{1}{2} = 14\frac{1}{2} - \sqrt{(210\frac{1}{4}-2x)}$, which reduced gives $9x^2 + 3926x = 355663$, whence $x = 77$ the miles travelled by the company of foot; therefore $217 - 77 = 140$ the distance for the horse.

Of FIGURATE NUMBERS.

140. These numbers result from the sums of arithmetical series, and are called *figurate* because the units in each term may be so disposed as to represent a geometrical figure or diagram, as a triangle, a square, or a pentagon, &c. Thus 1, 3, 6, 10, &c., are a series of triangular numbers:



1 3 6 10 15

Let any number of units be disposed as in *fig.* 1, where the perpendicular ranks are one more in number than the horizontal ones. And draw the diagonal line AB to divide the number of terms into two equal parts.

fig. 1.

A

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

B

fig. 2.

A

1	2	3	4	5
1	2	3	4	5
1	2	3	4	5
1	2	3	4	5

B

fig. 3.

A

1	3	6	10	15
1	3	6	10	15
1	3	6	10	15
1	3	6	10	15

B

fig. 4.

A

1	4	10	20	35
1	4	10	20	35
1	4	10	20	35
1	4	10	20	35

B

fig. 5.

A

1	5	15	35	70
1	5	15	35	70
1	5	15	35	70
1	5	15	35	70

B

Then 1, 1, 1, 1, &c. (*fig.* 1.) are called a series of the first order;

And the sums of the units on the left of the line AB form the progression 1, 2, 3, 4, &c. or a series of the second order. These being disposed as in *fig.* 2. the sums of the horizontal rows on the left of AB constitute the series 1, 3, 6, 10, &c. or the third order;

$$\begin{array}{ll} \text{Thus} & 1 \ldots\ldots\ldots\ldots = 1 \\ & 1+2 \ldots\ldots\ldots = 3 \\ & 1+2+3 \ldots = 6 \\ & 1+2+3+4 = 10, \text{ \&c.} \end{array}$$

Again, the sums of the horizontal ranks on the left of the line AB in *fig.* 3. form the fourth order:

$$\begin{array}{ll} \text{for} & 1 \ldots\ldots\ldots\ldots = 1 \\ & 1+3 \ldots\ldots\ldots = 4 \\ & 1+3+6 \ldots = 10 \\ & 1+3+6+10 = 20, \text{ \&c.} \end{array}$$

And so on, for the several orders.

	Order						
141.	1	1,	1,	1,	1,	1,	&c.
	2	1,	2,	3,	4,	5,	&c.
	3	1,	3,	6,	10,	15,	&c.
	4	1,	4,	10,	20,	35,	&c.
	5	1,	5,	15,	35,	70,	&c.

Hence it appears that the last term in any order is always equal to the sum of all the terms in the next inferior one;

Thus the 5*th*. term in the 3*d*. order is 15; which is equal to 1 + 2 + 3 + 4 + 5 in the 2*d*. order.

142. The line AB in *fig.* 1, divides the sum of all the terms into two equal parts; but in *fig.* 2, the sum of all the terms

$(1 + 3 + 6$, &c.) on the left of AB is $= \frac{1}{3}$ of the sum of all the terms in the figure; for each vertical row on the right hand of AB is double the corresponding horizontal row on the left; thus $3 + 3 =$ twice $1 + 2$; $4 + 4 + 4 =$ twice $1 + 2 + 3$, &c. In like manner, the sum of all the terms on the left of AB, *fig.* 3, is $= \frac{1}{4}$ of all the terms in that figure. And in *fig.* 4, the sum on the left is $\frac{1}{5}$ of the whole, &c.

143. Now, let n denote the number of terms in a vertical row, or the number of horizontal ranks; then $n + 1$ will be the number of terms in an horizontal rank, and in *fig.* 2, $n + 1$ is the last term in that rank or series: therefore (137) $\frac{n+1}{2} \times (n+2)$ will be the sum of the series $1 + 2 + 3$, &c, or of all the terms in that rank, which multiplied by n the number of horizontal ranks, is $\frac{n+1}{2} \times (n+2) \times n$, the sum of all the terms in the figure, and $\frac{1}{3}$ of that sum or $\frac{n+1}{2} \times (n+2) \times n \times \frac{1}{3} = n \times \frac{n+1}{2} \times \frac{n+2}{3}$ is the sum of all the terms on the left of AB, or the sum of the series $1 + 3 + 6 + 10$, &c. continued to n terms.

144. To find the sum of the series $1 + 3 + 6 + 10$, &c. to $n + 1$ terms, substitute $n + 1$ for n, and $n \times \frac{n+1}{2} \times \frac{n+2}{3}$ becomes $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3}$ the sum of all the terms in an horizontal rank (*fig.* 3), which multiplied by n the number of ranks, gives $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times n$ the sum of all the terms which compose the figure, and $\frac{1}{4}$ of this is $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, or $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$ the sum of all the terms on the left of AB, or sum of the series $1 + 4 + 10 + 20$, &c. continued to n terms.

Hence it appears that $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}$. &c. continued to 5 factors will be the sum of the series which is the *5th.* order of figurate numbers: and 6 factors give the sum in the *6th.* order, &c.

145. These series are useful in computing the number of cannon shot in a pile. The piles are usually triangular, square, or oblong. The triangular pile has an equilateral triangle for its base, and ends in a ball at top; and the several layers or courses of shot form the series 1, 3, 6, 10, 15, &c. from the top downward, the last term being the number of shot in the course next the ground.

Now the series $1 + 3 + 6 + 10$, &c. is the third order, and its sum or the number of shot in a triangular pile is $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}$, where n is the number of courses or the number of shot in the side of the base.

Suppose the number of courses in a triangular pile or pyramid is 40; then $\frac{40}{1} \times \frac{40+1}{2} + \frac{40+2}{3} = 11480$ the number of shot in such a pile.

146. The square pile is a pyramid having a square for its base, and a single ball at the top, this ball with the successive courses downward constitute the series of squares, 1, 4, 9, 16, 25, &c. the last term being the number of shot in the bottom course.

The series of squares	$1 + 4 + 9 + 16 + 25$, &c. to n terms
may be resolved into the two series.........	$1 + 3 + 6 + 10 + 15$, &c. to n terms
	$1 + 3 + 6 + 10$, &c. to $n - 1$ terms
sum	$1 + 4 + 9 + 16 + 25$, &c.

The sum of $1 + 3 + 6 + 10$, &c. to n terms is $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}$, and putting $n - 1$ for n, gives $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n+1}{3}$ the sum to $n - 1$ terms; and the sum of both these expressions, or $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n+1}{3}$ $= \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{2n+1}{3}$ is the sum of the series of squares $1 + 4 + 9 + 16$, &c. continued to n terms.

Suppose 30 shot in the side of the bottom course, then $n = 30$ the number of courses, and $\frac{n}{1} \times \frac{n+1}{1} \times \frac{2n+1}{3} = 9455$ the number of shot in the pile.

147. The oblong pile stands on a rectangular base; the number of shot in any course being found by multiplying the number of shot in one of its sides by the number of shot in the other side: and the whole pile is composed of a series of rectangular courses, the sides each diminishing by 1 from the base upwards; therefore if d be the difference of the shot in the sides of any course, the pile will end at top in a rank of $d+1$ balls.

Thus, if the sides of the bottom course contain 12 and 7 shot,

Then	$12 \times 7 = 84$	
	$11 \times 6 = 66$	
	$10 \times 5 = 50$	are the shot in
	$9 \times 4 = 36$	the successive courses.
	$8 \times 3 = 24$	
	$7 \times 2 = 14$	
	$6 \times 1 = 6$	

The whole sum may be found by resolving the series $6 + 14 + 24 + 36$, &c. into two other series,

thus	$1 + 4 + 9 + 16 + 25$ &c.
	$5 + 10 + 15 + 20 + 25$ &c.
sum	$6 + 14 + 24 + 36 + 50$ &c.

The sum of the squares $1 + 4 + 9$ &c. continued to n terms is $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{2n+1}{3}$. The last term of the series $5 + 10 + 15$ &c. continued to n terms is nd; and the sum of the same series to n terms is $n \times \frac{nd+d}{2}$. (137.)

Therefore the sum of both series, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{2n+1}{3} + n \times \frac{nd+d}{2}$, or $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{2n+1+3d}{3}$ is the sum of the series of products continued to n terms, where n is the number of courses, or the number of shot in the least side of the bottom course, and d the difference of the number of shot in that side and the other.

Let the sides of the bottom course contain 32 and 25 shot;

Then $n=25$, and $d=7$, and $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1+3d}{3}=7800$ the number of shot in the pile.

148. If the pile is broken, find the number of shot deficient, and also the whole number it would contain, supposing it complete, then the difference will be the shot remaining.

Suppose the shot in the sides of the bottom course of a broken pile are 32 and 25, and in the upper course 23 and 16, then the shot in the sides of the next course would be 22 and 15; therefore $n=15$ the number of courses deficient, and $d=7$; and $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1+3d}{3}=2080$ the number of shot deficient: now the number in the complete pile (found above) would be 7800; therefore $7800-2080=5720$ is the number of shot in the broken pile. And in this manner we may proceed when the broken pile is triangular, or square.

149. If s be the number of shot in a complete triangular pile; and we would find the number of courses or the number of shot in the side of the base; then $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$ $=s$, or $n^3+3n^2+2n=6s$; and if $n+1$ be added to each side of this equation, we get $n^3+3n^2+3n+1=6s+n+1$: now n^3+3n^2+3n+1 is a cube whose root is $n+1$; therefore $n+1=(6s+n+1)^{\frac{1}{3}}$; and since the value of n is restricted to an integer, $6s+n+1$ will be the integral cube next greater than $6s$.

Let the number of shot in a complete triangular pile be 11480; then $11480\times 6=68880$, and the cube next greater is 68921 whose root is $41=n+1$, therefore $n=40$.

150. In the complete square pile we have $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1}{3}=$ $\frac{2n^3+3n^2+n}{6}=s$, or $n^3+1\frac{1}{2}n^2+\frac{1}{2}n=3s$, and adding $1\frac{1}{2}n^2+2\frac{1}{2}n+1$ to each side of the equation, gives n^3+3n^2+

$3n+1 = 3s + 1\frac{1}{2}n^2 + 2\frac{1}{2}n + 1$, whence $n+1 = (3s + 1\frac{1}{2}n^2 + 2\frac{1}{2}n + 1)^{\frac{1}{3}}$; and $n+1$ will be equal to the root of the integral cube next greater than $3s$.

Suppose $s = 9455$; then $3s = 28365$, and the cube number next greater is 29791 whose root is $31 = n+1$; whence $n = 30$ the number of courses.

Of GEOMETRICAL PROGRESSIONS.

151. Let f be the first term of a geometrical series of quantities:

r the common ratio of the terms:
l the last term of the series:
n the number of terms:
s the sum of all the terms.

Then $f, rf, r^2f, r^3f, \ldots\ldots r^{n-1}f$, is a geometrical progression or series of quantities, (146 Arith.)

Hence it appears that the series will be ascending, or descending according as the ratio or multiplier r is greater, or less than 1:

And that the last term is always $= r^{n-1}f = l$.

152. To find the sum of the progression:

Let $f + rf + r^2f + r^3f + r^4f$ &c $= s$, this multiplied by r

gives $\underline{rf + r^2f + r^3f + r^4f + r^5f \text{ \&c}} = \underline{rs}$

$\underline{-f \quad . \quad . \quad . \quad . \quad + r^5f} = \underline{rs - s}$ the remainder

when the upper series &c. is subtracted from the lower:

But r^5f is the last term r^4f multiplied by r, or $r^{n-1}f \times r = r^nf$:

Therefore $-f + r^5f = r^nf - f = rs - s$

whence $\frac{r^nf - f}{r-1} = s$. the sum of the series. This process is exactly similar to the arithmetical operation, (Arith. *Art.* 152.)

Now, from the equations, $r^{n-1}f = l$;

$$\frac{r^n f - f}{r - 1} = s;$$

we obtain the following theorems or formulæ:

153. $$f = \frac{l}{r^{n-1}} = s \times \frac{r-1}{r^n - 1} = s + rl - rs.$$

$$l = f \times r^{n-1} = \frac{rs - s + f}{r} = s \times \frac{r^n - r^{n-1}}{r^n - 1}.$$

$$s = f \times \frac{r^n - 1}{r - 1} = \frac{rl - f}{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}.$$

$$r = \frac{s - f}{s - l} = \left(\frac{l}{f}\right)^{\frac{1}{n-1}}.$$

Put the *logarithm* of $\frac{l}{f} = Q$;

the *log.* of $\frac{sr - s + f}{f} = P$;

the *log.* of $r = R$; then

$$n = \frac{Q}{R} + 1 = \frac{P}{R}.$$

The *logarithmic* expressions for the value of n are derived from the method of raising powers by means of logarithms, explained in the Arithmetic, *Art.* 161, 187, thus;

Since $f \times r^{n-1} = l$, therefore $r^{n-1} = \frac{l}{f}$: and because the logarithm of any power of a number is equal to the logarithm of that number multiplied by the index or exponent denoting the power (187 Arith.) therefore $(n-1)\ log.\ r = log.\ \frac{l}{f}$, that is, $nR - R = Q$, whence $n - 1 = \frac{Q}{R}$, or $n = \frac{Q}{R} + 1$.

And the other expression for n is found in the same manner; for $s \times \frac{r-1}{r^n - 1} = f$, which gives $r^n = \frac{sr - s + f}{f}$, whence $n \times log.\ r = log.\ \frac{sr - s + f}{f}$, or $nR = P$, therefore $n = \frac{P}{R}$.

134. Some Examples explaining the use of the preceding Theorems.

1. What is the sum of the progression $2+6+18+54$, &c. continued to 10 terms?

Here $f=2$, $r=3$, and $n=10$:

And $s=f\times\frac{r^n-1}{r-1}=2\times\frac{3^{10}-1}{3-1}=59048$ the *sum required*.

2. Required the sum of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ &c. continued to 8 terms?

Here $f=\frac{1}{2}$, $r=\frac{1}{2}$, and $n=8$:

Then $s=f\times\frac{r^n-1}{r-1}=\frac{1}{2}\times\frac{(\frac{1}{2})^8-1}{\frac{1}{2}-1}=\frac{1}{2}\times\frac{-\frac{255}{256}}{-\frac{1}{2}}=\frac{255}{256}$ the *answer*.

3. What is the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c. continued *ad infinitum*?

If the terms are supposed to be infinite in number, the last term must be $=0$: therefore $f=1$, $r=\frac{1}{3}$, and $l=0$:

And $s=\frac{rl-f}{r-1}=\frac{0-1}{\frac{1}{3}-1}=\frac{-1}{-\frac{2}{3}}=1\frac{1}{2}$ the *sum of the series*.

4. What is the sum of the series $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}$ &c. infinitely continued?

In this progression $f=\frac{6}{10}$, $r=\frac{1}{10}$, and the last term $l=0$:

And $s=\frac{rl-f}{r-1}=\frac{0-\frac{6}{10}}{\frac{1}{10}-1}=\frac{-\frac{6}{10}}{-\frac{9}{10}}=\frac{6}{9}=\frac{2}{3}$ the *answer*.

This series is the decimal .6666 &c. or $\frac{2}{3}$ reduced to a decimal.

5. Required the vulgar fraction corresponding to the recurring decimal .363636, &c.

This decimal may be resolved into the series $\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}$ &c. where $f=\frac{36}{100}$, $r=\frac{1}{100}$, and $l=0$:

$$\text{Then } s=\frac{rl-f}{r-1}=\frac{-\frac{36}{100}}{-\frac{99}{100}}=\frac{36}{99}=\frac{4}{11} \text{ the answer.}$$

Hence, to find the vulgar fraction answering to a circulating decimal of this kind, make the figures which are repeated, the numerator, and the same number of nines, the denominator, and that will form the fraction.

Thus in the preceding example, 36 are the two figures repeated, which placed over two nines make $\frac{36}{99}$.

And if ,7142857142 &c. be the decimal proposed, then $\frac{714285}{999999}$ or $\frac{5}{7}$ is the equivalent vulgar fraction.

6. To find the vulgar fraction answering to the decimal ,41666, &c.

$$,41666 \text{ \&c.} = \left\{ \begin{array}{l} +\frac{41}{100} \\ +\frac{6}{1000}+\frac{6}{10000}, +\text{\&c.} \end{array} \right.$$

The sum of the series $\frac{6}{1000}+\frac{6}{10000}$ &c. is $\frac{6}{900}$

Therefore $\frac{41}{100}+\frac{6}{900}=\frac{375}{900}=\frac{5}{12}$ is the vulgar fraction sought.

7. What is the sum of the progression $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+$ &c. continued *ad infinitum?*

Here $f=1$, $r=-\frac{2}{3}$, and $l=0$:

And $s=\frac{rl-f}{r-1}=\frac{-1}{-1\frac{2}{3}}=\frac{3}{5}$ the answer.

8. Required the sum of the descending series $1-x+x^2-x^3+$ &c. infinitely continued?

In this progression $f=1$, $r=-x$, and $l=0$:

Therefore $s = \frac{rl - f}{r - 1} = \frac{-1}{-x - 1} = \frac{1}{1 + x}$ the answer.

For by actual division, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 -$ &c.

9. If the first term of a series be 2187, the last 128, and the ratio $\frac{2}{3}$; what is the number of terms?

Here $f = 2187$, $l = 128$, and $r = \frac{2}{3}$.

$\frac{l}{f} = \frac{128}{2187}$ .... *log.* 2.107210
.... *log.* 3.339849

$\frac{128}{2187}$ *log.* $-$ 2.767361 = Q.

$\frac{2}{3}$.... *log.* $-$ 1.823909 = R.

$\frac{Q}{R} = \frac{-2.767361}{-1.823909} = 7$; and $n = \frac{Q}{R} + 1 = 7 + 1 = 8$ the number of terms.

The learner must remember that the *indices only* of the logarithms are negative; whence, in dividing the *log.* of Q by that of *R*, the *positive* 5 which is carried to the *negative* 7 make 2 *negative*, and therefore one logarithm is contained in the other 7 times.

But in this case the use of negative indices may be avoided by making the last term the first, and *vice versâ*, and taking the reciprocal of the ratio *r*: thus;

Let $f = 128$, $l = 2187$, and $r = \frac{3}{2}$:

$\frac{l}{f} = \frac{2187}{128}$ .... *log.* 3.339849
.... *log.* 2.107210

1.232639 = Q

$r = \frac{3}{2}$ *log.* 0.176091 = R

and $\frac{Q}{R} = \frac{1.232639}{0.176091} = 7$, and $n = 7 + 1 = 8$ the number of terms as before.

When the last term is = 0, the number of terms (n) must evidently be infinite.

10. Suppose the first term of a series to be 4, the ratio $\frac{1}{2}$, and the sum of the series 8; what is the number of terms?

Here l (the last term) $= \frac{rs-s+f}{f} = \frac{8\times\frac{1}{2}-8+4}{4} = \frac{0}{4}$ an indefinite, or infinitely small quantity; therefore (n) the number of terms must also be infinite.

We may also remark that when the number of terms are infinite, the expression $r = \left(\frac{l}{f}\right)^{\frac{1}{n-1}}$ will not give the value of the ratio r.

155. When the numbers are too great for the logarithmic tables, the value of n or number of terms may be found from actual multiplication, or the powers of the ratio r, thus;

Suppose the first term of a progression to be 7, the ratio 3, and the sum of the series $= 36611236207$; then from the expression $l = \frac{rs-s+f}{r}$ we get l the last term $= 24407490807$, which divided by 7 the first term, gives $3486784401 = 3^{n-1}$, now 3486784401 is the 20*th*. power of 3, and therefore $n = 21$ the number of terms.

Of PERMUTATIONS and COMBINATIONS.

156. When a given number of things or quantities stand in any order or position, and that order is varied by changing the situation or place of any one of the quantities or things, it is called a Permutation.

Thus, one thing or quantity a is said to admit of one position only; But two things a and b can be varied, for a may stand first and b second, and *vice versâ*, thus ab
ba

And the variations or changes are 1×2.

157. If the number of things are three, as a, b, c, then each may stand first two times while the other two change places, therefore 3 things can be varied 2×3, or $1 \times 2 \times 3$ times;

Thus abc bac cab
acb bca cba

158. Four things a, b, c, d, are capable of 6×4 or $1 \times 2 \times 3 \times 4$ permutations; for each may stand the first, or the last, 6 times in a successive order, the other three being varied as above:

Thus	*abcd*	*abdc*	*adcb*	*bcda*
	acbd	*adbc*	*acdb*	*bdca*
	bacd	*bdac*	*dcab*	*cdba*
	bcad	*badc*	*dacb*	*cbda*
	cabd	*dabc*	*cadb*	*dcba*
	cbad	*dbac*	*cdab*	*dbca*

159. And 5 things will admit of 24×5 or $1 \times 2 \times 3 \times 4 \times 5$ changes; for each may occupy the 5*th*. place 24 times successively. Hence it appears that the permutations in n things are $1 \times 2 \times 3 \times 4$ &c. continued to n factors.

Examples.

1. How many changes can be rung on 8 bells?

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$ the *answer*.

2. If 6 columns of troops are in order of march; how many times can that order be varied?

$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, *answer*.

3. How many variations or changes can take place in the letters of the word *permutation?*

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 = 39916800$ the *answer*.

160. By the *combinations* or *elections* of quantities or things we understand the different collections that can be formed out of them, without any regard to their order, as in permutations.

Thus, suppose a, b, c, are the quantities, and that each collection or combination consists of two of them, then ab, bc, and ac are three different combinations, no two being alike.

161. *To investigate the number of combinations.* First, suppose the number of things in each combination to be two: then if the number of quantities or things are only two (a and b) it is evident there can be but 1 combination, ab.

Next, let the quantities be a, b, c; then since c can be combined with each of the two former letters a and b, the number of combinations will be increased by 2; therefore the number of combinations of 2 quantities in 3 will be $1+2$:

thus ab, ac, bc.

When the quantities or things are augmented to four, a, b, c, d, the number of combinations will be increased by 3; for the additional letter d may be combined with each of the former three, thereby forming three more combinations, the whole number being expressed by $1+2+3$:

thus ab, ac, bc, ad, bd, cd.

And by reasoning in the same manner, it will appear that the whole number of combinations of 2 in 5 quantities will be $1+2+3+4$: and in 6 quantities $1+2+3+4+5$, &c.

Therefore, if n be the number of things, the whole number of combinations, taken two by two, will be the series $1+2+3+4$, &c. continued to $n-1$ terms.

Now $1+2+3+4$, &c. is the 2*d*. order of figurate numbers (141), and the sum when continued to n terms is $\frac{n}{1}\cdot\frac{n+1}{2}$, (144), therefore substituting $n-1$ for n gives $\frac{n-1}{1}\times\frac{n}{2}$ or $\frac{n}{1}\times\frac{n-1}{2}$ the sum of $1+2+3+4$, &c. continued to $n-1$ terms.

Let us now suppose the number of quantities in each combination *to be three*. Then if the quantities are only three (a, b, c)

there can be but 1 combination *abc*: But if the quantities are four, *a*, *b*, *c*, *d*, the number of combinations will be increased by 3; for *d* may be combined with *ab*, *ac*, *bc*, the combinations of two in the preceding letters *a*, *b*, *c*; therefore the whole number of combinations of 3 in 4 things will be expressed by $1+3$;

thus *abc*
abd, *acd*, *bcd*.

And if the quantities are augmented to five *a*, *b*, *c*, *d*, *f*, the combinations will be increased by 6 (or $1+2+3$) the combinations of 2 in the 4 letters *a*, *b*, *c*, *d*; for *f* may be combined with every two of them; therefore the combinations in this case is denoted by $1+3+6$:

thus *abc*
abd, *acd*, *bcd*
abf, *acf*, *bcf*, *adf*, *bdf*, *cdf*.

It therefore appears that the combinations of 3 in 6 things will be $1+3+6+10$; in seven $1+3+6+10+15$; and in n quantities $1+3+6+10+15$, &c., continued to $n-2$ terms; which series is the 3*d*. order of figurate numbers (141); whence, by substituting $n-2$ for n in the general expression $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$ (144) we have $\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ for the combinations of 3 in n quantities.

And if the number of quantities in each combination be 4, we shall get the 4*th*. order of figurate numbers, or $1+4+10+20$, &c. continued to $n-3$ terms for the combinations in n quantities; whence, by putting $n-3$ for n in the same general expression (144) the result is $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ for the number of combinations in that case.

Hence the combinations of two things in n things is $\frac{n}{1} \times \frac{n-1}{2}$.

of three........ $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$.

of four...... $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$,

&c. &c.

Therefore *universally*, if m be the number of things in each combination, then $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}$, &c. continued to m factors, will give the whole number of combinations.

Examples.

1. How many combinations of 4 letters in the 24?

Here $n = 24$, and $m = 4$:

Therefore $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n-3}{4} = \frac{24 \times 23 \times 22 \times 21}{1 \times 2 \times 3 \times 4} = 23 \times 22 \times 21 = 10626$ the *answer*.

2. How many different hands can be held at the game of cribbage, if 6 cards is the deal?

Here $n = 52$, and $m = 5$.

And $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \frac{n-4}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{1 \times 2 \times 3 \times 4 \times 5} = 26 \times 17 \times 10 \times 49 \times 12 = 2598960$ the *answer*.

3. An old captain, who had often been successful in war, on being asked what reward he expected for his past services, desired a farthing only for every different file of 6 men he could make with his company which consisted of 100 men: what is the amount of his request?

$\frac{100 \times 99 \times 98 \times 97 \times 96 \times 95}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 1192052400$ the number of files or farthings, equal to 1241721*l*. 5*s*. the *answer*.

163. Beside the preceding, there are other kinds of combinations, as the *composition of quantities*, or when a given number of things are to be taken or combined from several setts, &c. The different cases however, are too numerous to be brought under any general rule.

We shall add a few miscellaneous examples with the methods of solution.

1. Suppose 4 ranks of men, 9 men in each rank; now how many ways can 4 men be chosen, 1 man being taken from every rank?

Since each man in one rank can be chosen with each man in another rank, the number of *twos* that can be formed out of two ranks will be 9 times 9 or 81: and because each man in a third rank can be taken, (or combined) with each of the 81 *twos*, the number of *threes* that can be chosen from three ranks is 81×9 or 729: again, each in the *4th* rank can be combined with each of the 729 *threes*; therefore 729×9 or $9^4 = 6561$ is the number of compositions, or the answer.

And if the ranks (or setts) are unequal, the number of compositions will be found exactly in the same manner; *ex. gr.* suppose 5, 6, 8, and 9, are in the respective ranks, then $5 \times 6 \times 8 \times 9$ (instead of $9 \times 9 \times 9 \times 9$ will be the number of compositions.

2. How many changes or chances are there in throwing 4 dice?

If we suppose 4 ranks, 6 in each rank, and each combination to be 1 from every rank, then, as in the preceding example, $6 \times 6 \times 6 \times 6$ or $6^4 = 1296$ is the number of different throws or chances.

3. Let there be 3 setts of different things, 4 in each sett, to find the compositions of 4, supposing 1, or more, is taken from each sett every time?

The combinations of 2 in 4 are 6, and since each in one sett can be combined with the twos in another, the compositions of the twos in one sett with the ones in another are 6×4 or 24, therefore the whole number of compositions of 3 in 2 setts is $24 \times 2 = 48$;

Again, each single one in the 3d. sett can be combined with each of the 48 threes in the other two, making 48×4 compositions; and as the combinations of 2 setts in 3 are 3, consequently $48 \times 4 \times 3$ or $576 = 12^2 \times 4$, *viz.* the square of the number of things multiplied by the number in each composition, is the *answer.*

4. How many changes can be rung with 4 bells out of 8?

The combinations of 4 in 8 are $\frac{8.7.6.5}{1.2.3.4} = 70$, which multiplied by $1 \times 2 \times 3 \times 4$, the changes in 4, make 1680 the *answer.*

5. How many different numbers can be made out of an unit, 2 twos, 3 threes, and 4 fours, taking four figures at a time?

To solve this problem it may be necessary to consider the changes or alternations that can take place in a form of this kind *aaabbc* where there are several things of one sort, and several of another.

If there are three things *aac*, two of them being alike, then *aac, aca, caa* are their variations; but when all are different, as *a, b, c,* the permutations will be $1 \times 2 \times 3$ which is 1×2 (the changes in 2 things) times greater than the changes in *aac*, the variations in *aac* are therefore expressed by $\frac{1 \times 2 \times 3}{1 \times 2}$.

And if *dddf* are 4 things where three are alike, all the variations are *dddf, ddfd, dfdd, fddd*, or $1 \times 2 \times 3$ (the changes in 3 things) times less than $1 \times 2 \times 3 \times 4$ the permutations when all four are different, consequently $\frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3}$ will denote the variations in *dddf*; and if these forms are combined, it follows that the variations in *aacdddf* will be truly expressed by $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{(1 \times 2)\ (1 \times 2 \times 3)}$, where the numerator is the permutations in 7 things (the number of letters), the denominator being the product of the respective changes in 2, and 3 things, the repetitions of *a* and *d*.

That any number of like things standing next to one another do not admit of a variation, is manifest from a repetition of any of the numeral digits; thus, the number 333 is not changed by any shifting of its figures.

Now let the proposed figures 1223334444 be represented by *abbcccdddd*; then combining them by fours, we get the following forms:

d^4	variations
d^3c, d^3b, d^3a, c^3d, c^3b, c^3a	4
d^2c^2, d^2b^2, b^2c^2	6
d^2bc, d^2ba, d^2ac, c^2db, c^2ba, c^2ad, b^2dc, b^2ca, b^2ad	12
$dcba$	24

The variations in d^3c, or d^3b, &c. are 4, in d^2bc, &c. 12;

$$\text{therefore}\quad \left.\begin{array}{r} 4\times 6 = 24 \\ 6\times 3 = 18 \\ 12\times 9 = 108 \\ 24\times 1 = 24 \\ d^4 \ldots\ldots\ldots\ 1 \end{array}\right\}\ \text{the variations multiplied by the number of combinations.}$$

sum $\overline{175}$ the *answer*, or all the combinations of 4 figures with their variations.

6 How many different numbers can be made with the same figures as in the preceding example (1223334444) supposing all the figures to be in every number?

By the last problem, $\dfrac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10}{(1\times 2)\ (1\times 2\times 3)\ (1\times 2\times 3\times 4)} = 12600$ the *answer*.

7. To find all the compositions or different integral numbers that can be formed by means of the nine digits, taking them by twos, by threes, &c. up to nines.

This is the same thing as finding the whole number of compositions in 9 ranks of the 9 digits when combined by 2, by 3, by 4, &c. up to 9 at a time:

Therefore, by the first example in this article, the compositions of 2 in two ranks 9 in each rank is 9×9 or 9^2, of 3 in three ranks is 9^3, of 4 in four ranks 9^4, &c.

Hence, if n be any number of things or quantities, the sum of all the possible compositions by twos, by threes, &c. up to n's, will be the sum $n + n^2 + n^3 + \ldots\ldots\ldots\ n^n$, which is a geometrical progression having n for the first term, for the ratio, and also for the number of terms; and the sum will be $n\times\dfrac{n^n-1}{n-1}$, (153), or, in the present case, $9\times\dfrac{9^9-1}{9-1} = 435848049$ the answer; being the number of different integers in which there is no cypher, from 1 to 999999999 both inclusive.

The doctrine of permutations, combinations, &c. is of considerable use in several parts of the mathematics; particularly in the calculation of annuities and chances.

OF NEWTON'S BINOMIAL THEOREM.

164. THIS is called the Binomial Theorem on account of its being a general formula for readily obtaining the powers, or roots, of any expression consisting of two terms. The method of denoting the coefficients admits of some variation; but one of the most commodious forms is the following:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3 + \text{\&c.}$$

If $(a-b)^n$ is the binomial and n is a positive integer, the 2*d*, 4*th*, 6*th*, &c. terms are negative (102).

In this theorem the index n may be any number, whole or fractional, positive or negative, and herein consists its principal excellence; because if n is a proper fraction, we obtain an approximating series for the root of the binomial denoted by that fractional exponent. A few examples will be sufficient to point out the method of substitution.

1. To find the cube or 3*d*. power of $a+b$.

Here $n=3$, or $(a+b)^n = (a+b)^3$.

And $n=3$ the coefficient of the 2*d*. term.

$\frac{n}{1}\cdot\frac{n-1}{2} = \frac{3}{1}\times\frac{3-1}{2} = 3$ the coefficient of the 3*d*. term.

$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} = \frac{6}{6} = 1$ the coefficient of the 4*th*. term.

Therefore $a^3+3a^{3-1}b+3a^{3-2}b^2+a^{3-3}b^3$ or $a^3+3a^2b+3ab^2+b^3$, (a^{3-3} being 1) is the required cube.

2. To find an approximating series for the square root of x^2+1. (107 *Examp.* 3).

The expression is $(x^2+1)^n$ or $(x^2+1)^{\frac{1}{2}}$, where $x^2=a$, $1=b$, and $n=\frac{1}{2}$.

$a^n=(x^2)^{\frac{1}{2}}=x$ the first term.

$na^{n-1}b=\frac{1}{2}\left(x^2\right)^{\frac{1}{2}-1}=\frac{1}{2}\left(x^2\right)^{-\frac{1}{2}}=\frac{1}{2}x^{-1}=\frac{1}{2x}$ the 2d. term.

$\frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^2=-\frac{1}{8}\left(x^2\right)^{\frac{1}{2}-2}=-\frac{1}{8}x^{-3}=-\frac{1}{8x^3}$ the 3d. term.

$\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^3=\frac{1}{16}\left(x^2\right)^{-2\frac{1}{2}}=\frac{1}{16}x^{-5}=\frac{1}{16x^5}$ the 4th term.

$\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^4=-\frac{5}{128}\left(x^2\right)^{-3\frac{1}{2}}=-\frac{5}{128x^7}$ the 5th. term, &c.

Hence $x+\frac{1}{2x}-\frac{1}{8x^3}+\frac{1}{16x^5}-\frac{5}{128x^7}+$ &c. is the series required.

3. Let it be required to convert $\frac{1}{(a+b)^2}$ or $(a+b)^{-2}$ into a series.

Here $n=-2$. And a^{-2} or $\frac{1}{a^2}$ is the first term.

$na^{n-1}b=-2a^{-3}b=-\frac{2b}{a^3}$ the second.

$\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2=+3a^{-4}b^2=+\frac{3b^2}{a^4}$ the third.

$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3=-4a^{-5}b^3=-\frac{4b^3}{a^5}$ the fourth, &c.

Therefore $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+$ &c. where the law of continuation is manifest.

4. To expand $\frac{1}{(a+b)^{\frac{1}{2}}}$ or $(a+b)^{-\frac{1}{2}}$ into a series.

In this example $n=-\frac{1}{2}$.

And $a^n=a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$ the first term.

$$na^{n-1}b = -\tfrac{1}{2}a^{-\frac{3}{2}}b = -\frac{b}{2a^{\frac{3}{2}}} \text{ the second.}$$

$$\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 = +\frac{3}{8}a^{-\frac{5}{2}}b^2 = +\frac{3b^2}{8a^{\frac{5}{2}}} \text{ the third.}$$

$$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3 = -\frac{5}{16}a^{-\frac{7}{2}}b^3 = -\frac{5b^3}{16a^{\frac{7}{2}}} \text{ the fourth.}$$

whence $$\frac{1}{(a+b)^{\frac{1}{2}}} = \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a^{\frac{3}{2}}} + \frac{3b^2}{8a^{\frac{5}{2}}} - \frac{5b^3}{16a^{\frac{7}{2}}} + \&c.$$

4. A trinomial may be raised to any given power by considering the sum or difference of two terms as one factor:

Thus $(a+b+c)^n = \left(a+\overline{b+c}\right)^n = a^n + na^{n-1}(b+c) + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}(b+c)^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}(b+c)^3 + \&c.$

Demoivre, *Miscell. Analyt.* and Dr. Cheyne, *Method. Inv.* have extended the rule to any multinomial.

163. Among the different investigations that have been given of the preceding theorem, the following, by means of the continued product of binomial factors, seems the most natural and easy when the exponent is a whole number.

If $a+b$, $a+c$, $a+d$, &c. are a series of binomial factors, to determine the coefficients of a in the product $(a-b)$ $(a+c)$ $(a+d)$ &c.

By actual multiplication we get

$$(a+b)(a+c) = a^2 + \left.\begin{matrix} b \\ c \end{matrix}\right\} a + bc$$

$$(a+b)(a+c)(a+d) = a^3 + \left.\begin{matrix} b \\ c \\ d \end{matrix}\right\} a^2 + \left.\begin{matrix} bc \\ bd \\ cd \end{matrix}\right\} a + bcd$$

$$(a+b)(a+c)(a+d)(a+f) = a^4 + \left.\begin{matrix} b \\ c \\ d \\ f \end{matrix}\right\} a^3 + \left.\begin{matrix} bc \\ bd \\ bf \\ cd \\ cf \\ df \end{matrix}\right\} a^2 + \left.\begin{matrix} bcd \\ bcf \\ bdf \\ cdf \end{matrix}\right\} a + bcdf, \ \&c.$$

Hence it appears, that the coefficient of a in the 2d. term is always the sum of the other quantities b, c, d, &c.

The coefficient in the 3*d*. term the sum of all their products or combinations two by two;

The coefficient in the 4*th*. term the sum of all their products combined three by three, &c. &c.

Therefore, if n be the number of factors $a+b$, $a+c$, $a+d$, &c. the number of letters in the coefficient of the 2*d*. term will also be denoted by n:

The number of their combinations two by two in the 3*d*. term, by $\frac{n}{1}\cdot\frac{n-1}{2}$, (161).

The combinations three by three in the 4*th*. by $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}$, &c.

Now suppose c, d, f, &c. to be each equal to b;

then $(a+b)\ (a+c)\ (a+d)\ (a+f)$ &c. $=(a+b)\ (a+b)\ (a+b)\ (a+b)$, &c. $=(a+b)^n$;

$$\text{that is, } (a+b)^4 = a^4 + \left.\begin{matrix} b \\ b \\ b \\ b \end{matrix}\right\} a^3 + \left.\begin{matrix} b^2 \\ b^2 \\ b^2 \\ b^2 \\ b^2 \\ b^2 \end{matrix}\right\} a^2 + \left.\begin{matrix} b^3 \\ b^3 \\ b^3 \\ b^3 \end{matrix}\right\} a + b^4, \text{ when } n=4.$$

It follows then, (n being the number of factors) that the coefficient of a in the 2*d*. term is nb:

in the 3*d*. $\frac{n}{1}\cdot\frac{n-1}{2}b^2$:

in the 4*th*. $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}b^3$;

in the 5*th*. $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}b^4$, &c.

And since a^n, a^{n-1}, a^{n-2}, a^{n-3}, &c. are the successive powers of a, we have $(a+b)^n = a^n + nba^{n-1} + \frac{n}{1}\cdot\frac{n-1}{2}b^2a^{n-2} + \frac{n}{1}\cdot\frac{n-1}{1}\cdot\frac{n-2}{3}b^3a^{n-3}$ + &c. where it is evident the series will terminate at $n+1$ terms, as in the first example.

166. But the preceding method of deriving the law of the coefficients has not been considered as sufficiently general, because the value of the index n is restricted to a positive integer. The following investigation however, by means of a trinomial $(1+a+b)^n$ is not liable to the same objection *. In this

* Manning's Algebra, vol. 2.

process we consider the trinomial as consisting of two terms only, $(\overline{1+a}+b)^n$, and $(1+\overline{a+b})^n$, then as these quantities are the same, the results from similar operations must be equal.

It is manifest from multiplication and the extraction of roots, that when a binomial $a+b$ is raised to the nth. power, the two first terms will always be $a^n+na^{n-1}b$, viz. $(a+b)^n=a^n+na^{n-1}b$, &c.

Thus if $n=2$, then $(a+b)^2=a^2+2a^{2-1}b$, &c. $=a^2+2ab$, &c.

$n=5$........ $(a+b)^5=a^5+5a^{5-1}b$, &c. $=a^5+5a^4b$, &c.

And when n is negative, or a fraction, the two leading terms are found exactly in the same manner:

Thus, let $n=-2$;

Then $(a+b)^{-2}=a^{-2}+(-2a^{-2-1}b)$ &c. $=\frac{1}{a^2}-\frac{2b}{a^3}$ &c. the result by actual division; for $(a+b)^{-2}=\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}$ &c.

Again, let $n=\frac{1}{2}$:

Then $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{\frac{1}{2}-1}b$ &c. $=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}$ &c. the terms which we get by extracting the square root of $a+b$. And other proofs are obtained in a similar manner with different values of n.

Hence, if the theorem is put in this form,

$$(a+b)^n=a^n+na^{n-1}b+Pa^{n-2}b^2+Ra^{n-3}b^3+Sa^{n-4}b^4 \text{ \&c.}$$

the thing to be determined is the law of the coefficients P, R, &c. independent of any numeral value of n: P, R, &c. being the coefficients of the 3*d*, 4*th*, &c. terms of the powers of any binomial.

Let $1+a$ be considered as one term;

$$\text{Then } (\overline{1+a}+b)^n=(1+a)^n+n(1+a)^{n-1}b, \text{ \&c.}$$

$$=(1+a)^n=1+na+Pa^2+Ra^3+Sa^4+\text{\&c.}$$

$$+(1+a)^{n-1}\times nb, \text{ \&c.}$$

Now, suppose p, r, &c. to denote the coefficients of the 3*d*, 4*th*, &c. terms of $(1+a)^{n-1}$ when expanded.

Then 1^{n-1}, or 1 is the first term;

$(n-1)\cdot 1^{n-1}a$, or $a(n-1)$ the 2d.

pa^2 the 3d.

ra^3 the 4th, &c.

Therefore $(1+a)^{n-1}\times nb = (1+a(n-1)+pa^2+ra^3$ &c.$)\,nb$
$= nb + n(n-1)\,ab + npa^2b + nra^3b$ &c.

And $(\overline{1+a}+b)^n = \left\{\begin{array}{l} 1+na+Pa^2+Ra^3+Sa^4, \text{ \&c.} \\ +nb+n(n-1)\,ab+npa^2b+nra^3b, \text{ \&c.}\end{array}\right.$

Again, let $a+b$ be taken as a single term;

Then $(1+\overline{a+b})^n = 1+n(a+b)+P(a+b)^2+R(a+b)^3+S(a+b)^4$, &c. now, to compare this with the preceding series, the terms of $(a+b)^2$, $(a+b)^3$, &c. which involve the 2d, 3d, &c. powers of b, may be omitted: and we shall have

$$\begin{aligned}(1+\overline{a+b})^n &= 1+na+nb+Pa^2+Pab \\ &\qquad +Ra^3+3ra^2b \\ &\qquad\qquad +Sa^4+4Sa^3b, \text{ \&c.} \\ &= 1+na+Pa^2+Ra^3+Sa^4, \text{ \&c.} \\ &\quad +nb+2Pab+3ra^2b+4Sa^3b, \text{ \&c.}\end{aligned}$$

this result must be equal to that obtained from $(\overline{1+a}+b)^n$, whence by omitting $1+na+Pa^2$, &c. which is common to both, we get

$$nb+n(n-1)\,ab+npa^2b+nra^3b, \text{\&c.} = nb+2Pab+3ra^2b+4Sa^3b, \text{\&c.}$$

Now the powers of a and b on one side of the equation being the same as on the other, their respective coefficients must be equal;

viz, $n=n$ $\qquad np=3R$

$n(n-1)=2P$ $\qquad nr=4S$, &c.

whence $P=\dfrac{n(n-1)}{2}$, that is, the coefficient of the 3d. term of $(1+a)^n$ is half the product of the index and the index *minus* 1; hence p or the coefficient of the 3d. term of $(1+a)^{n-1}$ will be $\dfrac{(n-1)\cdot(n-2)}{2}$, or half the product of the index and the index *minus* 1.

Now $\dfrac{n(n-1)\cdot(n-2)}{2}=3R$, whence $R=\dfrac{n(n-1)\cdot(n-2)}{2\cdot 3}$ the coefficient of the 4th. term: and $r=\dfrac{(n-1)\cdot(n-2)\cdot(n-3)}{2\cdot 3}$; and since $nr=4S$, S will be $=\dfrac{n(n-1)\cdot(n-2)\cdot(n-3)}{2\cdot 3\cdot 4}$ the coefficient of the 5th. term, &c.

This celebrated Theorem has been demonstrated various ways; but sometimes from principles which may be said to depend on the theorem itself. Newton seems to have found the law of the coefficients by *induction*: a method which has led to the most important discoveries in science.

Of CONTINUED FRACTIONS.

167. Continued Fractions have an integer and a fraction for the denominator, and the fraction in that denominator has also an integer and a fraction for its denominator; in like manner, the denominator of the last fraction is composed of an integer and a fraction, and so on. These fractions are generated by division after the manner of reducing a fraction to its lowest terms in Arithmetic:

Thus, to reduce $\frac{761}{2385}$ to a continued fraction, let both terms of the fraction be divided by 761

$$\text{and we have } \frac{761}{2385} = \cfrac{1}{3\,\frac{102}{761}}$$

Again, if both terms of the fraction $\frac{102}{761}$ are divided by the numerator 102

$$\text{then } \cfrac{1}{3\,\frac{102}{761}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{47}{102}}}$$

And both terms of the fraction $\frac{47}{102}$ divided by 47

$$\text{gives } \cfrac{1}{3+\cfrac{1}{7+\cfrac{47}{102}}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{8}{47}}}}$$

The next reduction is performed by dividing both terms of the fraction $\frac{8}{47}$ by 8,

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{8}{47}}}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{7}{8}}}}}$$

And $\frac{1}{a}$ is the first expression or approximation:

$$\cfrac{1}{a+\cfrac{1}{b}} \cdots\cdots = \frac{b}{ab+1} \text{ the 2d.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} \cdots\cdots = \frac{bc+1}{abc+a+c} \text{ the 3d.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}} \cdots\cdots = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4th, \&c.}$$

Hence it appears that the 3*d*. expression is found by multiplying the numerator and denominator of the 2*d*. by the third quotient c, and adding 1 the numerator of the first expression $\left(\frac{1}{a}\right)$ to the first of these products, and the denominator a to the second:

And, if the terms of the 3*d*. expression are multiplied by the fourth quotient d, and the products augmented in the same manner by the terms of the 2*d*. the result is the 4*th*. expression; and so on:

Thus

$$\frac{c \times b+1}{c \times (ab+1)+a} = \frac{bc+1}{abc+a+c} \text{ the 3d. expression}$$

$$\frac{d\,(bc+1)+b}{d\,(abc+a+c)+ab+1} = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4th.}$$

$$\frac{f\,(bcd+b+d)+bc+1}{f\,(abcd+ab+ad+cd+1)+abc+a+c} \text{ the 5th, \&c.}$$

170. These expressions are convenient for finding the vulgar fraction answering to a recurring decimal:

Thus, suppose the decimal .76923, &c.

$$\text{Then } \frac{76923}{100000} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3\frac{1}{7692}}}}$$

$$\cfrac{1}{3+\cfrac{1}{7}} = \frac{7}{22}$$ the 2*d*. too little, or less than $\frac{761}{2385}$;

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2}}} = \frac{15}{47}$$ the 3*d*. too great, but nearer than the last.

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5}}}} = \frac{82}{257}$$ the 4*th*. too little.

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1}}}}} = \frac{97}{304}$$ the 5*th*. too great, but nearer $\frac{761}{2385}$ than either of the preceding.

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{7}}}}}} = \frac{761}{2385}$$ the fraction proposed.

Here we may observe that the approximations are alternately too great and too little, but their values become nearer to the value of the given fraction as the numerators and denominators increase. We shall now give the method of obtaining all the other approximations or expressions by means of the two first, and the quotients that follow.

169. Let a, b, c, d, f, &c. be the quotients found in succession by reducing any fraction $\frac{n}{m}$ to a continued fraction.

Then $$\frac{n}{m} = \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{f+1\ \&c.}}}}}$$

And $\frac{1}{a}$ is the first expression or approximation:

$$\cfrac{1}{a+\cfrac{1}{b}} \cdots\cdots = \frac{b}{ab+1} \text{ the 2}d.$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} \cdots\cdots = \frac{bc+1}{abc+a+c} \text{ the 3}d.$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}} \cdots\cdots = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4}th\text{, \&c.}$$

Hence it appears that the 3*d*. expression is found by multiplying the numerator and denominator of the 2*d*. by the third quotient c, and adding 1 the numerator of the first expression $\left(\frac{1}{a}\right)$ to the first of these products, and the denominator a to the second:

And, if the terms of the 3*d*. expression are multiplied by the fourth quotient d, and the products augmented in the same manner by the terms of the 2*d*. the result is the 4*th*. expression; and so on:

Thus

$$\frac{c \times b+1}{c \times (ab+1)+a} = \frac{bc+1}{abc+a+c} \text{ the 3}d.\text{ expression}$$

$$\frac{d\,(bc+1)+b}{d\,(abc+a+c)+ab+1} = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4}th.$$

$$\frac{f\,(bcd+b+d)+bc+1}{f\,(abcd+ab+ad+cd+1)+abc+a+c} \text{ the 5}th\text{, \&c.}$$

170. These expressions are convenient for finding the vulgar fraction answering to a recurring decimal:

Thus, suppose the decimal .76923, &c.

$$\text{Then } \frac{76923}{100000} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3\frac{1}{7692}}}}$$

Here the three quotients are $1 = a$, $3 = b$, $3 = c$, these being substituted in the 3*d*. expression, give $\frac{9+1}{9+1+3} = \frac{10}{13}$ the required fraction.

Remark. It is not necessary to keep exactly to the preceding form in reducing a given fraction to a continued one. The process of division may be set down as it is in finding the greatest common measure, and the continued fraction formed afterwards by means of the quotients, as in the following example:

171. To find the ratio of the diameter of a circle to its circumference, nearly, in small integer numbers.

If the diameter is 1, the circumference will be 3.141593 nearly (*Art.* 266. Mensuration). Hence the given fraction is $\frac{1000000}{3141593}$.

```
1000000) 3141593 (3
         3000000
         -------
          141593) 1000000 (7
                   991151
                   ------
                     8849) 141593 (16 nearly
                            8849
                            -----
                            53103
                            53094
                            -----
```

Therefore $\frac{1000000}{3141593} = \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{16}}}$

The quotients are $3 = a$, $7 = b$, and $16 = c$; and if the two first are substituted in the 2*d*. expression, we have $\frac{7}{21+1} = \frac{7}{22}$ the ratio that Archimedes assigned, which is used for common purposes.

By taking in the quotient c, the 3*d*. expression gives $\frac{7 \times 16 + 1}{3 \times 7 \times 16 + 3 + 16}$ $= \frac{113}{355}$ an approximation nearer the truth than the fraction $\frac{1000000}{3141593}$.

The idea of these fractions seems to have originated with Lord Brouncker, the first President of the Royal Society. Afterwards Huygens extended their application: and since that time Lord Stanhope, Euler, and particularly Legrange, have greatly improved the theory, and shown their use in the extraction of roots, the summation of series, &c.

OF RECURRING SERIES.

172. RECURRING series are so constituted that each term has a constant relation to some given number of the preceding terms taken always in the same order:

thus if $1 + 2x + 3x^2 + 4x^3 + 5x^4 +$ &c. be the series,

then the 4*th*. term (for example) is $= 2x \times 3d.$ term $- x^2 \times 2d.$ term:

The 5*th*. term $= 2x \times 4th.$ term $- x^2 \times 3d.$ term, and so on.

And the expression $2x - x^2$ is called the *scale of relation* of the terms; this scale however, is sometimes exhibited by the coefficients only, (as $2 - 1$ the coefficients of $2x - x^2$).

Again, suppose $1 + 3x + 5x^2$ are the three first terms of a series, and assuming $3 - 2 + 4$ for the scale of the coefficients, or $3x - 2x^2 + 4x^3$ for the scale of relation of the terms;

Then $3x \times (5x^2$ the 3*d*. term$) - 2x^2 \times (3x$ the 2*d*. term$)$ $+ 4x^3 \times (1$ the 1*st*. term$) = 13x^3$ the 4*th*. term of the series:

Also $3x \times 13x^3 - 2x^2 \times 5x^2 + 4x^3 \times 3x = 41x^4$ the 5*th*. term of the series, &c.

And the series is $1 + 3x + 5x^2 + 13x^3 + 41x^4 + 117x^5$ &c.

$$\overset{A}{a} + \overset{B}{bx} + \overset{C}{cx^2} + \overset{D}{dx^3} + \overset{E}{fx^4}$$

173. Let $a + bx + cx^2 + dx^3 + fx^4 +$ &c. be a series, and $tx - sx^2$ the scale of relation of the terms;

Then $cx^2 = tx \times bx - sx^2 \times a = tbx^2 - sax^2$

$dx^3 = tx \times cx^2 - sx^2 \times bx = tcx^3 - sbx^3$

$fx^4 = tx \times dx^3 - sx^2 \times cx^2 = tdx^4 - scx^4$

&c. &c.

whence it is evident that all the terms after the two first, may be exhibited by means of those two terms and the scale of relation.

Now suppose it is required, to find the sum of the above series *infinitely continued*:

Let A, B, C, &c. denote the terms a, bx, cx^2, &c. respectively:

$$
\begin{aligned}
\text{Then } A &= A \\
B &= B \\
C &= Btx - Asx^2 \\
D &= Ctx - Bsx^2 \\
E &= Dtx - Csx^2 \\
\&c. &\quad \&c.
\end{aligned}
$$

here it is manifest that the sum $A + B + C + D + E$ (the first column) is equal to both the other columns added together:

Put $A + B + C +$ &c. $= S$; then $B + C + D$ &c. $= S - A$:

The sum of the terms in the 2*d*. column is $= A + B + (B + C + D$ &c.$)$ tx;

but $B + C + D$, &c. is $= S - A$,

therefore the 2*d*. column $= A + B + (S - A)\, tx = A + B + Stx - Atx$:

And since the series is supposed to be infinite,

$A + B + C +$ &c. in the 3*d*. column will be $= S$,

therefore the 3*d*. column, or $-(A + B + C)\, sx^2$ is $= -Ssx^2$;

And $A + B + Stx - Atx - Ssx^2$ is the aggregate of the second and 3*d*. columns, which sum is equal to the first,

or $A + B + Stx - Atx - Ssx^2 = S$,

whence $\dfrac{A + B - Atx}{1 - tx + sx^2} = S$ the sum of the series infinitely continued.

It is manifest this expression will not give the sum of any proposed number of terms, because if that number be *less* than infinite, the number of terms in the 3*d*. column will *always* be less by *two* than those in the first or second columns, and consequently $A + B + C +$ &c. in the 3*d*. column cannot in *that case* be $= S$. And since it is impossible to find the sum of an infinite number of any given quantities, it follows that $a + bx + cx^2 +$ &c. must be a converging series, that is, a series where

the terms constantly diminish or approach to the limit o, which may be considered as the least or last term.

If $a=1$, $b=2$, $t=2$, and $s=1$, the series $a+bx+cx^2+$ &c. becomes $1+2x+3x^2+$ &c.

$$\text{And } \frac{A+B-Atx}{1-tx+sx^2}=\frac{1+2x-2x}{1-2x+x^2}=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}=S.$$

Now $\frac{1}{(1-x)^2}$ expanded by division, gives the proposed series,

$$\begin{array}{l}
\text{Thus } 1-2x+x^2)\ 1 \qquad (1+2x+3x^2+4x^3+\&c.\\
\qquad\qquad\quad \underline{1-2x+x^2}\\
\qquad\qquad\qquad +2x-x^2\\
\qquad\qquad\qquad \underline{+2x-4x^2+2x^3}\\
\qquad\qquad\qquad\qquad +3x^2-2x^3\\
\qquad\qquad\qquad\qquad \underline{+3x^2-6x^3+3x^4}\\
\qquad\qquad\qquad\qquad\qquad +4x^3-3x^4\\
\qquad\qquad\qquad\qquad\qquad \underline{+4x^3-8x^4+4x^5}\\
\qquad\qquad\qquad\qquad\qquad\qquad +5x^4-4x^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \&c.
\end{array}$$

It therefore appears, that summing a recurring series is only discovering the *radix* or fraction from which it was, or might be derived, (72):

Thus, suppose the series to be $1+3x+4x^2+7x^3+11x^4+$ &c. where the scale of relation of the coefficients is $1+1$, *viz.* the coefficient of any term (after the two first) is the sum of the two preceding coefficients. Then $t=1$, and $s=1$, and the latter being positive, the expression $\frac{A+B-Atx}{1-tx+sx^2}$ becomes $\frac{1+2x}{1-x-x^2}=S$, the sum of the series infinitely continued. For $\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+$ &c. by actual division.

174. In order to find the sum of any number (n) of terms of the series $1+2x+3x^2+4x^3+$ &c. (for example), it is to be observed that the nth term is nx^{n-1}, and consequently the terms which follow will be $(n+1)x^n+(n+2)\,x^{n+1}+(n+3)\,x^{n+2}+$ &c. where the scale of relation of the coefficients is $2-1$ as before: therefore substituting $(n+1)x^n$ and $(n+2)x^{n+1}$, re-

spectively, for A and B in the expression $\frac{A+B-Atx}{1-tx+sx^2}$, and we have

$$\frac{(n+1)x^n+(n+2)x^{n+1}-(n+1)x^n tx}{1-tx+sx^2}=\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}$$ the

sum of the series $(n+1)x^n+(n+2)x^{n+1}+$ &c. *ad infinitum*:

Now it is evident that the difference of the expressions for the two sums will be the expression for the sum of n terms of the series,

that is, $$\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=\frac{1-(n-1)x^n+nx^{n+1}}{(1-x)^2}$$

which is the expression for the sum of n terms of the series $1+2x+3x^2+$ &c.

173. If the scale of relation of the coefficients consists of three terms $t+s+v$; then, proceeding as in *Art.* 173, we get the following expression for the sum of $a+bx+cx^2+$ &c. *in infin.*

viz. $$\frac{A+B+C-(A+B)tx-Asx^2}{1-tx-sx^2-vx^3}=S.$$

Hence it appears that when the scale consists of n terms, the expression will be

$$\frac{A+B+C+\ldots n-(A+B+\ldots\overline{n-1})tx-(A+\ldots\overline{n-2})sx^2\ \&c.}{1-tx-sx^2-vx^3-\ldots.(n+1)}=S;$$

Which is a general Theorem for the sum of an infinite recurring series; and from this the sum of any given number of the terms may be found as in the preceding article. But as the signs in the scale of relation are here supposed to be positive, care must be taken when negative signs occur, to make the substitution accordingly.

Of the DIFFERENTIAL METHOD.

176. This is a method of summing series, &c. by means of the successive differences of their terms.

Let 1 7 28 84 210 462 924 1716, &c. be a series of numbers; Then taking the difference of the first and second, of the second and third, of the third and fourth, &c. and again the differences of those differences, and so on, we shall have the following orders of differences:

	1		7		28		84		210		462		924		1716 &c.	
1st. order of differences		6		21		56		126		252		462		792		
2d. order			15		35		70		126		210		330			
3d. order				20		35		56		84		120				
4th. order					15		21		28		36					
5th. order						6		7		8						
6th. order							1		1							
								0								

Or suppose $a, b, c, d, f, g,$ &c. to be a series; then

1st. order of differences $b-a,\ c-b,\ d-c,\ f-d,\ g-f,$ &c.
2d. order .... $c-2b+a,\ d-2c+b,\ f-2d+c,\ g-2f+d,$ &c.
3d. order .... $d-3c+3b-a,\ f-3d+3c-b,\ g-3f+3d-c,$ &c.
4th. order .... $f-4d+6c-4b+a,\ g-4f+6d-4c+b,$ &c.
5th. order .... $g-5f+10d-10c+5b-a,$ &c.

177. Let $D^{\text{I}}, D^{\text{II}}, D^{\text{III}}, D^{\text{IV}}, D^{\text{V}}$, &c. denote the first terms of the several orders of differences, respectively,

that is, put
$$
\begin{aligned}
D^{\text{I}} &= b-a \\
D^{\text{II}} &= c-2b+a \\
D^{\text{III}} &= d-3c+3b-a \\
D^{\text{IV}} &= f-4d+6c-4b+a \\
D^{\text{V}} &= g-5f+10d-10c+5b-a \\
\text{\&c.} & \quad \text{\&c.}
\end{aligned}
$$

Then by transposition we get the values of $b, c, d,$ &c.

$$b = a + D^{\text{I}}$$
$$c = 2b - a + D^{\text{II}}$$
$$d = 3c - 3b + a + D^{\text{III}}$$
$$f = 4d - 6c + 4b - a + D^{\text{IV}}$$
$$g = 5f - 10d + 10c - 5b + a + D^{\text{V}}$$
&c. &c.

But $2b - a = b + (b - a) = a + D^{\text{I}} + D^{\text{I}} = a + 2D^{\text{I}}$, (because $b = a + D^{\text{I}}$ and $b - a = D^{\text{I}}$) therefore $c = a + 2D^{\text{I}} + D^{\text{II}}$.

Also, since $3b - a = 3\ (a + D^{\text{I}}) + a$;

we have $3c - 3b + a = 3\,(a + 2D^{\text{I}} + D^{\text{II}}) - 3\,(a + D^{\text{I}}) + a = a + 3D^{\text{I}} + 3D^{\text{II}}$,

whence $d = a + 3D^{\text{I}} + 3D^{\text{II}} + D^{\text{III}}$.

And in like manner we get

$$f = a + 4D^{\text{I}} + 6D^{\text{II}} + 4D^{\text{III}} + D^{\text{IV}}.$$

Hence
$$a = a$$
$$b = a + D^{\text{I}}$$
$$c = a + 2D^{\text{I}} + D^{\text{II}}$$
$$(A) \quad d = a + 3D^{\text{I}} + 3D^{\text{II}} + D^{\text{III}}$$
$$f = a + 4D^{\text{I}} + 6D^{\text{II}} + 4D^{\text{III}} + D^{\text{IV}}$$
$$g = a + 5D^{\text{I}} + 10D^{\text{II}} + 10D^{\text{III}} + 5D^{\text{IV}} + D^{\text{V}}$$
&c. &c.

where the law of continuation is evident.

Hence it appears that the coefficients of a, D^{I}, D^{II}, D^{III}, &c. in the expression for the $(n+1)\,th.$ term of the series $a, b, c,$ &c. are the coefficients of a binomial raised to the nth. power; that is, the $(n+1)\,th.$ term is $a + nD^{\text{I}} + n.\dfrac{n-1}{2}\,D^{\text{II}} + n.\dfrac{n-1}{2}.\dfrac{n-2}{3}\,D^{\text{III}}$, &c. (164)

Thus, for example, if the number of terms be 5, or $n = 5$, the $6th.$ or $(n+1)\,th.$ term is $a + 5D^{\text{I}} + 5.\dfrac{5-1}{2}\,D^{\text{II}} + 5.\dfrac{5-1}{2}.\dfrac{5-2}{3}\,D^{\text{III}}$ &c. or $a + 5D^{\text{I}} + 10D^{\text{II}}$ &c. the value of g the $6th.$ term.

Therefore substituting $n+1$ for n, the nth. term of the series $a, b, c,$ &c. will be

$$a + (n-1)\,D^{\text{I}} + \frac{n-1}{1}.\frac{n-2}{2}\,D^{\text{II}} + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}\,D^{\text{III}} + \text{\&c.}$$

178. A general expression for the sum of any number (n) of the terms of the series, is readily obtained from the aggregate sum of the perpendicular columns as they stand in the expressions (A):

Thus, the coefficients in the columns a, a, a, &c. D^{I}, $2D^{I}$, $3D^{II}$, &c. are the several orders of figurate numbers (141):

Now the sum of $a+a+a+$ &c. to n terms is na:

of $D^{I}+2D^{I}+3D^{I}+$ &c. to $n-1$ terms is $n.\frac{n-1}{2}D^{I}$: (144)

of $D^{II}+3D^{II}+6D^{II}+$ &c. to $n-2$ terms is $n.\frac{n-1}{2}.\frac{n-2}{3}D^{II}$:

of $D^{III}+4D^{III}+10D^{III}+$ &c. to $n-3$ terms is $n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D^{III}$:

&c. &c.

And the aggregate must be the sum of n terms of the series $a+b+c+$ &c. viz. $na+n.\frac{n-1}{2}D^{I}+n.\frac{n-1}{2}.\frac{n-2}{3}D^{II}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}$ $D^{III}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}D^{IV}+$ &c.

When the differences are finally $=0$, any term, or the sum of any number of the terms may be accurately determined; but if the differences do not vanish, the result is only an approximation: this approximate value however, will become nearer and nearer the truth as the differences diminish.

Examples.

1. What is the 17*th*. term of the series 1, 3, 6, 10, 15, &c.?

	1	3	6	10	15
1*st*. difference......	2	3	4	5	
2*d*....................	1	1	1		
	0	0			

Here $a=1$, $D^{I}=2$, $D^{II}=1$; these being substituted in the expression $a+(n-1)D^{I}+\frac{n-1}{1}.\frac{n-2}{2}D^{II}$ &c. give $1+(n-1)\times 2+\frac{n-1}{1}$ $\frac{n-2}{2}\times 1=\frac{n^2\times 4}{2}=$ (when $n=17$) $\frac{17^2+4}{2}=153$, the term required.

2. To find the nth. term of the series of rectangles 1×2, 3×4, 5×6, 7×8, 9×10, &c.

	2		12		30		56
1st. diff..........		10		18		26	
2d..................			8		8		
				0			

Here $a=2$, $D^{I}=10$, $D^{II}=8$

And $2+(n-1)\times 10+\frac{n-1}{1}\cdot\frac{n-2}{2}\times 8=4n^2-2n$ the required term.

3. To find the sum of n terms of the series of cubes $1^3+2^3+3^3+4^3+$ &c.

	1		8		27		64		125
1. diff....		7		19		37		61	
2. diff............			12		18		24		
3. diff............				6		6			
					0				

In this example $a=1$, $D^{I}=7$, $D^{II}=12$, $D^{III}=6$;

And $n+n\cdot\frac{n-1}{2}\times 7+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\times 12+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\times 6=\frac{n^4+2n^3+n^2}{4}=\left(\frac{n^2+n}{2}\right)^2$ the expression for the sum required.

Hence it appears that the aggregate of any number of the series of *cubes* $1^3+2^3+3^3+4^3$ &c. taken in succession from 1, is a *square* number.

4. When the series is descending, the differences will be alternately *minus* and *plus*. Thus, to find the sum of the *biquadrates* $10^4+9^4+8^4+$ &c. to 8 terms:

	10000		6561		4096		2401		1296		625
1. diff....		3439		2465		1695		1105		671	
2. diff.......			974		770		590		434		
3. diff..........				204		180		156			
4. diff............					24		24				
						0					

Here $a = 10000$, $D^{I} = -3439$, $D^{II} = +974$, $D^{III} = -204$, $D^{IV} = +24$, $n = 8$,

And $10000 \times 8 - 28 \times 3439 + 56 \times 974 - 70 \times 204 + 56 \times 24 = 25316$ the *sum required.*

And in the same manner, the sums of series of higher powers may be determined.

179. Hence we find

$$1^1 + 2^1 + 3^1 \ldots\ldots n^1 = \frac{n^2 + n}{2}.$$

$$1^2 + 2^2 + 3^2 \ldots\ldots n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

$$1^3 + 2^3 + 3^3 \ldots\ldots n^3 = \frac{n^4 + 2n^3 + n^2}{4}.$$

$$1^4 + 2^4 + 3^4 \ldots\ldots n^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}.$$

$$1^5 + 2^5 + 3^5 \ldots\ldots n^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}.$$

&c. &c.

If we suppose n to be infinite, all its inferior powers may be rejected as inconsiderable in respect of the greatest or highest power, because any power of an infinite quantity is the next inferior power taken an infinite number of times, and we shall get an expression for the sum of an infinite series of powers whose roots are in arithmetical progression, having an infinite or indefinitely small quantity for the common difference:

Thus, rejecting $3n^2 + n$ in the expression for the sum of a series of squares, gives $\frac{2n^3}{6}$ or $\frac{n^3}{3}$ = the sum of an infinite series of squares proceeding from o^2 the least, to n^2 the greatest.

And the sum of such a series of cubes will be $\frac{n^4}{4}$, the greatest term or cube being n^3;

of a series of biquadrates, $\frac{6n^5}{30}$ or $\frac{n^5}{5}$, the greatest being n^4;

of a series of 5th. powers, $\frac{2n^6}{12}$ or $\frac{n^6}{6}$, where the greatest is n^5.

&c. &c.

Hence it appears that the expression for the sum is found by adding 1 to the index, and dividing the power so increased by its index:

Thus, the sum of the series of squares is $\frac{n^3}{3}$ or $\frac{n^{2+1}}{2+1}$;

of the series of cubes.... $\frac{n^4}{4}$ or $\frac{n^{3+1}}{3+1}$;

of the biquadrates...... $\frac{n^5}{5}$ or $\frac{n^{4+1}}{4+1}$.

&c. &c.

We therefore conclude that the sum of an infinite series of n^r powers will be $\frac{n^{r+1}}{r+1}$; where n^r is the greatest, and o^r the least terms of the series.

The differential method is also applied to the interpolation of series, the quadrature of curves, &c.

On the REVERSION of SERIES.

180. When the value of the root or unknown quantity in the terms of an infinite series is expressed by another infinite series in which that root is not found, the series is said to be reversed.

Thus, suppose $ay + by^2 + cy^3 + dy^4 + \&c. = x$; and let it be required to revert the series, or, to find y in an infinite series expressed in powers of x with coefficients.

By transposition, $ay + by^2 + cy^3 + dy^4 + \&c. - x = o$.

Assume $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c.

Then $y^2 = (Ax + Bx^2 + Cx^3 \ \&c.)^2 = A^2x^2 + 2ABx^3 + 2ACx^4 + B^2x^4$ &c.

$y^3 = (Ax + Bx^2 + Cx^3 \ \&c.)^3 = A^3x^3 + 3A^2Bx^4$ &c.

$y^4 = (Ax + Bx^2 + Cx^3 \ \&c.)^4 = A^4x^4$ &c.

$$\begin{aligned}
\text{And } ay &= aAx + aBx^2 + aCx^3 + aDx^4 \ \&c. \\
+by^2 &= \ldots\ldots + bA^2x^2 + 2bABx^3 + \left.\begin{matrix} 2bACx^4 \\ + bB^2x^4 \ \&c. \end{matrix}\right\} \\
+cy^3 &= \ldots\ldots\ldots\ldots\ldots + cA^3x^3 + 3cA^2Bx^4 \ \&c. \\
+dy^4 &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + dA^4x^4 \ \&c. \\
-x &= -x
\end{aligned}$$

$$\left.\begin{matrix}(aA-1)x + (aB + bA^2)x^2 + (aC + 2bAB + cA^3)x^3 \\ + (aD + 2bAC + bB^2 + 3cA^2B + dA^4)x^4 \ \&c.\end{matrix}\right\} = o$$

Now this sum is $= ay + by^2 + cy^3 + dy^4$ &c. $- x = o$ (or equal to o); but to make the whole expression $= o$, the coefficients of x and its powers must vanish, or each become $= o$,

that is $aA - 1 = o$; whence $A = \frac{1}{a}$.

$$aB + bA^2 = o\text{; whence } B = -\frac{bA^2}{a} = -\frac{b}{a^3}.$$

$$aC + 2bAB + cA^3 = o\text{; whence } C = -\frac{2bAB}{a} - \frac{cA^3}{a} = \frac{2b^2 - ac}{a^5}.$$

$$aD + 2bAC + bB^2 + 3cA^2B + dA^4 = o\text{; and } D = \frac{5abc - 5b^3 - a^2d}{a^7}.$$

$$\text{Therefore } y = \frac{1}{a}x - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^5}x^3 - \frac{5b^3 - 5abc + a^2d}{a^7}x^4 \text{ \&c.}$$

To revert the series $y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c. $= x$.

Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, $f = \frac{1}{5}$, &c. these being substituted in the above series, we have

$$y = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 \text{ \&c.}$$

The law of this series for the value of y is not, perhaps, sufficiently evident from the coefficients $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}$; but by extending the powers of $(Ax + Bx^2 + Cx^3 + Dx^4$ &c.) to another term, we get $E = +\frac{1}{120}$ the coefficient of the $5th.$ term;

$$\text{hence } y = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 \text{ \&c.}$$

It therefore appears that the coefficient of the nth. term is the product $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$ &c. continued to n factors, in this *case*.

If the series to be reversed is $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c. $= x$;

then $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, $f = \frac{1}{5}$,

and $y = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$, &c.

181. Suppose it is required to revert the series $ay + by^3 + cy^5 + dy^7$, &c.

Let $ay + by^3 + cy^5 + dy^7$, &c. $= x$: then $ay + by^3 + cy^5 + dy^7$ &c. $- x = 0$.

Assume $y = Ax + Bx^3 + Cx^5 + Dx^7$ &c.

$$\begin{aligned}
\text{Then } ay &= aAx + aBx^3 + aCx^5 + aDx^7 \text{ \&c.} \\
by^3 &= \ldots\ldots + bA^3x^3 + 3bA^2Bx^5 + 3bA^2Cx^7 \\
&\qquad\qquad\qquad\qquad\qquad + 3bAB^2x^7 \text{ \&c.} \\
cy^5 &= \ldots\ldots\ldots\ldots\ldots + cA^5x^5 + 5cA^4Bx^7 \text{ \&c.} \\
dy^7 &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + dA^7x^7 \text{ \&c.} \\
-x &= -x
\end{aligned}$$

$$\text{sum}\left\{\begin{array}{l}(aA-1)x + (aB+bA^3)x^3 + (aC+3bA^2B+cA^5)x^5 \\ \qquad + (aD+3bA^2C+3bAB^2+5cA^4B+dA^7)x^7 \text{ \&c.}\end{array}\right.$$

Now making the coefficients each $= 0$, as in the preceding article, we have

$aA - 1 = 0$, whence $A = \frac{1}{a}$;

$aB + bA^3 = 0$, whence $B = -\frac{b}{a^4}$;

$aC + 3bA^2B + cA^5 = 0$ ..... and $C = \frac{3b^2 - ac}{a^7}$;

$aD + 3bA^2C + 3bAB^2 + 5cA^4B + dA^7 = 0$, whence $D = -\frac{a^2d - 8abc + 12b^3}{a^{10}}$;

consequently $y = \frac{1}{a}x - \frac{b}{a^4}x^3 + \frac{3b^2 - ac}{a^7}x^5 - \frac{a^2d - 8abc + 12b^3}{a^{10}}x^7$ &c.

Examp. To revert the series $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7$ &c. $= x$.

Here $a = 1$, $b = \frac{1}{6}$, $c = \frac{3}{40}$, $d = \frac{5}{112}$, these being substituted, give $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$ &c. where the law of continuation is manifest; the coefficient of any term being the product $\frac{1}{1} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4}$ &c. extended to as many factors as there are units in the index of x in that term: thus the continued product of the 5 factors $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{5}$ is $\frac{1}{120}$ the coefficient of x^5: And the coefficient of the 5*th*. term (or of x^9) will be the continued product of the 9 factors $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6} \times \frac{1}{7} \times \frac{1}{8} \times \frac{1}{9} = \frac{1}{362880}$, &c.

What we have given in the last ten articles respecting Series, may serve as an introduction to the study of those particular branches of the subject, which is one of the most copious and intricate in the science of Algebra.

OF CUBIC EQUATIONS.

182. AN Equation is said to have as many roots as there are units in the highest dimension of the unknown quantity:

Thus, if $x^2 = a^2$, then (129) x will have two values, for it may be either $+a$, or $-a$.

Let $x^3 - (a+b+c)\,x^2 + (ab+ac+bc)\,x = abc$,

or $x^3 - (a+b+c)x^2 + (ab+ac+bc)\,x - abc = 0$ be a cubic equation generated by the continued product $(x-a) \times (x-b) \times (x-c)$:

Then it is evident x may be taken equal to a, b, or c, for if either of those values be substituted for x, the whole expression will vanish; and consequently the equation must have three roots, or, there will be three values of x.

To illustrate this in numbers, let $a=2$, $b=3$, $c=5$; then the equation becomes $x^3 - 10x^2 + 31x = 30$,

or $x^3 - 10x^2 + 31x - 30 = 0$:

And if $x = 2$, then $2^3 - 10 \times 2^2 + 31 \times 2 = 30$

if $x = 3$........ $3^3 - 10 \times 3^2 + 31 \times 3 = 30$

if $x = 5$........ $5^3 - 10 \times 5^2 + 31 \times 5 = 30$

Therefore 2, 3, and 5 are the three roots or values of x.

In the preceding equations the coefficient of the second term is the sum of the three roots, $2 + 3 + 5$:

The coefficient of the third term, the sum of their products taken two by two, $2 \times 3 + 2 \times 5 + 3 \times 5$:

And the last term their continued product, $2 \times 3 \times 5$.

183. If $(x+a)\times(x+b)\times(x+c)=o$, or $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc=o$ be the equation,
then the three roots will evidently be negative,
that is, $x=-a$, $x=-b$, $x=-c$.

Hence also,

if $(x+a)\times(x-b)\times(x-c)=o$, the three roots are $-a$, $+b$, $+c$:
and $(x+a)\times(x+b)\times(x-c)=o$, its roots are $-a$, $-b$, $+c$.
&c. &c.

It therefore appears that *cubic equations* may have all the roots positive, or all negative, or two may be negative and one positive, or two may be positive and one negative.

184. When one of the roots is discovered, the others may be found by depressing the equation, thus,

Let $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=o$,
or $(x-a)\times(x-b)\times(x-c)=o$, be the equation, (182),
and suppose $+a$ is found to be one of the roots or values of x; subtract this from x and we have $x-a$, then if the whole equation, or $(x-a)\times(x-b)\times(x-c)=o$ be divided by $x-a$, the quotient is $(x-b)\times(x-c)=o$, or $x^2-bx-cx+bc=o$, a quadratic equation which will give the other two roots.

If the root first discovered is negative (suppose $-a$ for example), then subtracting $-a$ from x, the divisor becomes $x+a$ instead of $x-a$.

To exemplify this in numbers: Let the equation be $x^3-10x^2+31x=30$, or $x^3-10x^2+31x-30=o$; and suppose $+2$ is found to be one of the roots; subtract this from x, and we have $x-2$ the divisor:

$$
\begin{array}{l}
x-2)\ x^3-10x^2+31x-30\ (x^2-8x+15 \text{ a } \textit{quadratic}.\\
\quad\ \underline{x^3-\ 2x^2}\\
\qquad -8x^2+31x\\
\qquad \underline{-8x^2+16x}\\
\qquad\qquad +15x-30\\
\qquad\qquad \underline{+15x-30}\\
\qquad\qquad\qquad 0
\end{array}
$$

therefore $x^2-8x+15=0$, whence $x^2-8x=-15$, and $x=4\pm1=5$ and 3 the other two roots.

Again, if $x^3+2x^2-23x=60$, or $x^3+2x^2-23x-60=0$ be the equation, and -3 one of the roots; then $x+3$ is the divisor:

$$
\begin{array}{l}
x+3)\ x^3+2x^2-23x-60\ (x^2-x-20, \textit{ a quadratic.} \\
\qquad\ \ \underline{x^3+3x^2} \\
\qquad\quad -x^2-23x \\
\qquad\quad \underline{-x^2-\ \ 3x} \\
\qquad\qquad -20x-60 \\
\qquad\qquad \underline{-20x-60} \\
\qquad\qquad\qquad 0
\end{array}
$$

And $x^2-x-20=0$, or $x^2-x=20$, whence $x=\frac{1}{2}\pm4\frac{1}{2}=+5$ and -4 the other two roots.

185. *To take away the second term from an Equation.*

SUPPOSE $x^2+2ax=b$, where x is the unknown quantity; and let $z-a=x$:

Then $(z-a)^2+2a(z-a)=z^2-2az+a^2+2az-2a^2=z^2-a^2=b$, or $z^2=b+a^2$, a simple quadratic in which z is the unknown quantity. Now $z=\sqrt{(b+a^2)}$, and $z-a=\sqrt{(b+a^2)}-a=x$, the same value of x as is found by completing the square in the given equation $x^2+2ax=b$.

Again, if $x^3+3ax^2=b$; then putting $z-a=x$, we have $(z-a)^3+3a(z-a)^2=b$;

$$
\begin{array}{rl}
(z-a)^3= & z^3-3az^2+3a^2z-a^3 \\
3a(z-a)^2= & \underline{\qquad +3az^2-6a^2z+3a^3} \\
\text{sum} & z^3 \qquad\quad -3a^2z+2a^3=b
\end{array}
$$

an equation in which z^2 the second power of the unknown quantity z is wanting.

And if the equation was $x^4+4ax^3=b$, the assumed value of x will be $z-a$: *We therefore divide the coefficient of the second term of the equation by the index of the highest power of the unknown quantity, and the quotient is the second member of the assumed root;* but when the second term in the equation is negative, that quotient must be positive.

$$\left.\begin{array}{r}\text{Thus, if } x^2 - 2ax = b \\ x^3 - 3ax^2 = b \\ x^4 - 4ax^3 = b\end{array}\right\} \text{ then } z + a = x.$$

Let the proposed equation be $x^3 - 12x^2 + 3x = -72$:

Then 12 (the coefficient of the second term) divided by 3 (the index of the highest power of x) gives 4; therefore assume $z + 4 = x$:

$$\begin{array}{rl}\text{then } (z+4)^3 = & z^3 + 12z^2 + 48z + 64 \\ -12(z+4)^2 = & \quad\;\, - 12z^2 - 96z - 192 \\ 3(z+4) = & \qquad\qquad\quad + 3z + 12 \\ \hline & z^3 \qquad\quad - 45z - 116 = -72, \text{ an equation}\end{array}$$

where z^2, or the 2d. term is wanting.

186. In a simple cubic equation when one root is rational, the other two are imaginary or impossible:

Thus, if $x^3 = 1$, or $x^3 - 1 = 0$; then $x = 1$ the rational root:

To find the other roots, we have $x - 1$ for the divisor (184);

whence $\dfrac{x^3 - 1}{x - 1} = x^2 + x + 1$ a *quadratic equation:*

And $x^2 + x + 1 = o$, or $x^2 + x = -1$, and completing the square, we have $x^2 + x + \frac{1}{4} = -1 + \frac{1}{4} = -\frac{3}{4}$, whence $x + \frac{1}{2} = \sqrt{(-\frac{3}{4})}$, and $x = \sqrt{(-\frac{3}{4})} - \frac{1}{2}$ one of the impossible roots:

But $-x - \frac{1}{2}$ is also the square root of $x^2 + x + \frac{1}{4}$, whence $-x - \frac{1}{2} = \sqrt{(-\frac{3}{4})}$, or $x = -\sqrt{(-\frac{3}{4})} - \frac{1}{2}$ the other:

Therefore the three values of x, or the three cube roots of 1, are 1, $\sqrt{(-\frac{3}{4})} - \frac{1}{2}$, and $-\sqrt{(-\frac{3}{4})} - \frac{1}{2}$.

Here follows the operation of cubing the last of the preceding roots:

$$\begin{array}{l}-\sqrt{(-\frac{3}{4})} - \frac{1}{2} \\ \underline{-\sqrt{(-\frac{3}{4})} - \frac{1}{2}} \\ \quad -\frac{3}{4} + \frac{1}{2}\sqrt{(-\frac{3}{4})} \\ \quad\quad \underline{\frac{1}{2}\sqrt{(-\frac{3}{4})} + \frac{1}{4}} \\ \quad -\frac{3}{4} + \sqrt{(-\frac{3}{4})} + \frac{1}{4} \text{ the } \textit{square.} \\ \quad \underline{-\sqrt{(-\frac{3}{4})} - \frac{1}{2}} \\ \quad \frac{3}{4}\sqrt{(-\frac{3}{4})} - (-\frac{3}{4}) - \frac{1}{4}\sqrt{(-\frac{3}{4})} \\ \quad \underline{\frac{3}{8} - \frac{1}{2}\sqrt{(-\frac{3}{4})} - \frac{1}{8}} \\ \quad + \frac{3}{8} + \frac{3}{4} - \frac{1}{8} = 1 \text{ the } \textit{cube.}\end{array}$$

And in the same manner it may be shown that the cube of the other imaginary root is also $= 1$.

CARDAN's *method of solving Cubic Equations.*

186. IF the equation contains all the terms with coefficients, the whole equation must be divided by the coefficient of the highest power of the unknown quantity, and the second term taken away (185), it will then be reduced to this form $x^3 \pm ax = \pm b$.

Let the equation be $x^3 + ax = b$; and put $y + z = x$:

$$\text{Then } (y+z)^3 + a(y+z) = y^3 + 3y^2z + 3yz^2 + z^3 + ay + az$$
$$= y^3 + z^3 + 3yz(y+z) + a(y+z) = b:$$

Let $3yz = -a$ which substituted for $3yz$

$$\text{then } y^3 + z^3 + 3yz(y+z) + a(y+z) = b$$
$$\text{becomes } y^3 + z^3 - a(y+z) + a(y+z) = y^3 + z^3 = b:$$

And since $3yz = -a$, we have $z = \frac{-a}{3y}$, and $z^3 = \frac{-a^3}{27y^3}$ which put for z^3 in the equation $y^3 + z^3 = b$ gives $y^3 + \frac{-a^3}{27y^3} = b$, whence $y^6 - by^3 = \frac{1}{27}a^3$, an equation of the quadratic form: and by completing the square we get $y^3 = \frac{1}{2}b \pm \sqrt{\left(\frac{1}{27}a^3 + \frac{1}{4}b^2\right)}$, and $y = \left(\frac{1}{2}b \pm \sqrt{\left(\frac{1}{27}a^3 + \frac{1}{4}b^2\right)}\right)^{\frac{1}{3}}$.

And since $y^3 + z^3 = b$, z^3 is $= b - y^3$,

$$\text{that is, } z^3 = b - \left(\tfrac{1}{2}b \pm \sqrt{\left(\tfrac{1}{27}a^3 + \tfrac{1}{4}b^2\right)}\right)$$
$$\text{or } z^3 = \tfrac{1}{2}b \mp \sqrt{\left(\tfrac{1}{27}a^3 + \tfrac{1}{4}b^2\right)}$$
$$\text{whence } z = \left(\tfrac{1}{2}b \mp \sqrt{\left(\tfrac{1}{27}a^3 + \tfrac{1}{4}b^2\right)}\right)^{\frac{1}{3}}$$

therefore

$$x(=y+z) = \left(\tfrac{1}{2}b \pm \sqrt{\left(\tfrac{1}{27}a^3 + \tfrac{1}{4}b^2\right)}\right)^{\frac{1}{3}} + \left(\tfrac{1}{2}b \mp \sqrt{\left(\tfrac{1}{27}a^3 + \tfrac{1}{4}b^2\right)}\right)^{\frac{1}{3}}.$$

Remark. When a is negative and $\frac{1}{4}b^2$ less than $\frac{1}{27}a^3$, the solution by this rule cannot generally be obtained, because the

This method of cubic equations attributed to *Cardan* was discovered, it seems, by *Scipio Ferreus*, and *Nic. Tartalea*. All three were Italian mathematicians who flourished rather early in the 16*th*. century.

quantity $\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)$ becomes negative, and therefore its square root $\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}$ is impossible. This is called the *irreducible case.*

Examples.

1. To find x in the equation $3x^3+18x^2+6x=261$.

Divide the whole equation by 3 the coefficient of x^3, and we have $x^3+6x^2+2x=87$.

Next, to take away the 2*d.* term ($6x^2$), let $z-\frac{6}{3}$ or $z-2=x$ (183):

$$\begin{aligned}
\text{then } (z-2)^3 &= z^3-6z^2+12z-8\\
6(z-2)^2 &= +6z^2-24z+24\\
2(z-2) &= +2z-4\\
\hline
& z^3 -10z+12=87,\text{ or } z^3-10z=75.
\end{aligned}$$

Here $a=-10$, and $b=75$, these values being substituted in the general expression give

$$\left(\frac{75}{2}\pm\sqrt{\left(\frac{-10^3}{27}+\frac{75^2}{4}\right)}\right)^{\frac{1}{3}}+\left(\frac{75}{2}\mp\sqrt{\left(\frac{-10^3}{27}+\frac{75^2}{4}\right)}\right)^{\frac{1}{3}}=z$$

$$\text{or } \left(37\tfrac{1}{2}\pm\sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}}+\left(37\tfrac{1}{2}\mp\sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}}=z$$

where both the upper or both the lower signs may be taken in all cases, therefore retaining the former, we have

$$\left(37\tfrac{1}{2}+\sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}}+\left(37\tfrac{1}{2}-\sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}}=z$$

$$\text{or } 4.207+.793=5=z;$$

Now $z-2=x$, whence $x=3$ the value of x in the given equation.

2. Let the equation be $x^3+8x=-399$, to find x.

Here $a=+8$, and $b=-399$; whence, by substitution

$$x=\left(\frac{-399}{2}+\sqrt{\left(\frac{512}{27}+\frac{159201}{4}\right)}\right)^{\frac{1}{3}}+\left(\frac{-399}{2}-\sqrt{\left(\frac{512}{27}+\frac{159201}{4}\right)}\right)^{\frac{1}{3}}$$

$$=\left(-199\tfrac{1}{2}+\sqrt{39819\tfrac{23}{108}}\right)^{\frac{1}{3}}+\left(-199\tfrac{1}{2}-\sqrt{39819\tfrac{23}{108}}\right)^{\frac{1}{3}}=0.36$$

$-7.36=-7$ the root, or value of x.

The other two roots may be found as in *Art.* 184, thus, $x^3+8x+399=0$, and $\frac{x^3+8x+399}{x+7}=x^2-7x+57=0$, this quadratic gives $x=3\frac{1}{2}\pm\sqrt{(-44\frac{3}{4})}$; therefore both are imaginary or impossible.

187. Sometimes the method of solving an equation may be discovered by the addition, or subtraction, of a given quantity:

Thus, suppose $x^3+6x^2+12x=504$, where it appears that if 8 be added to each side of the equation, the sums will be complete cubes;

$$x^3+6x^2+12x+8=512$$

whence, $x+2=8$, and $x=6$.

Again, if $x^3-12x^2+48x=100$, then by subtracting 64 $\left(\text{the cube of } \frac{12}{3}\right)$ we have $x^3-12x^2+48x-64=36$, and taking the cube roots, $x-4=36^{\frac{1}{3}}$, or $x=4+36^{\frac{1}{3}}$.

THE RESOLUTION OF EQUATIONS BY APPROXIMATION.

188. THE preceding methods of solution are restricted to particular cases which seldom occur in practice. We shall therefore proceed to the resolution of equations by approximation: And for this purpose the rule of Double Position or Trial-and-error (Arith. 109.) seems the most general and expeditious of any.

Examples.

1. To find x in the cubic equation $x^3+x^2=500$.

From a trial or two it appears that the value of x is between 7 and 8; therefore let those two numbers be the first assumptions:

then $7^3=$	343	$8^3=$	512
$7^2=$	49	$8^2=$	64
	392		576
	500		500
error	108 too *little.*	*error*	76 too *great.*
			108
		sum of errors	184

$$\begin{array}{r}
108 \times 8 = 864 \\
76 \times 7 = \underline{532} \\
184)\ \underline{1396}
\end{array}$$

quotient nearly 7.6 the first approximation.

$$\begin{array}{rl}
7.6^3 = & 438.976 \\
7.6^2 = & \underline{57.76} \\
 & 496.736 \\
 & \underline{500} \\
\text{too } \textit{little} & 3.264
\end{array}$$

it therefore appears that the value of x is greater than 7.6

Now let 7.61 and 7.62 be the two suppositions:

$$\begin{array}{rlcrl}
\text{then } 7.61^3 = & 440.711081 & & 7.62^3 = & 442.450728 \\
7.61^2 = & \underline{57.9121} & & 7.62^2 = & \underline{58.0644} \\
 & 498.623181 & & & 500.515128 \\
 & \underline{500} & & & \underline{500} \\
\textit{error} & \underline{1.376819} \text{ too } \textit{little.} & & \textit{error} & \underline{0.515128} \text{ too } \textit{great.}
\end{array}$$

$$\begin{array}{rr}
1.376819 \times 7.62 = & 10.49136078 \\
.515128 \times 7.61 = & \underline{3.92012408}
\end{array}$$

$$\begin{array}{rl}
 & 1.376819 \\
 & \underline{0.515128} \\
\text{sum} & \underline{1.891947} \ldots\ldots\ldots\ldots\ldots\ 1.891947)\ \underline{14.41148486} \text{ sum}
\end{array}$$

quotient 7.617 second approximation, which is the value of x nearly.

2. Let the equation be $x^3 - 50x^2 + 3x = -4103$; to find x?

By a few trials x is found to be greater than 10 but less than 11; now assuming these numbers, then

$$\begin{array}{rrcrr}
10^3 = & 1000 & & 11^3 = & 1331 \\
3 \times 10 = & \underline{30} & & 3 \times 11 = & \underline{33} \\
+ & 1030 & & + & 1364 \\
50 \times 10^2 = & \underline{-5000} & & 50 \times 11^2 = & \underline{-6050} \\
 & -3970 & & & -4686 \\
 & \underline{-4103} & & & \underline{-4103} \\
\textit{error} & 133 \text{ too } \textit{little} & & \textit{error} & \underline{583} \text{ too } \textit{great.} \\
 & \underline{583} & & & \\
\text{sum} & \underline{716} & & &
\end{array}$$

$$\begin{array}{r}
10 \times 583 = 5830 \\
11 \times 133 = \underline{1463} \\
716)\ \underline{7293} \text{ sum}
\end{array}$$

quotient nearly $\underline{10.2}$ first approximation.

Now assume $x = 10.2$ again, suppose $x = 10.18$

$$\begin{array}{rr}
10.2^3 = & 1061.208 \\
3 \times 10.2 = & 30.6 \\
\hline
& +1091.808 \\
50 \times 10.2^2 = & -5202 \\
\hline
& -4110.192 \\
& -4103 \\
\hline
\textit{error} & 7.192 \text{ too } \textit{great} \\
& 6.897832 \\
\hline
\text{sum} & 14.089832 \\
\hline
\end{array}
\qquad
\begin{array}{rr}
10.18^3 = & 1054.977832 \\
3 \times 10.18 = & 30.54 \\
\hline
& +1085.517832 \\
50 \times 10.18^2 = & -5181.62 \\
\hline
& -4096.102168 \\
& -4103 \\
\hline
\textit{error} & 6.897832 \text{ too } \textit{little.} \\
\hline
\end{array}$$

$$\begin{array}{rr}
10.2 \times 6.897832 = & 70.3578864 \\
10.18 \times 7.192 \ldots = & 73.21456 \\
\hline
14.089832) & 143.5724464 \\
\hline
\text{quotient nearly} & 10.19 \text{ second approximation,}
\end{array}$$

which is very nearly the value of x.

To find the other two roots we have $x - 10.19$ for the *divisor* (184); and $x^3 - 50x^2 + 3x + 4103 = 0$, the *dividend*;

then

$$\begin{array}{l}
x - 10.19)\ x^3 - 50x^2 + 3x + 4103\ (x^2 - 39.81x - 402.6639 \text{ a } \textit{quadratic}. \\
\quad x^3 - 10.19x^2 \\
\hline
\quad -39.81x^2 + 3x \\
\quad -39.81x^2 + 405.6639x \\
\hline
\qquad\qquad -402.6639x + 4103 \\
\qquad\qquad -402.6639x + 4103.145141 \\
\hline
\end{array}$$

By this operation the learner will perceive that the true value of x is somewhat less than 10.19, the error in the whole equation being the decimal .145141.

Now $x^2 - 39.81x - 402.6639 = 0$, whence $x^2 - 39.81x = 402.6639$, and by completing the square, &c. $x = 19.905 \pm 28.26434 = 48.16934$, and -8.35934 the other two roots. Therefore the equation has three possible roots, two positive, and one negative.

And the sum of the three roots, $10.19 + 48.16934 - 8.35934 = 50$ the coefficient of the second term. (182).

3. To find x in the biquadratic equation $x^4 - 10x^3 + 100x^2 - 70x = 42676$?

The value of x appears to be between 15 and 16, therefore assuming those numbers, we shall have

$$\begin{array}{rrl} & 15^4 = & 50625 \\ 100 \times & 15^2 = & \underline{22500} \\ & + & 73125 \\ & - & \underline{34800} \\ & & 38325 \\ & & \underline{42676} \\ \textit{error} & & \underline{4351} \text{ too } \textit{little}. \end{array} \qquad \begin{array}{rrl} 10 \times & 15^3 = & 33750 \\ 70 \times & 15 = & \underline{1050} \\ & - & 34800 \end{array}$$

$$\begin{array}{rrl} & 16^4 = & 65536 \\ 100 \times & 16^2 = & \underline{25600} \\ & + & 91136 \\ & - & \underline{42080} \\ & & 49056 \\ & & \underline{42676} \\ \textit{error} & & 6380 \text{ too } \textit{great} \\ & & \underline{4351} \\ \text{sum} & & 10731 \end{array} \qquad \begin{array}{rrl} 10 \times & 16^3 = & 40960 \\ 70 \times & 16 = & \underline{1120} \\ & - & 42080 \\ \\ \\ 16 \times & 4351 = & 69616 \\ 15 \times & 6380 = & \underline{95700} \\ & \text{sum} & 165316 \end{array}$$

$\frac{165316}{10731} = 15.4$ the first approximation for the value of x; now let this be assumed:

$$\begin{array}{rrl} & 15.4^4 = & 56244.8656 \\ 100 \times & 15.4 = & \underline{23716} \\ & + & 79960.8656 \\ & - & \underline{37600.64} \\ & & 42360.2256 \\ & & \underline{42676} \\ \textit{error} & & 315.7744 \text{ too } \textit{little} \\ \textit{error} & & \underline{4351........} \text{ too } \textit{little}\text{, by assuming } x = 15 \text{ as before.} \\ \textit{diff.} \text{ of errors......} & & 4035.2256 \end{array} \qquad \begin{array}{rrl} 10 \times & 15.4^3 = & 36522.64 \\ 70 \times & 15.4 = & \underline{1078} \\ & - & 37600.64 \end{array}$$

$$\begin{array}{rl} 15. \times 315.7744 = & 4736.616 \\ 15.4 \times 4351 = & \underline{67005.4} \\ \textit{diff.} \text{ of products} & \underline{62268.784} \end{array} \quad \text{whence } \frac{62268.784}{4035.2256} = 15.43 \text{ the second}$$

approximation, which is the value of x nearly.

To depress the equation in order to approximate the other roots, the divisor is $x - 15.43$;

$$x-15.43)x^4-10x^3+100x^2-70x-42676=0(x^3+5.43x^2+183.7849x+2765.8$$

$$\begin{array}{l}
x^4-15.43x^3 \\
\hline
\quad +5.43x^3+100x^2 \\
\quad +5.43x^3-83.7849x^2 \\
\hline
\qquad +183.7849x^2-70x \\
\qquad +183.7849x^2-2835.801007x \\
\hline
\qquad\qquad +2765.801007x-42676 \\
\qquad\qquad +2765.801007x-42676.3 \text{ \&c.} \\
\hline
\qquad\qquad\qquad 0
\end{array}$$

Now $x^3+5.43x^2+183.7849x+2765.8=0$, or $x^3+5.43x^2+183.7849x=-2765.8$ where x is evidently negative; and from a few trials its value appears to be a little greater than 11; if therefore -11.1 and -11.2 are made the first suppositions, two operations will bring out $x=-11.163$ one of the roots. Consequently $x+11.163$ is the divisor for depressing the cubic to a quadratic equation:

Hence $\dfrac{x^3+5.43x^2+183.7849x+2765.8}{x+11.163}=x^2-5.733x+247.782379=0$, or $x^2-5.733x=-247.782379$, whence $x=2.8665\pm\sqrt{(-239.56 \text{ \&c.})}$ which values are both impossible or imaginary. The equation therefore, has a positive, a negative, and two impossible roots.

$$\begin{array}{l}
+15.43 \\
-11.163 \\
+\ 2.8665+\sqrt{(-239.56 \text{ \&c.})} \\
+\ 2.8665-\sqrt{(-239.56 \text{ \&c.})} \\
\hline
\ \ 10.0000
\end{array}$$

sum of roots, equal to the coefficient of the second term in the given equation.

4. Let the proposed equation be $(7x^3+4x^2)^{\frac{1}{3}}+(20x^2-10x)^{\frac{1}{2}}=28$; to find the value of x?

By trial x is found to be between 4 and 5, therefore let those numbers be the first suppositions:

$$\begin{array}{lrlr}
(7\times4^3+4\times4^2)^{\frac{1}{3}} = & 8 & \qquad (7\times5^3+4\times5^2)^{\frac{1}{3}} = & 9.916 \\
(20\times4^2-10\times4)^{\frac{1}{2}} = & 16.73 & \qquad (20\times5^2-10\times5)^{\frac{1}{2}} = & 21.213 \\
\hline
 & 24.73 & & 31.129 \\
 & 28 & & 28 \\
\hline
\textit{error too little} & 3.27 & \textit{error too great} & 3.129 \\
 & & & 3.27 \\
\hline
 & & \text{sum} & 6.399
\end{array}$$

$4 \times 3.129 = 12.516$
$5 \times 3.27 = 16.35$

sum of products........ 28.866 $\qquad \frac{28.866}{6.399} = 4.5$ the first approximation.

Next, assuming $x = 4.5$ and 4.51, and repeating the operation, we get 4.511, 4.5107, 4.51066 the successive values of x, the last being a very near approximation.

N. B. In resolving these complex equations, the student should make use of logarithms for raising powers and extracting roots, otherwise he will find the operation extremely tedious.

189. *Logarithms* are also peculiarly adapted to the resolution of *exponential* equations. We shall subjoin a few examples.

1. Suppose $3^x = 19683$; to find x?

It is evident from Arith. *art.* 187, that the *log.* of 3 multiplied by the exponent x gives the *log.* of 19683;

that is $x \times$ *log.* of $3 =$ *log.* of 19683,

therefore $x = \frac{\textit{log.}\text{ of } 19683}{\textit{log.}\text{ of } 3}$, or $\frac{4.294091}{0.477121} = 9$ the value of x: for $3^9 = 19683$.

2. To find x in the equation $24^x = 44620$?

$x = \frac{\textit{log.}\ 44620}{\textit{log.}\ 24} = \frac{4.649530}{1.380211} = 3.368709$ &c. This value of x however, is too great: the error arises in consequence of using logarithms to 6 places of decimals only; for the result by logarithms to 10 places will be 3.368708662 &c. and since 3.368709 &c. is correct in the 6*th.* figure, the other value from logarithms to 10 places will probably be so in the 10*th.*

3. Let the equation be $x^x = 46060$, to find the value of x?

$6^6 = 46656$, hence x appears to be a little less than 6;

Therefore let 6 and 5.9 be the two first assumptions for x:

$6^6 = 46656$
$\underline{46060}$
error $\quad 596$ too *great*

5.9 *log.* 0.770852
$5.9 \times 0.770852 = 4.5480268$ the *log.* of $35325 = 5.9^{5.9}$
$\underline{46060}$
error 10735 too *little.*
$\underline{596}$
sums of errors 11331

$6 \times 10735 = 64410$
$5.9 \times 596 = 3516.4$
sum of products $= 67926.4$

And $\frac{67926.4}{11331} = 5.99$ the first approximate value of x.

Again, let 6, and 5.99 be the next suppositions:

5.99 *log*. 0.777427

and $5.99 \times 0.777427 = 4.656787$ &c. the *log*. of $45372 = 5.99^{5.99}$

46060
error 688 too *little*
supposition 6......*error* 596 too *great*
sum of errors 1284

$6 \times 688 = 4128$
$5.99 \times 596 = 3570.04$
sum 7698.04 and $\frac{7698.04}{1284} = 5.995$ the 2*d*. approximation.

Next, suppose 6, and 5.995 are the assumptions:

5.995 *log* 0.777789

and $5.995 \times 0.777789 = 4.662845$ &c. the *log*. of $46009 = 5.995^{5.995}$

46060
error 51 too *little*
supposition 6, *error* 596 too *great*
sum 647

$6 \times 51 = 306$
$5.995 \times 596 = 3573.02$
sum products 3879.02. $\frac{3879.02}{647} = 5.9954$ the 3*d*. approximation.

Again, suppose 5.995 and 5.9954

5.9954 *log*. 0.777818

$5.9954 \times 0.777818 = 4.663330$ &c. the *log*. of $46060.6 = 5.9954^{5.9954}$

46060
error...... .6 too *great*
supposition 5.995, *error*...... 51.0 too *little*
sum 51.6

$51 \times 5.9954 = 305.7654$
$.6 \times 5.995 = 3.597$
sum 309.3624 $\frac{309.3624}{51.6} = 5.995395$ the 4*th*. approximation, which is very nearly the true value of x: for a table of logarithms to 10 places gives $5.995395^{5.995395} = 46060.1$.

190. But the operation in this method of approximation may be somewhat abridged in the following manner:

If S and s be the two suppositions, D and d the corresponding errors, and x the number sought:

Then (Art. 128, *examp.* 8.)

$x - S : x - s :: D : d$, and by division (90)

$s - S : x - s :: D - d : d$

or $D - d : d :: s - S : x - s$ when the errors are alike.

And

$d : D : s - x : x - S$, and by composition (89)

$d + D : d :: s - S : s - x$, when the errors are unlike.

To apply this in the last example, we have

$s = 6$.......... $d =$ 596 error too *great* } unlike
$S = 5.9$...... $D =$ 10735 error too *little* }
$s - S = \overline{0.1}$ sum $\overline{11331} = d + D$

As 11331 : 596 :: 0.1 : 0.005, and 6 − 0.005 = 5.995 first approximation.

Now let 6 and 5.995 be the two suppositions;

6................ *error* 596 too *great*
5.995......... *error* 51 too *little*.
diff. $\overline{0.005}$ sum $\overline{647}$

As 647 : 51 :: .005 : $\frac{51 \times .005}{647}$ = .00039, and 5.995 + .00039 = 5.99539 second approximation: &c.

Remark. It is to be observed that the correction or 4th. term of the proportion, must always be applied to that assumed number which gives the error that is made use of in finding the correction: thus in the first proportion, .005 is the correction, and 596 the error that is used, and therefore the correction must be applied to the supposition 6; now 6 is too great, consequently the correction is subtractive: but if we take the other error, the correction must be added to the other supposition because that is defective,

Thus 11331 : 10735 :: 0.1 : .095 the correction, and 5.9 + .095 = 5.995 the first approximation, as before.

The student will also perceive that when the errors are *alike*, their *difference* will be the first term of the proportion.

By raising 46060 to the 1000000th. power, the equation will be freed from the fractional index, thus, $9.995395 = \frac{9995395}{1000000}$, therefore $x^{\frac{9995395}{1000000}} = 46060$, and $x^{9995395} = 46060^{1000000}$, or $9.995395^{9995395} = 46060^{1000000}$.

191. *In particular cases, the unknown quantity in an equation may be found by summing a series.*

1. Thus, suppose $3x + 4x^2 + 7x^3 + 11x^4$ &c. *in infin.* $= 2$, where it is manifest the value of x must be less than 1; and by adding 1 to each side of the equation,

$1 + 3x + 4x^2 + 7x^3$ &c. $= 3$: now $1 + 3x + 4x^2 +$ &c. is a recurring series whose sum $= \frac{1+2x}{1-x-x^2}$ (172) $= 3$, whence $1 + 2x = 3 - 3x - 3x^2$, which quadratic equation gives $x = \frac{1}{3}$.

2. Again, if $3x + 6x^2 + 10x^3 + 15x^4$, &c. *in infin.* $= 100$, then adding 1 to each side of the equation, we have $1 + 3x + 6x^2 + 10x^3$ &c. $= 101$; and the sum of the recurring series $1 + 3x + 6x^2$, &c. $= \frac{1}{1 - 3x + 3x^2 - x^3}$ or $\frac{1}{(1-x)^3} = 101$, therefore $\frac{1}{1-x} = 101^{\frac{1}{3}}$, whence $x = 1 - \frac{1}{101^{\frac{1}{3}}}$.

3. Suppose $x - x^2 + x^3 - x^4 + x^5$ &c. *in infin.* $= \frac{1}{3}$: then reversing the series, (180)

gives $x = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$ &c. which is a geometrical series, and the sum *ad infin.* $= \frac{1}{2}$, or the value of x.

The value of x however, may be found without reverting the series.

4. Let the equation be $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$ &c. *in infin.* $= 1$.

By reverting the series, $x = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} +$ &c.

5. Suppose the given equation is $\frac{1}{2}x^2 + \frac{1}{2.3}x^3 + \frac{1}{2.3.4}x^4 + \frac{1}{2.3.4.5}x^5 +$ &c. *in infin.* $= 8$; to find the value of x?

The coefficients constitute the series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\frac{1}{5.6.7}\ \&c.$$

In order to investigate a theorem for the sums of such series, it may be observed, that in a series of quantities, the sum of all the differences of the terms taken in succession, is always equal to the difference between the first and last terms of the series:

Thus, if the series be $29+24+20+17+15$
differences $5+4+3+2=14=29-15$.

Or if $22+15+9+5+2+0$ be the series,
differences $7+6+4+3+2=22=22-0$: and so of others.

Now let $1+a+ab+abc+abcd+$ &c. be a series of quantities which continually decrease so that the last term becomes indefinitely small or equal to o; then taking the differences of the terms, we have

$$\begin{aligned}
1-a &= 1-a\\
a-ab &= a(1-b)\\
ab-abc &= ab(1-c)\\
abc-abcd &= abc(1-d)\ \&c.
\end{aligned}$$

sum of the differences $1-a+a(1-b)+ab(1-c)+abc(1-d)$ &c. $=1-0=1$ the difference of the first and last terms.

Now let a, b, c, &c. be expounded by fractional quantities,

viz. suppose $a=\frac{m}{v}$

$$b=\frac{m+p}{v+p} \qquad d=\frac{m+r}{v+r}$$

$$c=\frac{m+q}{v+q} \qquad f=\frac{m+s}{v+s}\ \&c.$$

then $1-a=1-\frac{m}{v}=\frac{v-m}{v+p}$

$$1-b=1-\frac{m+p}{v+p}=\frac{v-m}{v+p}$$

$$1-c=1-\frac{m+q}{v+q}=\frac{v-m}{v+q}$$

$$1-d=1-\frac{m+r}{v+r}=\frac{v-m}{v+r}\ \&c.$$

These several values being substituted in the equation

$1-a+a(1-b)+ab(1-c)+abc(1-d)$ &c. we have

$$\frac{v-m}{v}+\frac{m}{v}\left(\frac{v-m}{v+p}\right)\frac{m}{v}\cdot\frac{m+p}{v+p}\left(\frac{v-m}{v+q}\right)+\frac{m}{v}\cdot\frac{m+p}{v+p}\cdot\frac{m+q}{v+q}\left(\frac{v-m}{v+r}\right)$$

&c. $=1$;

and dividing the whole equation by $\frac{v-m}{v}$ gives

$$1+\frac{m}{v+p}+\frac{m}{v+p}\cdot\frac{m+p}{v+q}+\frac{m}{v+p}\cdot\frac{m+p}{v+q}\cdot\frac{m+q}{v+r}\text{ \&c.}=\frac{v}{v-m}:$$

If $q=2p$, $r=3p$, $s=4p$, &c. and $v+p=n$, then

$$1+\frac{m}{n}+\frac{m}{n}\cdot\frac{m+p}{n+p}+\frac{m}{n}\cdot\frac{m+p}{n+p}\cdot\frac{m+2p}{n+2p}+\frac{m}{n}\cdot\frac{m+p}{n+p}\cdot\frac{m+2p}{n+2p}\cdot\frac{m+3p}{n+3p}$$

&c. $=\frac{n-p}{n-p-m}$: let $p=1$, then

$$1+\frac{m}{n}+\frac{m(m+1)}{n(n+1)}+\frac{m(m+1)(m+2)}{n(n+1)(n+2)}+\frac{m(m+1)(m+2)(m+3)}{n(n+1)(n+2)(n+3)}$$

$+$ &c. *in infin.* $=\frac{n-1}{+n-1-m}$, which is a general theorem for summing a variety of series infinitely continued *.

To adapt this theorem to the series proposed, let $n=m+3$, then dividing the whole equation by $m(m+1)(m+2)$, we get

$$\frac{1}{m(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}+\frac{1}{(m+2)(m+3)(m+4)}+\text{\&c.}$$

$=\frac{1}{m(m+1)(2)}$, which, when $m=1$, becomes $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$ $+\frac{1}{4.5.6}$ &c. $=\frac{1}{4}$ the sum of the series infinitely continued.

Hence $\frac{1}{4}x^5=8$, and $x^5=32$, whence $x=2$.

192. Though this subject of series is rather misplaced, we shall subjoin a few examples to show the use of the preceding theorem in the summation of particular series.

If $n=m+4$, and the whole equation be divided by $m(m+1)(m+2)(m+3)$, we have

$$\frac{1}{m(m+1)(m+2)(m+3)}+\frac{1}{(m+1)(m+2)(m+3)(m+4)}\text{ \&c.}=\frac{1}{m(m+1)(m+2)(3)},$$

* Simpson's Algebra.

where the law of continuation for the sums of these kind of series is evident: and hence it appears, that if the difference of the first and last factors in the denominator of the first term be substituted for the last factor, the resulting fraction will be the sum of the whole infinite series:

Thus, in the expression for the sum of $\frac{1}{6}+\frac{1}{24}+\frac{1}{60}$ &c. the first term is $\frac{1}{m(m+1)(m+2)}$, and the difference of m and $m+2$, the first and last factors in the denominator, is 2, which substituted for $m+2$ the last factor, gives $\frac{1}{m(m+1)(2)}$ the sum of the series.

This will enable us very readily to find the sum of any given number of terms of the series: for example, to determine the sum of the 20 first terms of the series $\frac{1}{6}+\frac{1}{24}+\frac{1}{60}$ &c.

The 21*st.* term is $\frac{1}{21.22.23}$, therefore substituting $23-21$ for 23 gives $\frac{1}{21.22.2}$ or $\frac{1}{924}$ the sum of the series infinitely continued when $\frac{1}{21.22.23}$ is the first term; now $\frac{1}{4}$ being the *whole* sum when $\frac{1}{1.2.3}$ is the first term, we have $\frac{1}{4}-\frac{1}{924}=\frac{115}{462}$ the 20 first terms.

In the general theorem if $m=1$, then

$$1+\frac{1}{n}+\frac{1.2}{n(n+1)}+\frac{1.2.3}{n(n+1)(n+2)}+\frac{1.2.3.4}{n(n+1)(n+2)(n+3)}\text{ \&c.}=\frac{n-1}{n-2},$$

an expression for the several orders of the reciprocals of figurate numbers infinitely continued.

Thus, if $n=2$,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\text{ \&c.}=\frac{3}{0},\text{ therefore not summable.}$$

If $n=3$,

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\text{ \&c.}=2.$$

If $n=4$,

$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}\text{ \&c.}=\frac{3}{2}.$$

And so on for other values of n.

Of SIMPLE INTEREST.

193. All the computations which relate to Simple Interest may be wrought arithmetically (Arith. *art.* 106); but algebraic theorems for the different cases will facilitate the practice considerably.

Let r = the *rate* of interest, or the interest of one pound for one year.
p = any principal or sum bearing interest.
t = the time in years.
a = the amount in the time t, or sum of the principal and interest.

Then $1 : r :: p : rp$ = the interest of the sum p for 1 year, and trp = the interest for the time t; whence $p + trp$ (the sum of the principal and interest) $= a$ the amount in the time t: this equation gives the following theorems:

$$r = \frac{a - p}{pt} \text{ the rate.}$$

$$p = \frac{a}{1 + rt} \text{ the principal.}$$

$$t = \frac{a - p}{pr} \text{ the time.}$$

$$a = p + trp = (1 + tr)p \text{ the amount.}$$

A few examples will show the use of these expressions.

1. What is the amount of 400*l.* in 5 months at 4 *per cent.*?

To find r: as $100l. : 4l. :: 1l. : .04 = r$.

therefore $r = .04$
$p = 400$
$t = \frac{5}{12}$

And $a = p + trp = 400 + \frac{5}{12} \times .04 \times 400 = 406l.\ 13s.\ 4d.$ the sum required.

2. What is the interest of 1 pound for 1 day at 5 *per cent.*?

$$100 : 5 :: 1 : .05 = r:$$

Here then, we have $r = .05$

$$p = 1$$

$$t = \frac{1}{365};$$

And a (the amount) $= (1 + tr)p = 1 + \frac{.05}{365} = 1.000136986$, &c. the amount, from which deducting the principal 1, there remains .000136986 &c. of a pound, the *answer*.

3. What sum in ready money is equivalent to 600*l.* due nine months hence, allowing 5 *per cent.* discount?

In this case $r = .05$

$$t = \frac{9}{12}$$

$a = 600$. And $p = \dfrac{600}{1 + .05 \times \frac{9}{12}} = 578.313$ *l.* the *answer*.

4. At what rate will 400*l.* in 18 months amount to, or raise a stock of 440*l.*?

Here $t = 1\frac{1}{2}$

$$p = 400$$

$a = 440$. And $r = \dfrac{440 - 400}{400 \times 1\frac{1}{2}} = \dfrac{40}{600} = \dfrac{1}{15}$ the *rate* or the interest of 1 pound for 1 year; whence $\frac{1}{15} \times 100 = 6\frac{2}{3}$*l. per cent.* the *answer*.

5. In what time will 360*l.* raise a stock of 370*l.* at 4 *per cent.*?

Here we have given $r = .04$

$$p = 360$$

$a = 370$. And $t = \dfrac{370 - 360}{360 \times .04} = 253\frac{1}{2}$ *days*, nearly the time required.

Of COMPOUND INTEREST.

194. THEOREMS for the solution of the different cases of compound interest may be derived from a process similar to that for finding the amount of a given sum in a given time, Arith. *art.* 107.

In these computations the amount of 1 pound in 1 year at *simple interest* is usually called the *rate*:

Thus, the amount of 100*l.* in 1 year at 5 *per cent.* is 105*l.* then 100 : 105 :: 1 : 1.05 (the amount of 1 pound in 1 year) is the *rate*: and 1.04 is the *rate* when 4 *per cent.* is the interest.

Let $r =$ the rate,
$p =$ any principal,
$t =$ the time in years,
$a =$ the amount.

To find the amount of any principal sum (p) for the time t.

l. *l.*
As $1 : r :: p : rp$ the amount of p pounds at the end of 1 year.
And $1 : r :: rp : r^2p$ the amount of rp pounds at the end of the 2*d.* year.
Also $1 : r :: r^2p : r^3p$ the amount in three years, &c.

Hence it appears that r^tp will be the amount in t years,

viz, $r^tp = a$. Whence the following theorems are readily obtained:

$$a = r^tp. \qquad r = \left(\frac{a}{p}\right)^{\frac{1}{t}}.$$

$$p = \frac{a}{r^t}. \qquad t = \frac{\log. \frac{a}{p}}{\log. r}. \text{ (189. ex. 1.)}$$

By means of logarithms these expressions are simple in their application.

Examples.

1. What is the compound interest of 200*l.* in 15 years at 5 *per cent.*?

Here $r = 1.05$, $p = 200$, $t = 15$:

$$\begin{array}{rr}
r = 1.05 \ldots\ldots \text{log.} & 0.021189 \\
t = 15 \ldots\ldots\ldots\ldots & 15 \\
\hline
 & 105945 \\
 & 21189 \\
\hline
 & 0.317835 \ \text{log.}\ r^t \\
p = 200 \ \text{log.} & 2.301030 \\
\hline
\text{the amount} \ldots 415.8 = a \ \text{log.} & 2.618865 \ \text{log.}\ r^t p \\
p = 200 & \\
\text{the interest} = 215.8l. \ \text{nearly.} &
\end{array}$$

2. What will 50*l.* amount to in 10 years and 211 days at $4\frac{1}{2}$ *per cent.*?

In this case $r = 1.045$, $p = 50$, $t = 10\frac{211}{365}$.

$$\begin{array}{rr}
1.045 \ \text{log.} & 0.019116 \\
 & 10\frac{211}{365} \\
\hline
 & 0.202211 \ \text{log.}\ r^t \\
50 \ \text{log.} & 1.698970 \\
\hline
\text{Amount nearly } \overset{l.}{79.649} \ \text{log.} & 1.901181 \\
\hline
\end{array}$$

3. What is the compound interest of 242*l.* 10*s.* forborn $2\frac{1}{2}$ years, at 4 *per cent. per ann.* the interest payable half yearly.

As $100 : 102 :: 1 : 1.02$ (the amount of 1 pound in $\frac{1}{2}$ a year) $= r$, $t = 5$ ($\frac{1}{2}$ years), $p = 242.5$.

$$\begin{array}{rr}
r = 1.02 \ldots\ldots \text{log.} & 0.008600 \\
t = 5 \ldots\ldots\ldots\ldots & 5 \\
\hline
 & 0.043000 \\
p = 242.5 \ldots\ldots \text{log.} & 2.384712 \\
\hline
\text{amount } 267.74 \quad \text{log.} & 2.427712 \\
\hline
242.5 & \\
\text{Interest} = 15.24l. \ \text{nearly.} &
\end{array}$$

4. What principal will raise a stock of 1000*l.* in 15 years at 5 *per cent.*

Here $r = 1.05$, $t = 15$, $a = 1000$.

$r = 1.05$ *log.* 0.021189
15
0.317835 *log.* r^t
$a = 1000$... *log.* 3.000000

Ans. £481.02 $= p$ *log.* 2.682165 diff. *log.* $\frac{a}{r^t}$

5. At what rate of interest will 480*l.* raise a stock of 864*l.* 8*s.* in 15 years?

Here $a = 864.4$, $p = 480$, $t = 15$.

$a = 864.4$....*log.* 2.936715
$p = 480$....... *log.* 2.681241

15) 0.255474 *log.* $\frac{a}{p}$

rate 1.04 *log.* 0.017032 *log.* $\left(\frac{a}{p}\right)^{\frac{1}{t}}$

Ans. 4 *per cent.* nearly.

6. In what time would 575*l.* raise a stock of 756*l.* 14*s.* at 4 *per cent.*?

In this case $r = 1.04$, $p = 575$, $a = 756.7$.

$a = 756.7$.... *log.* 2.878924
$p = 575$....... *log,* 2.759668

$r = 1.04$ *log*....0.017033) 0.119256 *log.* $\frac{a}{p}$

7 quotient nearly, the value of t, or number of years required.

7. In what time would a sum double itself at 5 *per cent.*?

Here if p is the principal, $2p$ is the amount, and the expression $t = \frac{\log. \frac{a}{p}}{\log. r}$ becomes $t = \frac{\log. \frac{2p}{p}}{\log. r}$, or $t = \frac{\log. 2}{\log. r} = \frac{0.301030}{0.021189} = 14.2$ *years,* the time nearly.

And the time in which a sum would triple itself, is found by dividing the *log.* of 3 by the *log.* of the *rate,* &c.

Of ANNUITIES.

193. An Annuity, strictly speaking, is a yearly allowance or payment; the term however, is usually applied to any periodical income.

When the annuity is payable immediately, it is said to be in *possession*; but should its commencement depend upon a future event, or not become due till after a certain number of years have elapsed, it is then called an annuity in *reversion*.

If the annuity is not limited in respect of time but supposed to continue for ever, it is called a *perpetuity*.

All the computations relating to annuities are generally made according to compound interest.

Let $r =$ the *rate* or the amount of 1 pound in 1 year, as in compound interest.
$p =$ any annuity, pension, or yearly rent.
$t =$ the time.
$a =$ the amount of the annuity when it is forborn.
$v =$ its value or present worth.

To find the amount (a) in the time t:

The amount of the sum p in t years is pr^t (194)
in $t-1$ years .. pr^{t-1}
in $t-2$ years .. pr^{t-2}
in $t-3$ years .. pr^{t-3}
&c. &c.

Therefore the whole amount in t years will be

$$pr^t + pr^{t-1} + pr^{t-2} + pr^{t-3} + \dots pr^{t-t},$$ or

which is the same thing, $p + pr + pr^2 + pr^3 + \dots pr^t$, because $pr^{t-t} = p$, that is, supposing the amount includes the last payment, which bears no interest.

Now (153) $p + pr + pr^2 + \ldots . pr^t = p \times \frac{r^t - 1}{r - 1} = a$, from which theorem, the following expressions for the several cases of annuities in *arrear* are readily obtained:

$$a = p \times \frac{r^t - 1}{r - 1}.$$

$$p = \frac{a(r-1)}{r^t - 1}.$$

$$pr^t - ar = p - a.$$

$$t = \frac{\log.\left(\frac{a(r-1)}{p} + 1\right)}{\log. r}.$$

196. The present worth or value of an annuity (p) supposed to continue t years, is found in the following manner:

Since 1 pound is the present worth of the sum r due at the end of 1 year, we shall have,

$r : 1 :: p : \frac{p}{r}$ the present worth of p pounds due at the end of 1 year; therefore if the sum p becomes due at the end of 2 years, its value at the end of 1 year will also be $\frac{p}{r}$;

whence $r : 1 : \frac{p}{r} : \frac{p}{r^2}$ is the present value of $\frac{p}{r}$ due at the end of 1 year, or the present worth of p due at the end of 2 years:

In like manner we have $\frac{p}{r^3}$ for the present worth of p pounds due at the end of 3 years; hence the present worth of p due at the end of t years will be $\frac{p}{r^t}$: consequently $\frac{p}{r} + \frac{p}{r^2} + \frac{p}{r^3} + \ldots \frac{p}{r^t}$ (continued to t terms) the sum of all the present worths of the yearly payments, will be the present value of the annuity.

Now this series is a geometrical progression having $\frac{p}{r}$ for the first term, $\frac{1}{r}$ the ratio, and t the number of terms; and its sum

$$(153) \text{ is } = \frac{\frac{1}{r} \times \frac{p}{r^t} - \frac{p}{r}}{\frac{1}{r} - 1} = \frac{p}{r^t} \times \frac{r^t - 1}{r - 1} = v.$$

In the case of a perpetuity, where t or the number of years are supposed to be continued for ever, the last term $\frac{p}{r^t}$ becomes $= 0$,

and consequently $\frac{1}{r} \times \frac{p}{r^t} = 0$, and the expression is $\frac{-\frac{p}{r}}{\frac{1}{r} - 1}$ or

$\frac{p}{r - 1} = v$ the present worth.

From the theorem $v = \frac{p}{r^t} \times \frac{r^t - 1}{r - 1}$, we get the other three expressions which follow:

$$p = v \times \frac{r^{t+1} - r^t}{r^t - 1}.$$

$$t = \frac{\log. \frac{p}{p + v - vr}}{\log. r}.$$

$vr^{t+1} - (p + v)r^t + p = 0$: these four theorems relate to the valuation of annuities.

Examples.

1. If an annuity of $50l.$ be forborn 7 years, what will it amount to at 4 *per cent. per ann.* compound interest?

Here $p = 50$, $r = 1.04$, and $t = 7$; and the expression $p \times \frac{r^t - 1}{r - 1}$ becomes $50 \times \frac{1.04^7 - 1}{.04} = 394.95l. = a$ the amount sought

2. In how long time will 50*l.* annuity raise a stock of 395*l.* at 4 *per cent. per ann.* compound interest?

In this case $p=50$, $a=395$, and $r=1.04$,

and $t=\dfrac{log.\left(\dfrac{a(r-1)}{p}+1\right)}{log.\ r}=\dfrac{log.\ \dfrac{395\times.04}{50}+1}{log.\ 1.04}=\dfrac{log.\ 1.316}{log.\ 1.04}=$

$\dfrac{0.119256}{0.017033}=7$ *years*, the required time.

3. If 80*l.* annuity forborn 9 years amounts to 893*l.* what is the rate of interest?

Here $p=80$, $a=893$, and $t=9$; these substituted in the equation $pr^t-ar=p-a$, give $80r^9-893r=80-893$, or $r^9-11.1625r=-10.1625$:

To approximate the root r by the method of trial-and-error (188) let 1.05 and 1.06 be the first assumptions, because upon trial, its value appears to lie between those numbers:

Then $1.05^9-11.1625\times1.05=-10.1693$

-10.1625

error $.0068$

$1.06^9-11.1625\times1.06=-10.1427$

-10.1625

error $.0198$

$.0068$

sum $.0266$ errors unlike

$.0198\times1.05=.020790$

$.0068\times1.06=.007208$

sum $.027998$

$\dfrac{.027998}{.0266}=1.052$ first approximation.

Next, assuming 1.052, and 1.054; and the 2*d.* approximation will be 1.053 which is very nearly the true value of r: hence $1.053\times100=105.3$, and 5.3*l.* or 5*l.* 6*s.* *per cent.* is the rate required.

4. What is the value of a freehold estate which rents at 50*l.* *per ann.* allowing 5 *per cent.* compound interest?

If the yearly rent is considered as a perpetuity, then $p = 50$, and $r = 1.05$; and the expression $\frac{p}{r-1}$ becomes $\frac{50}{1.05-1} = 1000l.$ which is 20 years purchase.

5. What is the present worth of 100*l.* annuity to continue 10 years, allowing 5 *per cent. per annum* compound interest, supposing the payments are made quarterly, (viz. 25*l.* every quarter)?

Here $p = 25$, $t = 40$ (the quarters in 10 years), and $r = 1.0125$ the amount of 1 pound in a quarter of a year:

whence $v = \frac{p}{r^t} \times \frac{r^t - 1}{r-1} = \frac{25}{1.64362} \times \frac{1.64362 - 1}{1.0125 - 1} = 783.17l.$ nearly, the value sought.

6. What annuity or yearly income, to continue 20 years, may be purchased for 1000*l.* at $3\frac{1}{2}$ *per cent.*?

In this case $v = 1000$, $r = 1.035$, $t = 20$, whence, by substitution,
$p = v \times \frac{r^{t+1} - r^t}{r^t - 1} = 1000 \times \frac{1.035^{21} - 1.035^{20}}{1.035^{20} - 1} = \frac{69.64}{.98979} = 70.36l.$ nearly, the annuity required.

197. To calculate the present value of an annuity in *reversion*, let t denote the *whole* time till it expires (as before), and n the time *before* its commencement:

Then $\frac{p}{r^t} \times \frac{r^t - 1}{r-1} - \frac{p}{r^n} \times \frac{r^n - 1}{r-1}$, or (by reduction), $\left(\frac{1}{r^n} - \frac{1}{r^t}\right) \times \frac{p}{r-1}$ will evidently be the expression for its present worth. And from this theorem others may be derived for solving the different cases.

On the PROPERTIES of NUMBERS.

198. The sum of any number of even numbers is an even number.

199. Therefore an even number taken any number of times will make an even number. And consequently the continued product of any number of even numbers will also be even.

200. An even number of odd or of even numbers will be even.

201. The difference of two even, or of two odd numbers will be even.

202 The difference of an even and an odd number will be odd.

203. An odd number taken an odd number of times will make an odd number.

The last six articles may be considered as axioms rather than propositions requiring formal demonstration.

204. If an odd number measures an odd number, the quotient will be odd. This is evident from *art.* 203, because the product of the quotient and divisor is equal to the dividend.

205. If an odd, or an even number measures an even one, the quotient will be even. This follows from *art.* 200.

206. An even number cannot measure an odd number——In other words, if an odd number be divided by an even one, the quotient will always contain a fraction. For an even number taken any *whole* number of times whatever, cannot make an odd number.

207. If one number measures another, it will also measure any multiple of it.

Let d be the measure or divisor, and q the quotient; then dq will be the dividend, and mdq a multiple of it; and $\frac{mdq}{d} = mq$, that is, d measures the number dq, and also its multiple mdq.

208. If a number measures two other numbers, it will also measure their sum, and difference.

Let the measure be d, and a and b the other two numbers; then $\frac{a}{d}$ and $\frac{b}{d}$ are whole numbers (by hypothesis); therefore their sum $\frac{a+b}{d}$, and also their difference $\frac{a-b}{d}$, must be whole numbers.

Corol. 1. Hence, if a number d measures another number $a+b$, and also a part of it b, it will also measure the remaining part a. (Arith. *art.* 46).

Corol. 2. When $a=b$, then the number $a+b$ is an even number (200); therefore, if a number (d) measures an even number ($a+b$), it will also measure its half (a or b).

209. Every number having 0 or 5 in the units place, is divisible by 5.

210. All prime numbers (*i. e.* those which can only be measured by 1) are odd, except the number 2. And such numbers have 1, 3, 7, or 9 in the units place, 2 and 5 excepted. All other numbers are composite or the products of two or more numbers.

211. The least factors of every composite number are its prime divisors, Thus 1, 2, and 3 are the prime divisors of 6, or 12, or 18, &c.

212. The least common multiple of two or more numbers is the continued product of the highest powers of their unlike prime factors.

Let a^2bc, bcd, cd^2f be three numbers, a, b, c, d, f, being the prime factors, all unlike; then a^2d^2bcf is their least common multiple:

For if a^2d^2bcf be divided by either of its factors, the quotient is not divisible by all the three numbers: and whatever number is divisible by those numbers, it must contain a^2, d^2, and the factors bcf, because a, d, b, c, and f are primes, for which reason no number can contain a^2, d^2 and bcf except a^2d^2bcf or some multiple of it, therefore a^2d^2bcf is the least. From this expression, the rule in Arith. *art.* 46, is immediately obtained.

To give an example in numbers, let the least common multiple of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9, be required:

The numbers when resolved into their prime factors will be

$$1,\ 2,\ 3,\ 2^2,\ 5,\ 2\times 3,\ 7,\ 2^3,\ 3^2,$$

and the continued product of the highest powers of the unlike factors is $1\times 5\times 7\times 2^3\times 3^2=2520$ the multiple required.

The preceding rule is simple. But the great difficulty consists in resolving large numbers into their component factors: nor has any direct method been discovered for that purpose. When a number is composite, one of the factors must be its square root or a less number, and therefore if the number is odd (to which it should be reduced) the odd numbers less than its square root, are the most convenient divisors for resolving it into its factors.

213. The expression $\dfrac{1\times 2\times 3\times\ldots\ldots(n-1)+1}{n}$ will give an integer, or a fractional quotient, according as n is a prime, or a composite number: thus if $n=7$,

$$\text{then } \frac{1\times 2\times 3\times 4\times 5\times(7-1)+1}{7}=\frac{721}{7}=103.$$

$$\text{If } n=8, \text{ we have } \frac{1\times 2\times 3\times 4\times 5\times 6\times(8-1)+1}{8}=630\tfrac{1}{8}.$$

But if n be a large number, the operation of obtaining the continued product of all the inferior numbers will be too laborious to make the theorem useful in determining whether the number n be prime or composite. See Waring *Meditat. Algeb.* p. 218. and Legendre *Theorie des Nomb.* p. 183.

214. If n be put to denote any of the numbers 1, 2, 3, 4, &c. then $6n+1$, and $6n-1$ will give a series containing all the prime numbers greater than 3. But it must be remarked, that neither $6n+1$ nor $6n-1$ are *always* prime numbers: Thus if $n=8$, or 9, then $6n+1=49$, and 55, both composite: or if $n=6$, we have $6n-1=35$ a composite number. According to Fermat, the expression 2^x+1 should always be a prime number if any term of the series 1, 2, 4, 8, 16, 32, 64, &c. be substituted for x: Euler however, has found the theorem defective when $x=32$, for $2^{32}+1=641\times 6700417$.

215. If the sum of the digits of a number is divisible by 9, the number itself is also divisible by 9.

Let a, b, c, d, be the digits of a number consisting of 4 figures; then $1000a+100b+10c+d$ will express the number:

$$\begin{array}{rl} 9) & 1000a+100b+10c+d\ (111a+11b+c \\ & \underline{999a+\ \ 99b+\ \ 9c} \\ \text{remainder} & \underline{\ \ \ \ a+\ \ \ \ \ b+\ \ \ c+d} \end{array}$$

Hence it is evident, when the remainder or sum of the digits is divisible by 9, the number itself must be so too, whatever be the number of its figures.

On this property is founded the proof of multiplication by casting away the nines: Arith. *art.* 21.

216. The difference between a number consisting of two digits, and the number formed by the digits when in an inverted order, is always 9 times the difference of the two digits. *Art.* 128. *ex.* 11.

217. The sum of the odd numbers $1+3+5+7+\ldots n$ is $=n^2$. (139)

Hence the differences of the squares 1^2, 2^2, 3^2, 4^2, &c. will be 3, 5, 7, &c.

218. The sum of any number of the series of cubes $1^3 + 2^3 + 3^3 + 4^3$, &c. taken from the beginning, is a square number. (179, *ex.* 3.)

219. The sum of two numbers differing by 1, is equal to the difference of their squares.

Let n and $n+1$ be the numbers: then $2n+1 =$ their sum: and $(n+1)^2 - n^2 = 2n+1$.

220. The powers of prime numbers are prime to all numbers except their roots or powers of the roots. This is evident from *art.* 212.

221. If a and b be whole numbers, then $\frac{a^n + b^n}{a+b}$ and $\frac{a^n + b^n}{a-b}$ are both integers when n is an odd number: and $\frac{a^n - b^n}{a+b}$ and $\frac{a^n - b^n}{a-b}$ both integers if n is an even one. (54).

222. If twice a number is the sum of two squares, the number itself is the sum of two squares.

For suppose n to be the number, and let $2n = a^2 + c^2$; then $4n = 2a^2 + 2c^2$, and $n = \frac{2a^2 + 2c^2}{4} = \frac{a^2 + 2ac + c^2}{4} + \frac{a^2 - 2ac + c^2}{4}$.

223. The product of the sum of two squares by the sum of two squares, is also the sum of two squares.

For $(a^2 + b^2) \times (c^2 + d^2) = (db + ac)^2 + (ad - bc)^2$.

224. The product of the sum of four squares by the sum of four squares, is the sum of four squares. This theorem has been demonstrated by Euler and Lagrange.

*224. Neither the sum nor difference of two cube numbers is a cube.

225. If n be any prime number, and N any number not divisible by n, then $N^{n-1} - 1$ is divisible by n.

226. If $4n + 1$ be a prime number, it is the sum of two squares. And when $8n + 1$ is a prime number, it is the sum of two, and also of three squares.

227. Every prime number is the sum of four squares.

228. Every number is the sum of four, or of a less number of squares.

Euler, Lagrange, and others have investigated these latter properties; the demonstrations however, are too long to be admitted in this place.

229. A *perfect number* is equal to the sum of all its aliquot parts.

Thus 6 is a perfect number, its aliquot parts being 1, 2, and 3, whose sum $1 + 2 + 3 = 6$. And 28 is also a perfect number, for $28 = 1 + 2 + 4 + 7 + 14$ the aliquot parts of 28. In the last proposition of Euclid's 9*th*. book it is proved, that when the sum of the geometrical series $1 + 2 + 4 + 8 + 16 +$ &c. is a prime number, the said sum multiplied by the last term of the series will be a perfect number. If therefore, n is put to denote the number of terms, $2^n - 1$ will be the sum, and 2^{n-1} the last term; consequently $(2^n - 1)\ 2^{n-1}$ is a perfect number when $2^n - 1$ is prime. Thus, if $n = 5$, then $(2^5 - 1)\ 2^{5-1} = 31 \times 16 = 496$ the third perfect number.

230. *Amicable numbers*, are pairs of numbers having this property, that each is equal to the sum of all the aliquot parts of the other:

Thus 220 and 284 are amicable numbers; for the sum of 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 which are the aliquot parts of 220, is $= 284$; and 1, 2, 4, 71, 142, the aliquot parts of 284, when added together, make 220: those two numbers are the least of the kind.

The two next amicable numbers are 6232 and 6368 according to Euler, who has treated the subject at very considerable length, and given a table containing 61 pair of these numbers, in a miscellaneous tract published in 1750. In this we are informed that Stifelius was the first who took notice of such numbers.

Many curious investigations relative to the properties of numbers are to be found in Legendre *Essai sur la Théorie des Nombres.*

231. A Collection of Exercises *in the several* Rules *of* Algebra, *beginning with Multiplication.*

1. $(x-a) \times (x+a) = x^2 - a^2$.
2. $(5x-7) \times (7x-5) = 35x^2 - 74x + 35$.
3. $(-a-b-1) . (a-1) = b - a^2 - ab + 1$.
4. $(3a^2-2b) . (4a+3b) = 12a^3 + 9a^2b - 8ab - 6b^2$.
5. $(a^2+2ab+b^2) . (a+b) = a^3 + 3a^2b + 3ab^2 + b^3$.
6. $(x^2+2xy+y^2) . (x-y) = x^3 + x^2y - xy^2 - y^3$.
7. $(x^3+x^2y+xy^2+y^3) . (x-y) = x^4 - y^4$.
8. $(x^4-x^3y+x^2y^2-xy^3+y^4) . (x+y) = x^5 + y^5$.
9. $(\frac{1}{3}x^2y - \frac{2}{5}) . (\frac{1}{2}x + \frac{1}{4}) = \frac{1}{6}x^3y + \frac{1}{12}x^2y - \frac{1}{5}x - \frac{1}{10}$.
10. $(\frac{2}{3}a^3 - \frac{3}{5}a^2b + \frac{1}{2}b^3) . (\frac{3}{4}ab - 2b^2) = \frac{1}{2}a^4b - \frac{116}{75}a^3b^2 - \frac{3}{5}a^2b^3 + \frac{3}{10}ab^4$.
11. $(5a^2b - 2ab^2 + 4a^2c^2) . (2a^3b - ab^3 + 3a^2c^2) = 10a^5b^2 - 9a^4b^4 + 23a^4bc^2$
$+ 2a^2b^5 - 10a^3b^3c^2 + 12a^4c^4$.

Division.

Quotients.

1. $x+c)\ x^2 - c^2\ (x - c$.
2. $x^2-16)\ x^6 - 8x^4 - 124x^2 - 64\ (x^4 + 8x^2 + 4$.
3. $4a^2+5ab+b^2)\ 8a^4 - 2a^3b - 13a^2b^2 - 3ab^3\ (2a^2 - 3ab$.
4. $a^2+b^2+c^2)\ a^4 + 2a^2b^2 + b^4 - c^4\ (a^2 + b^2 - c^2$.
5. $\frac{2}{3}a^2 - \frac{2}{5}b^2)\ \frac{2}{9}a^3 - \frac{2}{15}ab^2 + \frac{1}{6}a^2b - \frac{1}{10}b^3\ (\frac{1}{3}a + \frac{1}{4}b$.
6. $a-x)\ a^3 - x^3 - 1\ (a^2 + ax + x^2 + \frac{-1}{a-x}$.
7. $b-y)\ b^4 - 3y^4\ (b^3 + b^2y + by^2 + y^3 - \frac{2y^4}{b-y}$.
8. $b^2 - 5a^2 - 3ab)\ 13a^3b + 19a^2b^2 - 20a^4 - 5ab^3\ (4a^2 - 5ab$.

9. $\dfrac{20a^5-41a^4b+a^3b^2-45a^2b^3+25ab^4-6b^5}{5a^3-4a^2b+5ab^2-3b^3}=4a^2-5ab+2b^2.$

10. $\dfrac{4x^8+32x^7+96x^6+144x^5+132x^4+80x^3+32x^2+8x+1}{2x^4+8x^3+8x^2+4x+1}=2x^4+8x^3$
$+8x^2+4x+1.$

11. $\dfrac{x^6-y^6}{x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5}=x+y.$

The divisors are under the dividends in the three last examples.

Fractions reduced to their lowest terms.

1. $\dfrac{x^2-4xy+4y^2}{x^3-6x^2y+12xy^2-8y^3}$ .... $=\dfrac{1}{x-2y}.$

2. $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$ .... $=\dfrac{x^2-bx}{x+b},$

3. $\dfrac{a^2-ab-2b^2}{a^2-3ab+2b^2}$ .... $=\dfrac{-a-b}{b-a}.$

4. $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ .... $=\dfrac{a^2+x^2}{a-x}.$

5. $\dfrac{10a^5+20a^4b+10a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$ .... $=\dfrac{10a^4+10a^3b}{a^2b+ab^2+b^3}.$

6. $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$ .... $=\dfrac{3a^2+b^2}{4a-b}.$

7. $\dfrac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$ .... $=\dfrac{7a-2b}{5a^2-3ab+2b^2}.$

Improper fractions reduced to whole or mixt quantities.

1. $\dfrac{12xy-6x^2y^2-x^3y}{8xy}$ .... $=1\frac{1}{2}-\frac{3}{4}xy-\frac{1}{8}x^2.$

2. $\dfrac{x^2-2x+2}{x-1}$ .... $=x-1+\dfrac{1}{x-1}.$

3. $\dfrac{1-x^3}{1-x}$ .... $=1+x+x^2$

4. $\dfrac{1-5x+10x^2-10x^3+5x^4-x^5}{1-2x+x^2}$ .... $=1-3x+3x^2-x^3.$

5. $\dfrac{\frac{1}{2}x^2-\frac{7}{12}xy+\frac{1}{6}y^2}{\frac{2}{3}x-\frac{1}{2}y}$ .... $=\frac{3}{4}x-\frac{1}{3}y.$

6. $\dfrac{a^2-b^2+a-2b}{a-b}$ .... $=a+b+1-\dfrac{b}{a-b}$

Mixt quantities brought to improper fractions.

1. $4+x+\dfrac{1}{x-4}$ .... $=\dfrac{x^2-15}{x-4}$.

2. $a-b+\dfrac{b}{a+1}$ .... $=\dfrac{a^2-ab+a}{a+1}$.

3. $3x-7+\dfrac{3}{3x+7}$ .... $=\dfrac{9x-46}{3x+7}$.

4. $\frac{1}{3}-\frac{1}{3}x+\dfrac{x-y}{y-x}$ .... $=\dfrac{\frac{1}{3}x^2-\frac{1}{3}xy+\frac{2}{3}x-\frac{2}{3}y}{y-x}$.

5. $x^3+x^2y+xy^2+y^3+\dfrac{y^4}{x-y}$ .... $=\dfrac{x^4}{x-y}$.

6. $1+x+x^2+x^3+x^4+\dfrac{x^5}{1-x}$ .... $=\dfrac{1}{1-x}$.

7. $1+x-x^3-x^4+x^6+\dfrac{x^7-x^8}{1-x+x^2}$ .... $=\dfrac{1}{1-x+x^2}$.

Fractions reduced to common denominators.

1. $\dfrac{x}{2}$, $\dfrac{x}{3}$, and $\dfrac{x}{4}$ .... $=\dfrac{6x}{12}$, $\dfrac{4x}{12}$, and $\dfrac{3x}{12}$.

2. $\dfrac{z}{3}$, $\dfrac{z}{4}$, $\dfrac{z}{5}$, and $\dfrac{z}{6}$ .... $=\dfrac{20z}{60}$, $\dfrac{15z}{60}$, $\dfrac{12z}{60}$, and $\dfrac{10z}{60}$.

3. $\dfrac{n}{a}$, $\dfrac{n}{b}$, and $\dfrac{n}{c}$ .... $=\dfrac{bcn}{abc}$, $\dfrac{acn}{abc}$, and $\dfrac{abn}{abc}$.

4. $\dfrac{a}{bc}$, $\dfrac{b}{ac}$, and $\dfrac{c}{ab}$ .... $=\dfrac{a^2}{abc}$, $\dfrac{b^2}{abc}$, and $\dfrac{c^2}{abc}$.

5. ab, $\dfrac{bc}{a}$, and $\dfrac{cd}{ab}$, ... $=\dfrac{a^2b^2}{ab}$, $\dfrac{b^2c}{ab}$, and $\dfrac{cd}{ab}$.

6. $\dfrac{2x^2-18}{x+3}$, $\dfrac{1}{x-3}$, and $\dfrac{1}{x}$ .... $=\dfrac{2x(x-3)^2}{x^2-3x}$, $\dfrac{x}{x^2-3x}$, and $\dfrac{x-3}{x^2-3x}$.

Addition of fractions.

1. $\dfrac{n}{2}+\dfrac{2n}{3}+\dfrac{3n}{4}+\dfrac{4n}{5}$ .... $=\dfrac{163n}{60}$.

2. $\frac{1}{2}x+\frac{2}{3}x+\frac{5}{6}x+\frac{1}{3}x$ .... $=2\frac{1}{3}x$.

3. $\dfrac{2x}{3}+\dfrac{7x}{4}+\dfrac{2x+1}{3}$ .... $=3\frac{1}{12}x+\frac{1}{3}$.

4. $\dfrac{a^2+ax}{a-x}+a$ .... $=\dfrac{2a^2}{a-x}$.

5. $\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}+1$ .... $=\dfrac{3a^2+b^2}{a^2-b^2}$.

6. $a-b+c+\dfrac{b-a-c}{1+a}$ .... $=\dfrac{a^2-ab+ac}{1+a}$.

7. $7\frac{1}{2}x+\dfrac{x-5}{6}+4\frac{1}{2}x+\dfrac{2x+7}{4}+\dfrac{3x}{2}$ .... $=14\frac{1}{6}x+\dfrac{11}{12}$.

8. $\dfrac{-a}{x}+\dfrac{-x-b}{5}+\dfrac{\frac{1}{2}x-1+a}{3}$ .... $=\dfrac{(5a-3b-5)x-\frac{1}{2}x^2-10a}{15x}$.

9. $\dfrac{n+x}{n-x}+\dfrac{x-a}{c-x}$ .... $=\dfrac{(n+x)\,(c-x)+(n-x)\,(x-a)}{(n-x)\,(c-x)}$.

10. $1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\dfrac{x^3}{a^3}+\dfrac{x^4}{a^4-a^3x}$ .... $=\dfrac{a}{a-x}$.

11. $\dfrac{1}{a}-\dfrac{x}{a^2}+\dfrac{x^2}{a^3}-\dfrac{x^3}{a^4}+\dfrac{x^4}{a^5+a^4x}$ .... $=\dfrac{1}{a+x}$.

Subtraction of fractions.

1. $\dfrac{x-7}{8}-\dfrac{7-x}{6}$ .... $=\dfrac{7x-49}{24}$.

2. $\dfrac{x-7}{-8}-\dfrac{7-x}{-6}$ .... $=\dfrac{49-7x}{24}$.

3. $\dfrac{x}{m}-\dfrac{x}{n}$ .... $=\dfrac{(n-m)x}{mn}$.

4. $\dfrac{9x+1}{2}-\left(x-\dfrac{x-1}{5}\right)$ .... $=\dfrac{37x+3}{10}$.

5. $\dfrac{a+c}{a-c}-\dfrac{a-c}{a+c}$ .... $=\dfrac{4ac}{a^2-c^2}$.

6. $\dfrac{a+b}{x}-\dfrac{a+b}{z}$ .... $=\dfrac{(a+b)\,(z-x)}{xz}$.

7. $(a+c)\,(a-c)-\dfrac{-2a^2c^2}{(a+c)(a-c)}$ .... $=a^2+c^2$.

Multiplication of fractions.

1. $\dfrac{a}{b}\times\dfrac{b}{c}\times\dfrac{c}{d}\times\dfrac{d}{f}$ .... $=\dfrac{a}{f}$.

2. $\dfrac{a^2-c^2}{n}\times\dfrac{n^2}{a+c}$ .... $=(a-c)n$.

3. $\frac{x+y}{a+b}\times\frac{x-y}{a+b}$ .... $=\frac{x^2-y^2}{(a+b)^2}$.

4. $\frac{\frac{1}{2}x-6z}{y}\times\frac{2xy}{3}$ .... $=\frac{1}{3}x^2-4xz$.

5. $(3a-6x)\times\frac{3x-1}{a^2-4ax+4x^2}$ .... $=\frac{9x-3}{a-2x}$

6. $\frac{5a-x}{4x}\times\frac{-x^2}{1-x}$ .... $=\frac{x^3-5ax^2}{4x-4x^2}$.

7. $\frac{a}{x^3}\times\frac{x^3}{b}\times\frac{x^4}{c}$ .... $=\frac{ax^3}{bc}$.

8. $\frac{x^n}{a}\times\frac{x^r}{b}\times\frac{x^s}{c}$ .... $=\frac{x^{n+r+s}}{abc}$.

9. $\frac{2x^n}{a(c-b)^2}\times\frac{ax^{-2m}}{d(c-b)^n}$ .... $=\frac{2ax^{n-2m}}{ad(c-b)^{n+2}}$.

Division of fractions.

	Divisors.	Dividends.	Quotients.
1.	$\frac{a}{b}$	$\frac{c}{d}$	$=\frac{bc}{ad}$.
2.	m	$\frac{x+c}{a+b}$	$=\frac{x+c}{m(a+b)}$.
3.	$\frac{x}{a+b}$	m	$=\frac{m(a+b)}{x}$.
4.	$\frac{a+b}{c}$	$\frac{a^2-b^2}{c^2}$	$=\frac{a-b}{c}$.
5.	$\frac{5x}{6}$	$\frac{x+1}{6}$	$=\frac{x+1}{5x}$.
6.	$\frac{a^2-b^2}{d}$	$\frac{a^3+b^3}{c}$	$=\frac{a^2d-abd+b^2d}{ac-bc}$.
7.	$\frac{a+c}{x-1}\times(n-1)$	$\frac{a-c}{b}$	$=\frac{(a-c)(x-1)}{b(a+c)(n-1)}$.
8.	$\frac{2x-2y}{a+b}$	$\frac{8x^3-8y^3}{(a+b)^2}$	$=\frac{4x^2+4xy+4y^2}{a+b}$.
9.	$\frac{(a+b)^{-n}}{(c-d)^m}$	$\frac{(a+b)^m}{(c-d)^{-n}}$	$=\frac{(a+b)^{m+n}}{(c-d)^{-(m+n)}}$.
10.	$\frac{y^{-m}}{a^{-n}}$	$\frac{a^n}{y^m}$	$=\frac{a^0}{y^0}=\frac{1}{1}$.

Fractions resolved into Infinite Series.

1. $\frac{1+x}{1-x}$ .... $= 1+2x+2x^2+2x^3+$ &c.

2. $\frac{ax}{a-x}$ .... $= x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+$ &c.

3. $\frac{6}{10-1}$ .... $= \frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+$ &c.

4. $\frac{1}{1-a+a^2}$ .... $= 1+a-a^3-a^4+a^6+a^7-a^9$ &c.

5. $\frac{c^2}{(c+x)^2}$ .... $= 1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+$ &c.

6. $\frac{1+x}{(1-x)^2}$ .... $= 1+3x+5x^2+7x^3+$ &c.

7. $\frac{a^2+x^2}{a^2-x^2}$ .... $= 1+\frac{2x^2}{a^2}+\frac{2x^4}{a^4}+\frac{2x^6}{a^6}+$ &c.

SIMPLE EQUATIONS.

1. Given $\frac{2x}{3}+4=\frac{4x}{5}+12-\frac{5x}{7}$; required x? .. *Ans.* $x=13\frac{47}{61}$.

2. Given $z+\frac{1}{3}z-\frac{1}{7}z+\frac{2z}{5}=7z^2$; req. z? ...... *Ans.* $z=\frac{167}{735}$.

3. Given $\sqrt{x}+\sqrt{(10+x)}=\frac{20}{\sqrt{(10+x)}}$; req. x? *Ans.* $x=3\frac{1}{3}$.

4. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$; req. x? *Ans.* $x=a\sqrt{\frac{1}{3}}$.

5. Given $x^2+a^2=\frac{a^4}{x^2+a^2}$; req. x? .... *Ans.* $x=\sqrt{\left(-\frac{a^2}{2}\right)}$.

6. Given $\frac{ax}{b}+b=\frac{cx}{d}+\frac{ab}{c}$; req. x? .... *Ans.* $x=\frac{(a-c)b^2d}{acd-bc^2}$.

7. Given $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$; req. x? .. *Ans.* $x=\frac{4b^3+8ab^2-12a^2b}{3a^2+ab-ac+bc}$.

8. Given $\left.\begin{array}{l} ax^2+bx=c \\ dx^2-fx+n=0 \end{array}\right\}$ req. x? .... *Ans.* $x=\frac{dc+an}{af+db}$.

9. Given $\left.\begin{array}{l} 5x-3y=24 \\ 11y-7x=14 \end{array}\right\}$ req. x and y? ...... *Ans.* $x=9$, $y=7$.

10. Given $\left.\begin{array}{l} ax+by=n \\ cx+dy=m \end{array}\right\}$ req. x and y? ...... *Ans.* $x=\frac{bm-dn}{bc-ad}$.

$y=\frac{am-cn}{ad-bc}$.

11. Given $\left.\begin{array}{r} ax+by=d \\ cx-y=x \end{array}\right\}$ req. x and y? .... *Ans.* $x=\frac{d}{bc+a-b}$.

$y=\frac{cd-d}{bc+a-b}$.

12. Given $\left.\begin{array}{r} a:x+y::x-y:b \\ x^2+y^2=c \end{array}\right\}$ req. x and y? .. *Ans.* $x=\sqrt{\frac{c+ab}{2}}$.

$y=\sqrt{\frac{c-ab}{2}}$.

13. Given $\left.\begin{array}{r} x+y+\frac{y^2}{x}=20 \\ x^2+y^2+\frac{y^4}{x^2}=140 \end{array}\right\}$ req. y? ...... *Ans.* $y=6\frac{1}{2}$

14. Given $\left.\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z=42 \\ \frac{1}{4}x+\frac{1}{2}y-\frac{1}{4}z=36 \\ \frac{1}{3}z+\frac{3}{4}x-\frac{1}{9}y=47 \end{array}\right\}$ req. x, y, and z? .. *Ans.* $x=60$, $y=54$, $z=24$.

Involution.

1. What is the square of $ax+bx$? *Ans.* $(a^2+2ab+b^2)x^2$.

2. What is the square of $\frac{1}{2}x-\frac{z}{x}$? *Ans.* $\frac{1}{4}x^2-z+\frac{z^2}{x^2}$.

3. Required the cube of $1-\frac{1}{2}x^2$? *Ans.* $1-1\frac{1}{2}x^2+\frac{3}{4}x^4-\frac{1}{8}x^6$.

4. What is the 4th. power of $\frac{x}{y}+\frac{y}{x}$? *Ans.* $\frac{x^4}{y^4}+\frac{y^4}{x^4}+\frac{4x^2}{y^2}+\frac{4y^2}{x^2}+6$.

or $x^4y^{-4}+y^4x^{-4}+4x^2y^{-2}+4y^2x^{-2}+6$.

5. What is the square of x^m+y^{-n}? *Ans.* $x^{2m}+2y^{-n}x^m+y^{-2n}$.

or $x^{2m}+\frac{2x^m}{y^n}+\frac{1}{y^{2n}}$.

Evolution *or the Extraction of Roots.*

1. What is the square root of $x^4-4x^3+6x^2-4x+1$?

Ans. x^2-2x+1.

2. Required the square root of $4z^4+12z^3y+13z^2y^2+6zy^3+y^4$?

Ans. $2z^2+3zy+y^2$.

3. What is the square root of $\frac{1}{9}x^2-\frac{1}{2}xy-\frac{4}{9}xz+\frac{9}{16}y^2-yz+\frac{4}{9}z^2$?

Ans. $\frac{1}{3}x+\frac{3}{4}y-\frac{2}{3}z$.

4. What is the square root of a^2+x^2?

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$&c.

5. What is the square root of x^2+a^2?

Ans. $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-$&c.

6. Required the cube root of $-\frac{27}{125}a^6z^3$? *Ans.* $-\frac{3}{5}a^2z$.

7. What is the cube root of $x^6+6x^5-40x^3+96x-64$?
Ans. x^2+2x-4.

8. What is the cube root of $\frac{1}{8}x^3+\frac{1}{4}x^2y-\frac{3}{4}x^2+\frac{3}{2}x-xy+\frac{1}{6}xy^2+y-\frac{1}{3}y^2+\frac{1}{27}y^3-1$? *Ans.* $\frac{1}{2}x+\frac{1}{3}y-1$.

9. Required the cube root of a^3-b?
Ans. $a-\frac{b}{3a^2}-\frac{b^2}{9a^5}-\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}$ &c.

10. What is the 5*th.* root of a^2-x^2?
Ans. $a^{\frac{2}{5}}\left(1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}, \text{ \&c.}\right)$

11. What is the square root of $a+b$?
Ans. $a^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{b}{a}-\frac{1}{8}\frac{b^2}{a^2}+\frac{1}{16}\frac{b^3}{a^3} \text{ \&c.}\right)$

Surds.

1. Reduce $7\frac{1}{2}$ to the form of the square root? *Ans.* $\sqrt{56\frac{1}{4}}$.

2. Reduce $\frac{1}{3}yz$ to the form of the cube root? *Ans.* $\left(\frac{1}{27}y^3z^3\right)^{\frac{1}{3}}$.

3. Reduce $(a+b)(a-b)$ to the form of the square root?
Ans. $\left(a^4-2a^2b^2+b^4\right)^{\frac{1}{2}}$

4. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to equivalent quantities having a common index?
Ans. $256^{\frac{1}{12}}$ and $125^{\frac{1}{12}}$.

5. Reduce $a^{\frac{2}{3}}$ and $b^{\frac{1}{2}}$ to equivalent quantities having a common index?
Ans. $\left(a^4\right)^{\frac{1}{6}}$ and $\left(b^3\right)^{\frac{1}{6}}$.

6. Let $3^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ be reduced to equivalent quantities having the common index $\frac{1}{4}$? *Ans.* $\left(81^{\frac{1}{3}}\right)^{\frac{1}{4}}$ and $25^{\frac{1}{4}}$.

7. Reduce $a^{\frac{2}{3}}$ and $b^{\frac{1}{4}}$ to equivalent quantities having the common index $\frac{1}{2}$? *Ans.* $\left(a^{\frac{4}{3}}\right)^{\frac{1}{2}}$ and $\left(b^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Multiplication of Surds.

1. What is the continued product of $\sqrt{4}$, $\sqrt{5}$, and $\sqrt{7}$? *Ans.* $140^{\frac{1}{2}}$.

2. Required the product of $\sqrt{a^3}$ and $\sqrt{b^3}$? *Ans.* $ab\sqrt{ab}$.

3. What is the product $ax^{\frac{1}{2}} \times bx^{\frac{2}{3}} \times cx^{\frac{1}{3}}$? *Ans.* $abcx^{\frac{3}{2}}$.

4. Required the product $5 \times 4^{\frac{1}{3}} \times 18^{\frac{1}{3}}$ *Ans.* $10 \times 9^{\frac{1}{3}}$.

5. What is the product $\frac{2}{3}\sqrt{\frac{1}{2}} \times \frac{3}{4}\sqrt{\frac{7}{10}}$? *Ans.* $\frac{1}{20}\sqrt{35}$.

6. What is the product $(a^2b^3)^{\frac{1}{5}} \times (a^3b^2)^{\frac{1}{5}}$? *Ans.* ab.

7. Required the product $(\sqrt{a}+\sqrt{b})\,(\sqrt{a}-\sqrt{b})$? *Ans.* $a-b$.

8. Required the product $\sqrt{(2a^2x-8ax+8x)} \times \sqrt{2x}$?
Ans. $2ax-4x$.

9. What is the product $x^2 \times \frac{1}{x^{2\frac{1}{2}}}$? *Ans.* $\frac{1}{x^{\frac{1}{2}}}$ or $x^{-\frac{1}{2}}$.

10. Required the product $(4+2\sqrt{2}) \times (2-\sqrt{2})$? *Ans.* 4.

11. Required the product $x^{\frac{1}{3}} \times (x-b)^{\frac{2}{3}}$? *Ans.* $(x(x-b)^2)^{\frac{1}{3}}$.

Division of Surds.

1. Divide $a^{\frac{1}{m}}$ by $a^{\frac{1}{n}}$? *Quotient* $a^{\frac{n-m}{mn}}$.

2. Divide $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$? *Quot.* $x^{\frac{1}{6}}$.

3. Divide $a^{\frac{2}{3}}$ by $b^{\frac{3}{4}}$? *Quot.* $\left(\frac{a^8}{b^9}\right)^{\frac{1}{12}}$.

4. Divide $\frac{1}{2}\sqrt{\frac{1}{15}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$? *Quot.* $\frac{1}{4}\sqrt{3}$.

5. Divide $\frac{5}{7} \times \left(\frac{2}{3}\right)^{\frac{1}{2}}$ by $\frac{3}{5} \times \left(\frac{3}{4}\right)^{\frac{1}{3}}$? *Quot.* $\frac{25}{21} \times 3^{\frac{1}{2}}$.

6. Divide $x^2-dx-b+d\sqrt{b}$ by $x-\sqrt{b}$? *Quot.* $x+\sqrt{b}-d$.

7. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}-\sqrt{3}$? *Quot.* $8+2\sqrt{15}$.

8. Divide $(a+x)^{\frac{1}{m}}$ by $(a+x)^{-\frac{1}{n}}$? *Quot.* $(a+x)^{\frac{m+n}{mn}}$.

9. Divide $8-5\sqrt{2}$ by $3-2\sqrt{2}$? *Quot.* $4+\sqrt{2}$.

Surds reduced to their simplest terms.

1. Reduce $\sqrt{121b^2x}$ to its most simple terms? *Ans.* $11b\sqrt{x}$.

2. Reduce $875^{\frac{1}{3}}$ to its simplest terms? *Ans.* $5 \times 7^{\frac{1}{3}}$.

3. Reduce $\left(\frac{16}{81}\right)^{\frac{1}{3}}$ to its most simple terms? *Ans.* $\frac{2}{9} \times 18^{\frac{1}{3}}$.

4. Reduce $(x^4-a^2x^3)^{\frac{1}{3}}$ to its most simple terms? *Ans.* $x(x-a^2)^{\frac{1}{3}}$.

5. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to more simple terms? *Ans.* $\sqrt{5}+\sqrt{2}$.

6. Let $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ be reduced to its most simple terms?
Ans. $8+2\sqrt{15}$.

Addition and Subtraction of Surds.

1. What is the sum of $500^{\frac{1}{3}}$ and $108^{\frac{1}{3}}$? *Ans.* $8 \times 4^{\frac{1}{3}}$.
2. .... of $27^{\frac{1}{2}}$ and $147^{\frac{1}{2}}$? *Ans.* $10\sqrt{3}$.
3. .... of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$? *Ans.* $(2a+24x^2)\sqrt{b}$.
4. .... of $9\sqrt{243}$ and $10\sqrt{363}$? *Ans.* $191\sqrt{3}$.
5. .... of $\sqrt{27a^4x}$ and $\sqrt{3a^2x}$? *Ans.* $(3a^2+a)\sqrt{3x}$.
6. .... of $5^{\frac{1}{2}}$ and $5^{\frac{1}{3}}$? *Ans.* $(5^{\frac{1}{6}}+1)5^{\frac{1}{3}}$.
7. What is the difference of $\sqrt{448}$ and $\sqrt{112}$? *Ans.* $4\sqrt{7}$.
8. .... of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$? *Ans.* $(4a^2 \backsim 2ax)\sqrt{5x}$.
9. .... of $8(a^3b)^{\frac{1}{3}}$ and $(a^6b)^{\frac{1}{3}}$? *Ans.* $(8a \backsim a^2)\,b^{\frac{1}{3}}$.
10. .... of $\left(\frac{2}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{9}{32}\right)^{\frac{1}{3}}$? *Ans.* $\frac{1}{12} \times 18^{\frac{1}{3}}$.

Powers, and Roots of Surds.

1. What is the square of $ax^{\frac{m}{n}}$? *Ans.* $a^2x^{\frac{2m}{n}}$.
2. Required the cube of $4^{\frac{1}{3}}x^{\frac{2}{3}}$? *Ans.* $4x^2$.
3. What is the $\frac{n}{m}$th. power of $(a+b)^{\frac{r}{s}}$? *Ans.* $(a+b)^{\frac{nr}{ms}}$.
4. What is the square of $5-\sqrt{5}$? *Ans.* $30-10\sqrt{5}$.
5. What is the cube of $3x-2\sqrt{x}$? *Ans.* $27x^3-54x^2\sqrt{x}+36x^2-8x\sqrt{x}$.
6. Let $\dfrac{ax^2}{bx^{-3}}$ be raised to the nth. power? *Ans.* $\dfrac{a^n}{b^n}x^{5n}$.
7. Required the 4th. power of $\frac{1}{a}\sqrt{a}$? *Ans.* $\dfrac{1}{a^2}$.
8. What is the nth. root of a^m? *Ans.* $a^{\frac{m}{n}}$.
9. Required the $\frac{n}{m}$ root of $x^{\frac{r}{s}}$? *Ans.* $x^{\frac{mr}{ns}}$.
10. What is the square root of $9x-6a\sqrt{x}+a^2$? *Ans.* $3\sqrt{x}-a$.
11. What is the square root of $1\frac{4}{25}-2\frac{2}{5}\sqrt{\frac{1}{5}}$? *Ans.* $\frac{3}{5}-2\sqrt{\frac{1}{5}}$.
12. Required the square root of $12-\sqrt{140}$? *Ans.* $\sqrt{7}-\sqrt{5}$.
13. What is the $-n$th. root of $(x+y)^{-m}$? *Ans* $(x+y)^{\frac{m}{n}}$.

Questions producing SIMPLE EQUATIONS.

1. The difference of two numbers being $\frac{1}{4}$, and the difference of their squares 2, then what are the numbers?

$3\frac{7}{8}$ and $4\frac{1}{8}$. *Ans.* $3\frac{7}{8}$ and $4\frac{1}{8}$

2. The whole number of troops in two companies are 180, and the number in one troop to the number in the other as 8 to 7. What is the strength of each? *Ans.* 96, and 84 *men.*

3. What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 20?

Ans. $\frac{20}{21}$ and $\frac{1}{21}$.

4. A General having detached 400 men to take possession of a strong post, and $\frac{1}{4}$ of the remainder of his troops to watch the motions of the enemy, finds that he has only $\frac{1}{17}$ of his army left; what was his whole force? *Ans.* 850 *men.*

5. Three battalions of unequal force are in column of march; the extent of the first battalion is 210 *paces*, the extent of the second is equal to that of the first and $\frac{1}{4}$ of the third together, and the extent of the third is equal to that of the first and half the second; what is the length of the column?

Ans. 1302 *paces.*

6. A company of foot are 1165 of their own paces a head of a troop of horse; now if the foot take 5 paces to every 4 of the horse, but 3 paces of a horse are equal in extent to 4 paces of the foot; how many paces will the horse have marched before they overtake the foot? *Ans.* 13980.

7. If a person buys a certain number of eggs at 2 for a penny, and the like number at 3 a penny, and by selling the whole together at 5 for 2 pence, loses 1 penny; what was the number bought? *Ans.* 30.

8. If the agents A and B acting separately, produce a like effect *a* in the times *b* and *c*, respectively, and A, B, and C

together produce the same effect (a) in the time d; in what time would C alone produce the effect m?

$$\textit{Ans.}\ \frac{bcdm}{abc - adc - adb}.$$

9. A labourer agreed to serve 10 weeks upon these conditions, that for every day he worked he was to receive 2*s*. 4*d*, but to forfeit 7*d*. for every day he absented himself; now at the end of the time he had to receive 4*l*. 19*s*. 2*d*. What number of days did he work? *Ans.* 46.

10. The weight of a cubic foot of *copper* is 9000, of *tin* 7320, of *gun metal* 8784 *ounces*.

Those numbers also denote the *specific gravities* of the metals: hence the quantity of copper and of tin in the mixture which is gun metal, is required?

Ans. $7842\frac{6}{7}$ ounces of *copper*,

$941\frac{1}{7}$ ounces of *tin*; or $8\frac{1}{3}$*lb*. of *copper* to 1 of *tin*, nearly.

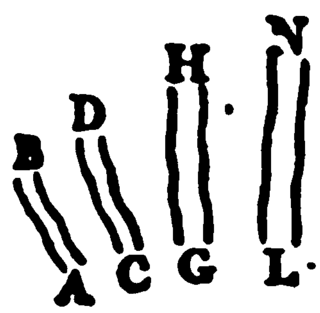
11. Suppose the weight of a brass 12 pounder is 18 *hundred weight*, and that of another brass 12 pounder exactly of the same dimensions is 16 *hundred weight*; now if the former is gun metal whose specific gravity is 8784, it is required to find the weight of *copper* and also of *tin* in the latter piece?

Oz.

Ans. 9600 *copper*

19072 *tin*.

12. Suppose 25 battalions of troops have to march through 4 roads or defiles,



viz.

AB = 1 mile, in this the length of a battalion on the march is 243 paces of $2\frac{1}{2}$ *feet* each, and the rate of marching 65 paces per minute.

CD = $1\frac{1}{4}$ miles, in this a battalion extends 210 paces, and the rate of marching is 50 paces per minute.

GH = $1\frac{1}{2}$ miles, here a battalion is 204 paces in length, and the rate of marching 60 paces per minute.

LN = 2 miles, in this the extent of a battalion is 232 paces, and the rate of marching 80 paces per minute.

Now how should the 23 battalions be divided that the whole march through the 4 roads may be made in the least time, supposing the 4 divisions begin the march together at A, C, G, and L, respectively? *Ans.* 10 battal. through AB.

4 through CD.

3 through GH.

6 through LN.

13. To divide a given number n into two such parts that the difference of their squares shall be equal to a given number d?

Ans. the two parts are $\frac{n^2+d}{2n}$, and $\frac{n^2-d}{2n}$.

14. A body of 1905 troops consists of three battalions; now $\frac{1}{2}$ the first battalion is to $\frac{1}{3}$ of the second, as 7 to 5; and $\frac{2}{3}$ of the second battalion is to $\frac{5}{6}$ of the third as 9 to 10. Required the strength of each battalion? *Ans.* 630, 675, 600 *men.*

15. A waterman finds that he can row 5 miles *with* the tide in $\frac{3}{4}$ of an hour, and that it takes him $1\frac{1}{4}$ hours to row the same distance back *against* the tide when it is but $\frac{1}{5}$ as strong; hence the velocity of the strongest tide is required?

Ans. $2\frac{2}{9}$ *miles per hour.*

16. A Garrison had provisions sufficient for 20 months, but at the end of 4 months the number of troops were doubled, and 3 months after that it was reinforced with 400 men more, by which means the provisions lasted but 13 months in the whole. Required the strength of the garrison before any augmentation took place? *Ans.* 800 *men.*

17. The weight of a cubic foot of rain water is 1000 *ounces avoirdupois*, and that of a cubic foot of sea water 1031 *ounces*; now how much of each must be taken that a cubic foot of the mixture shall weigh 1008 *ounces*? Oz.

Ans. $741\frac{29}{31}$ rain water.

$266\frac{2}{31}$ sea water.

together f ... ingots of metal composed of gold, silver, would C ... weighing 16 ounces; the first contains 7 ounces ... and 1 of copper; the second 5 ounces of ... and 4 of copper; and the third 2 ounces of ... and 5 of copper; now what quantity of each ... to make another mixture of 16 ounces that ... ounces of gold, $7\frac{1}{7}$ of silver, and $3\frac{1}{11}$ of ...

Ans. 4 *ounces* of the first ingot, 9 of the second, and 3 of the third.

9. A ... that for ... forfeit ... of the ... days did ...

10. Th... divided into 3 parts such, that the sum of the ... the sum of the second and third as 7 is to 9; ... of the first and second is to the sum of the ... as 1 to 3, the three results will be the number ... three companies of foot, respectively, 40 rounds ... Hence the strength of each company is required?

Ans. 72, 96, and 120, *men.*

Those ... hence th... is gun ... *Ans.* 7 ... 91

11. Sa... weight ... same ... great ... the weight ... number of men in three companies of foot are such, ... company with ; the other two, the second with ... and the third with ... of the other two, are the ... 119 men. Hence the respective numbers are

Ans. 35 men in the first company.
77 in the second.
91 in the third.

12. S... a man ... ... his wife being with child, ordered by will, ... proved a daughter, then his wife should have ... of his estate; but if it was a son, then he ... the mother $\frac{1}{4}$ thereof; now it happened that ... delivered of a son and a daughter; how must ... be divided to answer the father's ...

Ans. 900 the daughter's share.
1800 the mother's.
3600 the son's.

22. Suppose 4 footmen were to start together to travel the same way round an island which is 124 miles in circumference, and that the first went 11 miles *per day*, the second 15, the third 19, and the fourth 23: when would they come together again? *Ans.* in 31 *days*.

23. Several detachments of Artillery divided a certain number of cannon shot in the following manner:

The first detachment took 72 and $\frac{1}{9}$ of the remainder.

The second took 144 and $\frac{1}{9}$ of the remainder.

The third took 216 and $\frac{1}{9}$ of those that were left.

The fourth took 288 and $\frac{1}{9}$ of those left; and so on.

Now at last it was found that the shot had been equally divided. Hence the whole number of balls, and the number of detachments are required? *Ans.* No. of shot 4608.
Detachments 8.

Quadratic Equations.

1. Given $x^2 - x - 400 = 1700$; to find x.
Ans. $x = (2100\frac{1}{4})^{\frac{1}{2}} + \frac{1}{2}$.

2. Given $9x^2 + 6x - 27 = 228$; to find x. *Ans.* $x = 5$.

3. Given $ax^2 + x = b$; required x. *Ans.* $x = \frac{1}{2a}(4ab + 1)^{\frac{1}{2}} - \frac{1}{2a}$.

4. Given $x - \sqrt{x} = b$; required x. *Ans.* $x = (\frac{1}{2} \pm (b + \frac{1}{4})^{\frac{1}{2}})^2$.

5. Given $ax^{2n} - bx^{n} - c = -d$; to find x.
Ans. $x = \left(\frac{b}{2a} \pm \left(\frac{4ac - 4ad + b^2}{4a^2}\right)^{\frac{1}{2}}\right)^{\frac{1}{n}}$.

6. Given $x + \sqrt{(5x + 10)} = 8$; required x. *Ans.* $x = 3$.

7. Given $x - \frac{x - y}{2} = 4$
$y - \frac{x + 3y}{x + 2} = 1$; required x and y. *Ans.* $x = 2$ or 3.
$y = 6$ or 5.

8. Given $xy = 125x + 300y$
$y^2 - x^2 = 90000$; required x and y.
Ans. $x = 400$, $y = 500$.

18. Suppose three ingots of metal composed of gold, silver, and copper, each weighing 16 ounces; the first contains 7 *ounces* of gold, 8 of silver, and 1 of copper; the second 5 *ounces* of gold, 7 of silver, and 4 of copper; and the third 2 *ounces* of gold, 9 of silver, and 5 of copper; now what quantity of each ingot must be taken to make another mixture of 16 *ounces* that shall contain $4\frac{15}{16}$ *ounces* of gold, $7\frac{10}{16}$ of silver, and $3\frac{7}{16}$ of copper?

Ans. 4 *ounces* of the first ingot, 9 of the second, and 3 of the third.

19. If 11520 be divided into 3 parts such, that the sum of the first and second is to the sum of the second and third as 7 is to 9; and the difference of the first and second is to the sum of the first and third, as 1 to 8, the three results will be the number of cartridges for three companies of foot, respectively, 40 rounds to each man. Hence the strength of each company is required?

Ans. 72, 96, and 120, *men*.

20. The number of men in three companies of foot are such, that the first company with $\frac{1}{2}$ the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, are the same, each being 119 men. Hence the respective numbers are required?

Ans. 35 men in the first company.
77 in the second.
91 in the third.

21. A man dying, his wife being with child, ordered by will, that if the child proved a daughter, then his wife should have $\frac{2}{3}$ and the child $\frac{1}{3}$ of his estate; but if it was a son, then he should have $\frac{2}{3}$ and the mother $\frac{1}{3}$ thereof; now it happened that the mother was delivered of a son and a daughter; how must the estate, which was 6300*l*. be divided to answer the father's intention?

l.
Ans. 900 the daughter's share.
1800 the mother's.
3600 the son's.

22. Suppose 4 footmen were to start together to travel the same way round an island which is 124 miles in circumference, and that the first went 11 miles *per day*, the second 18, the third 19, and the fourth 23: when would they come together again? *Ans.* in 31 *days*.

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4. Given $x - \sqrt{x} = b$; required x. *Ans.* $x = (\frac{1}{2} \pm (b + \frac{1}{4})^{\frac{1}{2}})^2$.

5. Given $ax^n - bx^{\frac{n}{2}} - c = -d$; to find x.
Ans. $x = \left(\frac{b}{2a} \pm \left(\frac{4ac - 4ad + b^2}{4a^2}\right)^{\frac{1}{2}}\right)^{\frac{2}{n}}$.

6. Given $x + \sqrt{(5x + 10)} = 8$; required x. *Ans.* $x = 3$.

7. Given $x - \frac{x - y}{2} = 4$

$y - \frac{x + 3y}{x + 2} = 1$; required x and y. *Ans.* $x = 2$ or 5.
$y = 6$ or 3.

8. Given $xy = 125x + 300y$

$y^2 - x^2 = 90000$; required x and y.
Ans. $x = 400$, $y = 500$.

Questions producing QUADRATIC EQUATIONS.

1. To find a number such, that if you subtract it from 30, and multiply the remainder by the number itself, the product shall be 209? *Ans.* 11 or 19.

2. The difference of two numbers is 5, and the difference of their cubes is 1685; what are those numbers?

Ans. 8 and 13.

3. When 962 men were drawn up in two square columns (i. e. the number of ranks equal to the number of men in front) it was found that one column consisted of 18 ranks more than the other: hence the strength of each column is required?

Ans. 841, and 121 men.

4. To find two numbers whose product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes?

Ans. $\frac{1}{2}\sqrt{5}$, and $\frac{5+\sqrt{5}}{4}$.

5. Two partners A and B gained 140*l.* by trade; A's money was 3 months in trade, and his gain was 60*l.* less than his stock; and B's money, which was 50*l.* more than A's, was in trade 5 months; what was A's stock? *Ans.* 100*l.*

6. A and B take, in trade, 5940*l.* *per annum* each, but A, whose profits are 2 *per cent.* greater than those of B, clears 100*l.* *per annum* more than B. What are the profits of each, *per cent.* and what do they clear *per annum*?

Ans. A gains 10 *per cent.* and clears 540*l.* *per ann.*
B gains 8 *per cent.* and clears 440*l.* *per ann.*

7. What two numbers are those whose difference multiplied by the difference of their squares will produce 576; and whose sum multiplied by the sum of their squares is 2336?

Ans. 11 and 5.

8. What number is that which being multiplied by the sum of its two digits, the product shall be 1012, and if 63 be subtracted from the number, its digits will be inverted?

Ans. 92.

9. When 732 men were drawn up in column, the number in front and the number of ranks together made 73. How many were the rank? *Ans.* 61 or 12.

10. Two detachments of foot are ordered to a station distant 39 miles, they begin their march at the same time, but one party by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrives 1 hour sooner: hence the rates of marching are required?

Ans. $3\frac{1}{4}$, and 3 miles *per hour.*

11. To find two numbers whose product shall be 320, and the difference of their cubes to the cube of their difference, as 61 is to 1? *Ans.* 20 and 16.

12. Given the sum of three numbers in harmonic proportion = 191, and the product of the first and third = 4032; to find the numbers? *Ans.* 72, 63, 56.

13. Given the sum of 3 numbers in geometrical progression = 91, and the sum of their squares = 4459; what are the numbers? *Ans.* 7, 21, 63.

14. If the sum of two numbers is 11, and the sum of their 5*th.* powers 17831; what are the numbers? *Ans.* 4 and 7.

INDETERMINATE PROBLEMS.

1. To find the least whole number which being divided by 17 shall leave a remainder of 7, but when divided by 26 the remainder shall be 13? *Ans.* 143.

2. Required the least possible integer that being divided by 28, 19, and 15, the respective remainders shall be 19, 15, and 11? *Ans.* 7691.

3. When a company of foot was drawn up in column with 11 men in front, it was found that 5 men were wanting to form complete ranks; but when they were drawn up with 7 men in front, only 1 was required; what was the strength of the company, supposing the number less than 100? *Ans.* 83 *men*.

4. To find the year when the Roman Indiction was 4, the Golden Number 2, and Cycle of the Sun 12? *Ans.* in 1711.

5. A regiment of foot (less than 1000) when put in column with 13 men in front, wanted 9 men to complete the last rank; when 15 were in front then 14 men were wanting; but with 17 in front the ranks were complete: what was the strength of the regiment? *Ans.* 901 *men*.

6. How many different ways is it possible to pay 20*l*. without any other coin than *half guineas* and *half crowns*? *Ans.* 7.

7. If $17x + 19y + 21z = 400$; how many positive integral values are there of x, y, and z? *Ans.* 10 of each.

8. To find two whole numbers having 77 for the difference of their squares? *Ans.* 2 and 9, or 38 and 39.

9. To find that number which being any how divided into two unequal parts, the greater part added to the square of the less, shall be equal to the less part added to the square of the greater? *Ans.* 1.

10. To find the two least whole numbers whose difference, the difference of their squares, and the difference of their cubes, are all square numbers? *Ans.* 6 and 10.

11. To find a square number to which if you add 7, or subtract 7, the sum, and difference, shall also be square numbers? *Ans.* $7\frac{12769}{14400}$.

12. To divide 10 into 4 such parts, that the sum of every three shall be a square? *Ans.* 6, 1, $\frac{6906}{4225}$, and $\frac{5769}{4225}$.

13. To divide $\frac{1}{4}$ into 4 parts such, that either part when added to the cube of $\frac{1}{4}$ shall be a square?

Ans. $\frac{1679}{40000}$, $\frac{1875}{40000}$, $\frac{2975}{40000}$, and $\frac{3471}{40000}$.

Arithmetical Progressions.

1. If the first term $= \frac{1}{2}$, number of terms $= 20$, and the sum of all the terms $= 100$; what is the common difference?

Ans. $\frac{1}{2}$.

2. A detachment of foot have to occupy a post distant 159 miles; the first day they march 16 miles, the second day $15\frac{1}{2}$, the third day 15, and so on, lessening each day's march $\frac{1}{2}$ a mile: in what time will the journey be performed?

Ans. 12 *days.*

3. A detachment marched 198 miles in 16 days, and the first day they travelled 18 miles; now supposing each day's march was diminished by the same distance, how far did they travel the last day? *Ans.* $6\frac{3}{4}$ *miles.*

4. If the first term of a progression is $= 0$, common difference $= 1\frac{1}{2}$, and sum of all the terms $= 1170$; what is the number of terms? *Ans.* 40.

5. A party of foot begin their march at 6 in the morning, and travel $3\frac{1}{2}$ miles an hour; 3 hours after a troop of horse follow them from the same place, and march $3\frac{1}{2}$ miles the first hour, 4 miles the next, $4\frac{1}{2}$ the third, &c. increasing their march $\frac{1}{2}$ a mile every hour; in what time will they overtake the foot?

Ans. 7 *hours.*

6. Given the sum of the squares of the two means $= 346$, and the sum of the squares of the two extremes $= 410$; to determine the four numbers. *Ans.* 7, 11, 15, 19.

7. If a complete square pile of cannon balls contains just $7\frac{7}{40}$ times the number in the bottom layer; then how many are there in the pile? *Ans.* 2870.

8. The cannon shot of a complete triangular pile when placed in rows that touched one another on the ground, formed an exact square. What was the whole number of balls, the number being greater than 4?

Geometrical Progressions.

1. What is the sum of the first 11 terms of the series, $9, 4\frac{1}{2}, 2\frac{1}{4}, 1\frac{1}{8}$, &c. *Ans.* $17\frac{1015}{1024}$.

2. What is the 13*th.* term of the progression $21, 7, 2\frac{1}{3}, \frac{7}{9}$, &c. *Ans.* $\frac{7}{177147}$.

3. Required the sum of the progression $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}$, &c. infinitely continued, r being greater than 1?

Ans. $\dfrac{ar}{r-1}$.

4. If the first term $= 6$, the ratio $= \frac{1}{2}$, and sum of the progression $= 12$? what is the number of terms?

5. There are 4 numbers in geometrical progression, and the sum of the two least $= 20$, and that of the two greatest $= 45$; what are the numbers? *Ans.* 8, 12, 18, 27.

6. To find 4 numbers in arithmetical progression which being increased by 4, 7, 48, and 224, respectively, the sums shall be in geometrical progression? *Ans.* 7, 29, 51, 73.

7. From a vessel containing 10 gallons of brandy, 1 gallon was drawn out, and a gallon of water poured into the vessel; a gallon of the mixture was then drawn out, and another gallon of water poured in; now the like process being repeated 10

times, it is required to find how much brandy remained in the vessel, supposing the two fluids were thoroughly mixed every time? *Ans.* $3\frac{486784401}{1000000000}$ *gall.*

8. The sum of three numbers in geometrical progression is 91, and their continued product 9261; what are the numbers? *Ans.* 7, 21, 63.

9. If the first term of a series be 90, the last term 2, and the number of terms 20; what is the ratio?

Ans. .8184438, nearly.

10. Suppose the first term is 1, the last 0, and the sum of the series $\frac{3}{3-x}$; what is the ratio? *Ans.* $\frac{x}{3}$.

Permutations, Combinations, &c.

1. How many changes or variations can take place in the letters of the word *change?* *Ans.* 720.

2. Suppose 7 men stand in a rank; how many times can their order be varied? *Ans.* 5040.

3. If a company consisting of 30 men are drawn up in column, with how many different fronts can that be done, when 5 men are always in front? *Ans.* 142506.

4. How many different hands can be held at the game of whist? *Ans.* 635013559600.

5. How many variations may be made of the letters in the word *Bacchanalia?* *Ans.* 831600.

6. How many different numbers can be made out of an unit, 2 twos, 3 threes, 4 fours, and five fives, taken 5 at a time?

Ans. 2111.

7. How many different numbers can be made with the same figures as in the last example, supposing all the 15 figures to be in every number? *Ans.* 37837800.

8. Let there be 5 ranks of men, and suppose the first rank consists of 7 men, the second of 10, the third of 12, the fourth of 14, and the fifth of 15; now how many ways can 5 men be chosen from the ranks, one man being taken from each rank every time? *Ans.* 176400.

9. How many words, significant and insignificant, can be made out of the 24 letters?

Ans. 1391724288887252999425128493402200.

Recurring, and other SERIES.—*Differential Method.*

1. What is the sum of the infinite series $\frac{3}{4}-\frac{9}{16}+\frac{27}{64}-\frac{81}{256}+$ &c.

Ans. $\frac{3}{7}$.

2. Required the sum of the series $1+3x+6x^2+10x^3+15x^4+$ &c. infinitely continued?

Ans. $\frac{1}{(1-x)^3}$.

3. What is the sum of the infinite series $1+4x+9x^2+16x^3+$ &c.?

Ans. $\frac{1+x}{(1-x)^3}$.

4. What is the sum of the infinite series $\frac{x}{a}-\frac{x^2}{a^2}+\frac{x^3}{a^3}-\frac{x^4}{a^4}+$ &c.?

Ans $\frac{x}{a+x}$.

5. Required the sum of the infinite series $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+$ &c.

Ans. $\frac{3}{4}$.

6. What is the sum of the first 10 terms of the series $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}$ $+\frac{5}{81}$ &c. *Ans.* $2\,\frac{4915}{19683}$.

7. Required the sum of the series $1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+$ &c. *in infin.*

Ans. $\frac{4}{3}$.

8. Required the sum of the infinite series $\frac{3}{5}+\frac{3.4}{5.6}+\frac{3.4.5}{5.6.7}+\frac{3.4.5.6}{5.6.7.8}$ &c.

Ans. 3.

9. What is the sum of $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$ &c. infinitely continued?

Ans. $\frac{1}{18}$.

10. Required the sum of the infinite series $\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$ &c.

Ans. 1.

11. Required the sum of 20 terms of the series of products $1.2.3+3.4.5+5.6.7+$ &c.

Ans. 352380.

12. If the top row of a complete oblong pile of cannon shot consists of 10 balls, and the number of courses are 11; then how many shot are in the pile?

Ans. 1100.

13. The sides of the top course of a broken rectangular pile of shot are 12 and 7, and the number of courses 9: required the number in the pile?

Ans. 1728.

14. What is the 20*th.* term of the series 1, 6, 21, 56, 126, 252, &c.?

Ans. 42504.

Reversion of Series.

1. To revert the series $ay+by^2+cy^3+dy^4+fy^5$, &c. $=x$. (180)

$$Ans.\ y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{(2b^2-ac)x^3}{a^5}-\frac{(5b^3-5abc+a^2d)x^4}{a^7}$$

$$+\frac{(14b^4-21ab^2c+6a^2bd+3a^2c^2-a^3f)x^5}{a^9}-\&c.$$

2. To revert the series $ay+\frac{a^2y^2}{2}+\frac{a^3y^3}{2.3}+\frac{a^4y^4}{2.3.4}+\frac{a^5y^5}{2.3.4.5}+$ &c. $=x$.

$$Ans.\ y=\frac{x}{a}-\frac{x^2}{2a}+\frac{x^3}{3a}-\frac{x^4}{4a}+\frac{x^5}{5a}-\&c.$$

3. To revert the series $y-2y^2+\frac{8}{3}y^3-\frac{13}{3}y^4+$ &c. $=x$.

$$Ans.\ y=x-2x^2+\frac{16}{3}x^3+\frac{53}{3}x^4\ \&c.$$

4. Let the series $y + \frac{y^3}{6a^2} + \frac{3y^5}{40a^4} + \frac{5y^7}{112a^6} + \&c. = x$, be reverted.

Ans. $y = x - \frac{x^3}{6a^2} + \frac{x^5}{120a^4} - \frac{x^7}{5040a^6} + \&c.$

5. To revert the series $y^{\frac{1}{2}} + by^4 + cy^{7\frac{1}{2}} + dy^{11} + \&c. = x$.

Ans. $y = x^6 - 2bx^7 + (17b^2 - 2c)x^{10} + (18bc - 200b^3 - 2d)x^{13}$ &c.

Cubic, *and higher* Equations.

1. If $2x^3 - 24x^2 + 96x = 378$; what is the value of x? Ans. $x = 9$.

2. Let $x^3 + 9x = 1430$; required the value of x? Ans. $x = 11$.

3. Given $x^3 + 21x^2 - 196x = 4116$; to find x?

Ans. $x = 14, -14, -21$, the three roots.

4. Given $x^3 + 7x^2 - 43x = 301$; to find x?

Ans. $x = -7, \sqrt{43}, -\sqrt{43}$, the three roots.

5. Suppose $x^3 - 171.91x^2 + 7905.6x = 71256$; required the value of x?

Ans. $x = 11.862, 60.106, 99.942$, the three roots, nearly.

6. The sum of 4 numbers in geometrical progression being 140, and their continued product $= 109395\frac{9}{16}$; what are the numbers?

Ans. $3\frac{1}{2}$, $10\frac{1}{2}$, $31\frac{1}{2}$, $94\frac{1}{2}$.

7. The sum of 3 numbers in harmonic proportion is 191, and their continued product 254016. Required the numbers.

Ans. 72, 63, and 56.

8. Given the sum of three numbers $= 32$, the sum of their squares $= 350$, and the sum of their cubes $= 3926$. What are the numbers?

Ans. 9, 10, and 13.

9. A company of foot can be drawn up in column with 34220 different fronts having always 3 men in front: what is its strength?

Ans. 60 *men*.

10. The number of cannon shot in a complete triangular pile is 9139; then how many are in the bottom course? Ans. 703.

11. The number of cannon shot in a complete square pile or pyramid exceeds the number in a complete triangular one by 2300 when the sides of the two bases are equal: how many balls are in each pile?

Ans. 4900 and 2600.

12. If $\frac{x}{a} - \frac{x^2}{a^2} + \frac{x^3}{a^3} - \frac{x^4}{a^4} + \&c.$ *in infin.* $= m$; what is the value of x?

Ans. $x = \frac{ma}{1 - m}$.

13. Suppose $1+3x+6x^2+10x^3+15x^4+$ &c. *in infin.* $=10$; required the value of x?

Ans. $x=1-\frac{1}{10^{\frac{1}{3}}}$.

14. If the sum of the series of biquadrates $1^4+2^4+3^4+4^4+$ &c. be equal to 6367 times the number of terms; what is the sum of the series?

Ans. 89271.

15. If $x^4-.611977x^3+.755698x^2-.376366x=.26106285$. Required the value of x?

Ans. $x=.791207$.

16. Suppose $3^{2x}+3^{x}=4785156$: what is the value of x?

Ans. 7.

17. If $2^{x}-2^{\frac{1}{2}x}=12$; required the value of x?

Ans. 2.33985 *nearly.*

18. Given $1+2x^2+3x^4+4x^6+$ &c. (*in infin.*) $=y^4$
and $2x=y$.
Required values of x and y?

Ans. $x=\sqrt{\frac{1}{2}}$.
$y=2\sqrt{\frac{1}{2}}$.

19. Suppose $\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+$ &c. (*in infin.*) $=\frac{1}{4}$; what is the value of x?

Ans. $x=\frac{1}{2}-\frac{1}{16}+\frac{1}{96}-\frac{1}{512}+\frac{1}{2560}-$ &c.

20. Given $x^4+y^4=10000$, and $x^5-y^5=25000$; to find x and y?

Ans. $x=8.87017$, and $y=7.85585$ *nearly.*

INTEREST, *and* ANNUITIES.

1. A sum of money put out at simple interest amounts to 297*l.* 12*s.* in 8 months; and the amount of the same sum in 15 months is 306*l.* Required that sum: also the rate of interest?

Ans. 288*l.* the sum.
5 *per cent.* the interest.

2. Two notes, one of 120*l.* payable in 6 months, and the other of 130*l.* payable in 9 months, were discounted for 8*l.* 10*s.* what was the rate of interest?

Ans. 5*l.* 1*s.* 10½*d.* *per cent.* nearly

3. There is 320*l*. due to me at this time, and 96*l*. more will be due at the end of 5 years (both from the same person); now we make an agreement that the whole shall be discharged at one payment, at the time when the interest of the 320*l*. becomes equal to the discount of the 96*l*. Hence the time of payment is required: the calculation being made at 5 *per cent. per ann.* simple interest? *Ans.* at the end of 1 year.

4. At what rate of compound interest will 481*l*. raise a stock of 1000*l*. in 15 years? *Ans.* 5 *per cent.*

5. What is the amount of 217*l*. forborn 2¼ years at 5 *per cent. per annum* compound interest, supposing the interest payable quarterly? *Ans.* 241*l*. 13*s*. 4½*d*.

6. If 356*l*. be payable at the end of 7 years, what is it worth in ready money, discompting after the rate of 7 *per cent. per ann.* compound interest? *Ans.* 221*l*. 14*s*. *nearly*.

7. The compound interest of a certain sum of money amounted to 344.61*l*. in 4 years; but the simple interest of the same sum, at the same rate in 4 years, would have been only 320*l*. Hence the principal, and the rate of interest are required?

Ans. 1600*l*. the principal.
5 *per cent.* the rate.

8. What is the present worth of an annuity or rent of 50*l*. *per ann.* payable yearly for 21 years, reckoning compound interest at the rate of 6 *per cent. per annum*.

Ans. 588*l*. 4*s*. ¾*d*. *nearly*.

9. For how long time will 600*l*. purchase an annuity of 100*l*. at 4 *per cent.* compound interest? *Ans.* 7 *years*.

10. To determine at what rate of interest an annuity of 50*l*. to continue 10 years, may be purchased for 400*l*.

Ans. 4.2775*l*. *per cent.* nearly.

11. Suppose an annuity of 175*l.* is to commence 9 years hence, and then continue 11 years: to find the present value, allowing 6 *per cent. per ann.* compound interest,

Ans. 816*l.* 18*s.* 9*d.* nearly.

12. A young man sinks 1000*l.* in purchasing an income of 100*l. per ann.* to continue till he is 60 years of age; now if he be 24 years old when the purchase money is paid, at what age will he begin to receive the annuity, allowing 5 *per cent. per ann.* compound interest? *Ans.* 32.127 *years*, nearly.

APPLICATION OF ALGEBRA TO GEOMETRY. *With the Solutions by* GEOMETRICAL CONSTRUCTION.

232. In the preceding Articles we have considered Algebra as independent of Geometry, and demonstrated its operations from its own principles. We shall now explain the use of Algebra in resolving Geometrical Problems which depend on the properties of right lines and the circle. The student should therefore be master of the geometry in the first volume before he enters on this part.

Though the algebraic method is concise, and admirably adapted to the discovery of general properties and theorems, yet constructions purely geometrical claim the preference in point of elegance and perspicuity: this however, is to be understood of *plane problems* only, or such that may be resolved by a simple, or a quadratic equation. When the equation rises to a *cubic*, the problem is called a *solid* one, and the *actual* description of those lines by which the construction is effected in *that case*, becomes a matter of some difficulty.

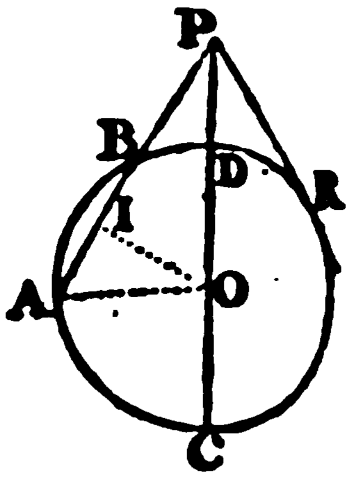
Different problems will require different methods of solution: and consequently it would not be easy to frame precepts for

general reference. The student must therefore acquire his knowledge and dexterity in this branch from examples, and his own practice.

233. We shall begin with

The construction of the three forms of adfected Quadratic Equations:

$$\text{viz.}\quad x^2 + ax = bc.$$
$$x^2 - ax = bc.$$
$$ax - x^2 = bc.$$



Construction of the first and second forms. With a radius equal to $\frac{1}{2}a$ let a circle be described, its centre being O. In this circle draw a chord AB $= b - c$ (b being supposed greater than c) and produce AB till BP $= c$; and from P draw PC through the center O. Then will $x =$ DP in the *first form*; and $x =$ CP in the *second*.

For AB $= b - c$, and BP $= c$, therefore AP $= b - c + c = b$. And since the radius of the circle is $= \frac{1}{2}a$, the diameter DC $= a$.

Now (DP + DC) × DP = AP × BP, (*Geom.* 98)

that is $(x + a) \times x = b \times c$, or $x^2 + ax = bc$, the first form.

And in the second form (PC − DC) × PC = AP × BP,

or $(x - a) \times x = b \times c$, that is $x^2 - ax = bc$.

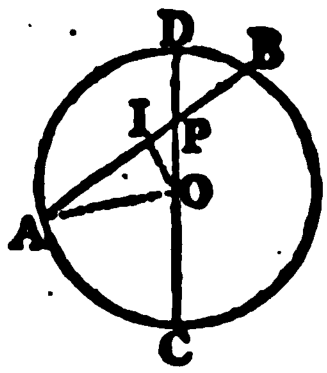
If the rectangle $bc = n^2$, or n is taken a geometrical mean between b and c, then the distance of the point P from the circle is found by drawing a tangent (RP) $= n$ from any point (R) in the circumference: for AP × BP = PR² (*Geom.* 99) $= bc = n^2$. This latter method of coustruction (by means of the tangent) may be adopted when $b - c$ is greater than a the diameter of the circle:

Method of calculation. Draw the radius OA, and on AP let fall the perpendicular OI which will bisect the chord AB (*Geom.* 63).

Then $OI^2 = OA^2 - AI^2$ (*Geom.* 83); and $OP^2 = OI^2 + IP^2$, or $OP^2 = AO^2 - AI^2 + IP^2$; therefore $OP = \sqrt{(AO^2 - AI^2 + IP^2)}$, and consequently DP or $x = \sqrt{(AO^2 - AI^2 + IP^2)} - OD$ in the first form: and PC or $x = \sqrt{(AO^2 - AI^2 + IP^2)} + OC$ in the second.

Suppose $x^2 + 9x = 8 \times 4\frac{1}{2}$ (the first form).

Then $AO = \frac{1}{2}a = 4\frac{1}{2}$, $AI = \frac{b-c}{2} = 1\frac{3}{4}$, $IP = 6\frac{1}{4}$, OD or $OC = 4\frac{1}{2}$: and $x = \sqrt{(20\frac{1}{4} - 3\frac{1}{16} + 39\frac{1}{16})} - 4\frac{1}{2} = 7\frac{1}{2} - 4\frac{1}{2} = 3 = DP$; and $7\frac{1}{2} + 4\frac{1}{2} = 12 = PC$, the values of x in the two first forms.



Construction of the third form. Let a circle whose diameter is a be described as in the preceding forms; and take the chord $AB = b + c$; make $AP = b$, and $PB = c$, and through P draw the diameter DC. Then PD, and PC will be the two roots or values of x.

For $(DC - PD) \times PD = AP \times PB$, (*Geom.* 97)
that is $(a - x) \times x = b \times c$, or $ax - x^2 = bc$, (x being $=$ PD).
And $(DC - PC) \times PC = AP \times PB$,
that is $(a - x) \times x$, or $ax - x^2 = bc$, (x being denoted by PC).

When $b + c$ is greater than the diameter a, take $n^2 = bc$, or let $ax - x^2 = n^2$, then AP and PB will be equal, and $AB = 2n$.

Method of calculation. From the centre O let fall the perpendicular OI upon AB, and join OA.

Then $OI^2 = AO^2 - AI^2$ (*Geom.* 83) $= \frac{1}{4}a^2 - \left(\frac{b+c}{2}\right)^2$;

And $IP = AP - AI = b - \frac{b+c}{2} = \frac{b-c}{2}$;

therefore $OP^2 = OI^2 + IP^2 = \frac{1}{4}a^2 - \left(\frac{b+c}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2 = \frac{1}{4}a^2 - bc$;

and $OP = \sqrt{(\frac{1}{4}a^2 - bc)}$; whence PD, and PC the two values of x are $OD - OP$, and $OC + OP$, or $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - bc)}$; and $\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - bc)}$.

Let $14x - x^2 = 8 \times 3$ be the equation. Then $a = 14 = DC$, $b = 8 = AP$, $c = 3 = PB$; and PD and PC will be 2, and 12 the values of x.

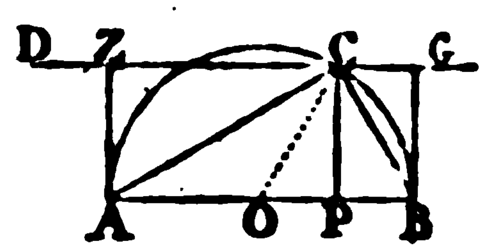
The same conclusions result from completing the square: thus, taking the third form $ax - x^2 = bc$, or $x^2 - ax = -bc$, whence $x^2 - ax + \frac{1}{4}a^2 = \frac{1}{4}a^2 - bc$, and $x = \frac{1}{2}a \pm \sqrt{.}(\frac{1}{4}a^2 - bc)$.

From the preceding constructions it appears, that when a geometrical problem can be solved by an equation not exceeding a quadratic, it also admits of a construction by means of right lines and the circle.

234. *The area* (a) *of a right angled triangle, and the hypotenuse* (h) *being given, to find the other two sides.*

Suppose AB is the hypotenuse, and CP the perpendicular let fall from the right angle ACB upon the hypotenuse.

Since AB (or h) $\times$ CP $= 2a$ (*Mensur.* 257) we get $\frac{2a}{h} = CP$ the perpendicular.



Now let one of the segments AP or PB be denoted by x, then the other will be $h - x$. And because CP is a mean proportional between the segments AP, PB, (*Geom.* 164) we have $(h - x)x = \frac{4a^2}{h^2}$, or $x^2 - hx = -\frac{4a^2}{h^2}$; which equation gives $x = \frac{1}{2}h \pm \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)}$, the two segments AP and PB. Whence the sides AC, BC will be found by *Geom.* 83.

Suppose the hypotenuse $AB = 13 = h$, and the area $= 39 = a$: then $\frac{1}{2}h \pm \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)} = 6\frac{1}{2} \pm \sqrt{(42\frac{1}{4} - 36)} = 9$ and 4 the two segments. Now CP being $= \frac{2a}{h} = 6$, we have $AC = \sqrt{(81 + 36)}$, and $BC = \sqrt{(16 + 36)}$.

If the perpendicular $\frac{2a}{h} = \frac{1}{2}h$, the expression $\sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)}$ is $= o$, and $x = \frac{1}{2}h$, or the point P coincides with the center of the circle: therefore should the given area exceed $\frac{1}{4}h^2$ the problem is impossible.

Otherwise thus:

Suppose O is the middle of AB; and put $x = $ OP, half the difference of the segments AP, PB; then $\frac{1}{2}h + x$ and $\frac{1}{2}h - x$ will denote the two segments; whence (*Geom.* 164), $(\frac{1}{2}h + x) \times (\frac{1}{2}h - x) = \frac{4a^2}{h^2}$ $(=$ CP$^2)$, which gives $x^2 = \frac{1}{4}h^2 - \frac{4a^2}{h^2}$, and $x = \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)} = $ OP, which added to, and subtracted from $\frac{1}{2}h$, give the two segments AP and PB, as before.

Geometrically.

On the given hypotenuse AB describe a semicircle; also on the same line AB let a rectangle ZB be made equal to twice the area of the triangle (*Geom.* 179): from C draw CA, CB; and ACB is the triangle. For the triangle ACB is $= \frac{1}{2}$ the parallelogram ZB (*Geom.* 82. *corol.* 2), and the angle ACB a right one (*Geom.* 72.)

Method of calculation. Draw the radius OC; then the perpendicular CP being found as before; we have $\sqrt{(\text{OC}^2 - \text{CP}^2)} = $ OP (*Geom.* 83), or half the difference of the segments AP and PB: which is the same expression as that found by the last of the preceding methods.

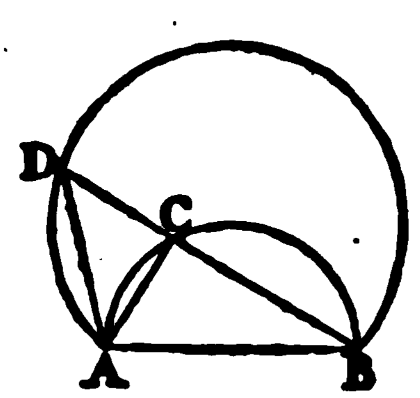
In this construction, the vertex (C) of the triangle is determined by the intersection of two *loci*. Thus the circular arc ACB is called the *locus* or *place* of the angle ACB, because two lines drawn from A and B to meet any where in the arc ACB will form the same angle (*Geom.* 70). Therefore when the base, and vertical angle of a triangle are given, the *locus* of the vertex is the arc of a circle. But when the base and area are given,

the *locus* of the vertex is a right line parallel to the base. Thus all triangles standing on the base AB, and having equal areas, will have their vertices in the indefinite line DG: *Geom.* 82. *corol.* 2.

233. *In a right-angled triangle* ACB, *let there be given the hypotenuse* AB, *and the sum of the other sides* AC + BC; *to determine those sides.*

Suppose AB $= h$, AC + BC $= s$, and BC $= x$.

Then $s - x =$ AC, and $(s-x)^2 + x^2 = h^2$ (*Geom.* 83) or $s^2 - 2sx + 2x^2 = h^2$; which equation solved gives $x = \frac{1}{2}s \pm \sqrt{\left(\frac{2h^2 - s^2}{4}\right)}$; and these two values of x are BC and AC.



If s^2 is greater than $2h^2$, the problem is impossible.

Suppose AB $= 15 = h$, and AC + BC $= 21 = s$:
Then $\frac{1}{2}s \pm \sqrt{\left(\frac{2h^2 - s^2}{4}\right)} = 10\frac{1}{2} \pm \sqrt{\frac{9}{4}} = 12$ and 9, the required sides.

Geometrically.

On the hypotenuse AB describe a semicircle, and also a segment of a circle to contain half a right angle; make the chord BD = the sum of the sides; then from the point of intersection C, draw CA; and ACB is the triangle.

For ACB is a right angle (*Geom.* 72), and therefore ACD is also a right angle; and since ADC is = half a right angle, the angle DAC must be half a right angle, and consequently AC = CD; therefore AC + CB = BD the sum of the sides by construction.

Method of calculation. By trigonometry, as AB : *sine* ADB :: BD : *sine* DAB; whence the angle DBA becomes known; and then the sides BC and AC are readily calculated from the given side AB.

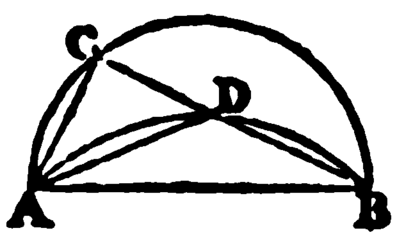
236. *Let the hypotenuse* AB, *and the difference of the sides* AC *and* BC, *be given; to find* AC *and* BC.

Put h = the hypotenuse, d = the difference of the required sides, and x = the less side. Then $x+d$ will denote the greater.

And $(x+d)^2 + x^2 = h^2$ (*Geom.* 83), or $2x^2 + 2dx + d^2 = h^2$; this equation gives $x = \sqrt{(\frac{1}{2}h^2 - \frac{1}{4}d^2)} - \frac{1}{2}d$.

Suppose $h = 15$, and $d = 3$; then $\sqrt{(\frac{1}{2}h^2 - \frac{1}{4}d^2)} - \frac{1}{2}d = 10\frac{1}{2} - 1\frac{1}{2} = 9$ the least side; and $9 + 3 = 12$ the greater.

Geometrically.



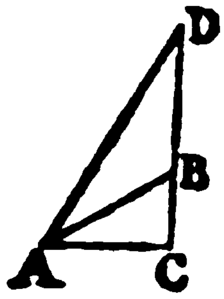
On the hypotenuse describe a semicircle, and also a segment to contain $1\frac{1}{2}$ right angles; and make the chord BD = the given difference of the sides: produce BD to C, and join CA; then ACB is the triangle.

For ACB being a right angle, and the angle ADC = $\frac{1}{2}$ a right one, the angle DAC must be $\frac{1}{2}$ a right one, and therefore AC = DC, and consequently DB is the difference of AC and CB.

Method of calculation. By trigonometry, as AB : *sine* ADB :: BD : *sine* DAB; this being determined, all the other angles in the figure become known.

237. *Having the base* (AC) *of a right angled triangle, and the sum of the hypotenuse and perpendicular* (AB + BC), *to find each of the latter sides.*

Let the base AC = b, AB + BC = s, and CB = x: Then the hypotenuse AB = $s - x$. And $s^2 - 2sx + x^2 = b^2 + x^2$ (*Geom.* 83); whence $s^2 - 2sx = b^2$, and $x = \frac{s^2 - b^2}{2s}$ the perp. CB: and $s - \frac{s^2 - b^2}{2s} = \frac{s^2 + b^2}{2s}$ the hypotenuse AB.



If $AC = 8 = b$, and $AB + CB = 16 = s$; then $\frac{s^2 - b^2}{2s} = 6 = CB$; and $\frac{s^2 + b^2}{2s} = 10 = AB$.

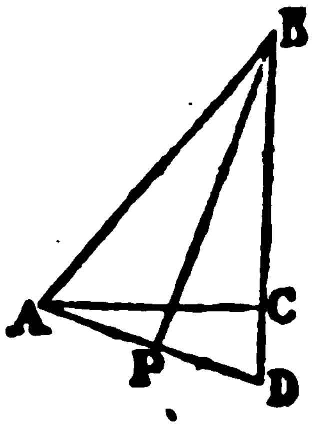
Geometrically.

At C the extremity of the given base, erect CD perpendicular to AC, and equal to the sum of the other sides; join AD, and make the angle BAD = BDA: Then ACB is the triangle. For the angle BAD being = the angle BDA, the opposite sides BD, BA must also be equal (*Geom.* 46. *cor.* 2) and therefore $AB + BC = CD$.

Method of calculation. By trigonom. DC : CA :: *radius* : *tang.* angle ADC; double this is the angle ABC (*Geom.* 41); therefore in the triangle ACB, the base AC, and all the angles will be given.

238. *Let the base* AC, *and difference of the hypotenuse* AB *and perpendicular* CB, *be given; to determine those sides separately.*

Suppose b = the base, d = the difference $AB - BC$, and $x = BC$. Then $x + d = AB$ the hypotenuse.



And $(x + d)^2 = b^2 + x^2$, or $x^2 + 2dx + d^2 = b^2 + x^2$ (*Geom.* 83); whence $2dx = b^2 - d^2$, and $x = \frac{b^2 - d^2}{2d} = CB$: and therefore $\frac{b^2 - d^2}{2d} + d = \frac{b^2 + d^2}{2d} = AB$ the hypotenuse.

Geometrically.

Make CD perpendicular to the given base AC, and equal to the difference $AB - BC$; join AD which bisect in P, and draw PB perpendicular to AD, meeting DC produced in B; draw AB: and ACB is the triangle. For the angles at P being right ones, and PA = PD, therefore the triangle ABD is isosceles, and

AB = BD; therefore CD is the difference of CB and AB, the given difference by construction.

Method of calculation. By trigonom. AC : CD :: *radius* : *tang.* angle DAC; and its complement is the angle ADC = DAB, whence all the angles in the triangle ACB become known.

239. *To divide a given line* AB *into mean and extreme proportion.*

Let AB = a, and x = AD the greater part, then $a-x$ = DB the less.

And AB : AD :: AD : DB
or $a : x :: x : a-x$, whence $x^2 = a^2 - ax$,
and $x^2 + ax = a^2$; which gives $x = \left(\frac{5a^2}{4}\right)^{\frac{1}{2}} - \frac{1}{2}a$ = AD.

For the *geometrical construction*, see *Geom.* 137.

* 239. *To divide a given line* AB *into two parts* AP *and* PB *so that their rectangle* AP × PB *shall be of a given magnitude* (m^2).

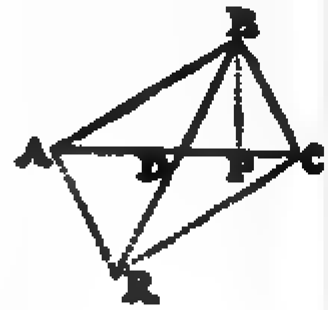
Let a denote the given line, and x one of the parts; then the other will be $a-x$; and $(a-x)\,x = m^2$, whence $x^2 - ax = -m^2$, which gives $x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - m^2)}$; and these two roots or values of x are the parts AP and PB.

If m is greater than $\frac{1}{2}a$, the problem is evidently impossible.

The construction and method of calculation will be exactly the same as that in Art. 234, by taking PC = m.

240. THEOREM. *If a line* BD *be drawn from the vertex of any triangle* ABC *to bisect the base: Then* $2BD^2 + 2AD^2 = AB^2 + BC^2$.

Let BP be perpendicular to AC; and put BD $= b$, DP $= d$, and $h =$ AD or DC: then $h + d =$ AP, and $h - d =$ PC:



And $BD^2 - DP^2 = BP^2$

$AP^2 + BP^2 = AB^2$ (*Geom.* 83.)

$CP^2 + BP^2 = BC^2$

That is $b^2 - d^2 = DP^2$,

$(h + d)^2 + b^2 - d^2$ or $h^2 + 2hd + b^2 = AB^2$

$(h - d)^2 + b^2 - d^2$ or $h^2 - 2hd + b^2 = BC^2$

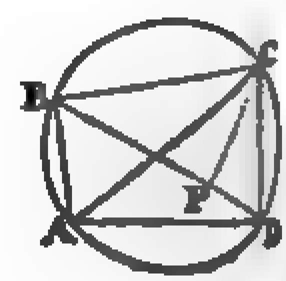
whence by addition $2h^2 + 2b^2 = AB^2 + BC^2$,

or $2AD^2 + 2BD^2 = AB^2 + BC^2$.

Corol. If AR, CR be parallel to BC, AB, respectively; then BDR and AC will be the diagonals of the parallelogram ABCR, and DR = DB (*Geom.* 81. *corol.* 2), and therefore $2DC^2 + 2RD^2 = CR^2 + RA^2$; but DC = AD, and BD = RD; consequently $4DC^2 + 4RD^2 = CR^2 + RA^2 + AB^2 + BC^2$: but 4 times the square on half a line is equal to the square on the whole line; therefore $4DC^2 = AC^2$, and $4RD^2 = RB^2$; whence we have $AC^2 + RB^2 = CR^2 + RA^2 + AB^2 + BC^2$; that is, *the sum of the squares of the two diagonals of a parallelogram, is equal to the squares on the four sides taken together.*

241. THEOREM. *The rectangle under the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides:*

That is, AC × BD = AB × CD + AD × BC.



Suppose CP is drawn to make the angle PCD = BCA:

Then because the angle PDC = BAC, (*Geom.* 70), the triangles CPD, CBA are equiangular; whence AC : AB :: DC : DP (*Geom.* 94. *corol.* 1), therefore AC × DP = AB × DC.

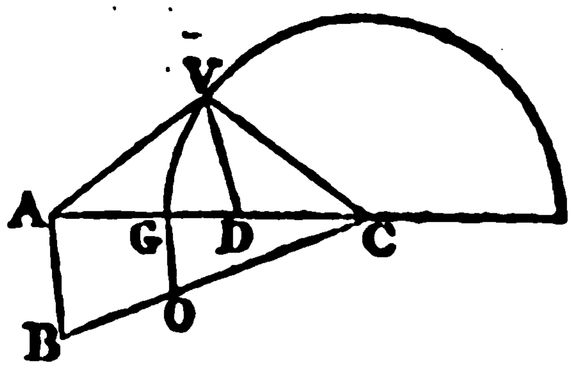
Again; because the angle $CPD = CBA$, and $CPD + CPB$ make two right angles, and $CBA + CDA$ also make two right angles, the angle $CPB = CDA$, and the angle $PBC = CAD$ (*Geom.* 70), therefore the triangles CPB, CDA are equiangular; consequently $AC : AD :: BC : BP$, whence $AC \times BP = AD \times BC$; now by adding this equation and the former together, we have

$$AC \times BP + AC \times DP = AD \times BC + AB \times DC,$$

that is $(BP + DP) \times AC$, or $BD \times AC = AD \times BC + AB \times DC$.

242. THEOREM. *Let* ABC *be any triangle, and suppose* $AG = AB$, *and* GO *parallel to* AB; *then if* GD *be taken* $= GO$, CG *will be a mean proportional between* AC and CD.

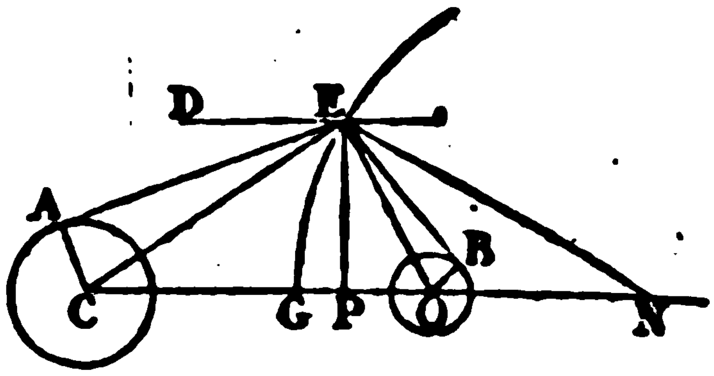
By similar triangles, $CA : AB\ (AG) :: CG : GO\ (GD)$,
and by division, $CA : CA - AG :: CG : CG - GD$,
or $CA : CG :: CG : CD$.



From this Theorem the *locus* of the vertex of a triangle may be determined when the base AD and the *ratio* of the sides AV, DV are given. Thus, if a circle be described about the center C with the radius CG, then any two lines drawn from A and D to meet in the circumference will have the ratio of AG to GD.

For $CA : CG\ (CV) :: CG\ (CV) : CD$; that is, the sides of the triangles CVA, CVD about the common angle at C, are proportional, and therefore the triangles are similar (*Geom.* 94. *cor.* 1), consequently the other sides are proportional, *viz.* $AV : DV :: CA : CV\ (CG) :: AB\ (AG) : GO\ (GD.)$ Hence, to describe the circle which is the required *locus*, divide the base in the given proportion of the sides, and find the center (C) as above.

243. *If a 9lb. iron shot, and another 48lb. are on the ground at 20 yards distance from each other; where must I stand in a line between them so that each may appear of the same magnitude; the height of the eye being $5\frac{1}{2}$ feet?*



Suppose O to be the center of the less shot, C the center of the greater, and E the place of the eye; also let EA, EB be tangents at A and B; then CAE, OBE will be right angles.

Now it is evident that when the two shot appear of the same magnitude, the diameters of the circles which bound the visible surfaces will be seen under equal angles, or, which amounts to the same thing, the angles CEA, OEB will be equal; therefore the triangles CAE, OBE will be similar, and consequently CE, OE will have the same ratio as the radii CA, OB, or as $3\frac{1}{2}$ *inches* to 2 *inches*, which are the radii of a 48*lb.* shot, and a 9*lb.* shot, nearly.

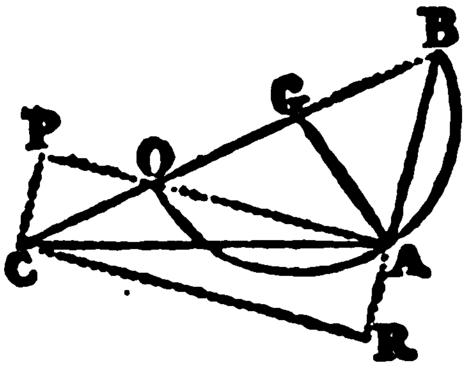
Construction. Divide the distance CO = 20 yards into two parts CG, GO having the ratio of $3\frac{1}{2}$ to 2, and describe the *locus* of the vertex of the triangle CEO, as directed in the preceding article: then if DE be drawn parallel to, and at the distance of $5\frac{1}{2}$ *feet* from CO, the point E where it intersects the circle, will be the place of the eye.

Calculation. CG = 38.182, and GO = 21.818 *feet*, nearly.

And 38.182 — 21.818 : 38.182 :: 38.182 : 89.089 = CN, from this take CG, and there remains 50.907 = GN = NE the radius; then the perpendicular PE being = $5\frac{1}{2}$ *feet*, we get PN = 50.6, which taken from CN gives CP = 38.189 *feet*, the distance from the greatest ball where a person must stand to see them both under the same angle.

N. B. The point P is between 2 and 3 *inches* from the ground, and consequently PE is taken that quantity too great in the computation; the conclusion however, is not materially affected on that account.

244. *Having the sides of a plane triangle, to find the angles.*



Let CGA be the triangle. About G as a center with the radius GA describe a semicircle; produce CG to B; draw BAR, and AOP; make CP perpendicular to AP, and CR perpendicular to BR; then the angle OAB in a semicircle is a right one, consequently PA, CR are parallel, and PC = AR; also GO = GA = GB; and CB is the sum, and OC the difference of the sides GC, GA; and because the triangle AGB is isoceles, the angle GBA is = $\frac{1}{2}$ the angle CGA.

Now if AB be the base of the triangle ACB, and CR the perpendicular on the base (produced), we have (Trigonom. art. 228) $AB : BC + AC :: BC - AC : \frac{BC^2 - AC^2}{AB}$, and $\frac{BC^2 - AC^2}{2AB} - \frac{AB}{2} = AR = CP.$

Again, suppose AO is the base of the triangle ACO, and CP the perpendicular on the base (produced),

then $AO : AC + CO :: AC - CO : \frac{AC^2 - CO^2}{AO}$, and $\frac{AC^2 - CO^2}{2AO} - \frac{AO}{2} = OP.$

But the triangles CPO, BAO are similar,

that is $CP : AB :: OP : AO$;

or $\frac{BC^2 - AC^2}{2AB} - \frac{AB}{2} : AB :: \frac{AC^2 - CO^2}{2AO} - \frac{AO}{2} : AO$;

or $\frac{BC^2 - AC^2}{AB} - AB : AB :: \frac{AC^2 - CO^2}{AO} - AO : AO$;

and by composition, $\frac{BC^2 - AC^2}{AB} : AB :: \frac{AC^2 - CO^2}{AO} : AO$;

whence $BC^2 - AC^2 : AB^2 :: AC^2 - CO^2 : AO^2$, (by equimultiples), or $BC^2 - AC^2 : AC^2 - CO^2 :: AB^2 : AO^2$, that is $(BC+AC)(BC-AC) : (AC+CO)(AC-CO) :: AB^2 : AO^2$:

But if AB be made *radius*, AO will be the *tangent* of the angle OBA or half the angle CGA of the proposed triangle CGA.

Therefore let s and d respectively denote the *sum* (BC) and *difference* (CO), of the sides including the required angle (CGA), and b the other side (AC):

Then $(s+b)(s-b) : (b+d)(b-d) :: rad.^2 : \frac{(b+d)(b-d)}{(s+b)(s-b)} \times rad.^2 =$ the square of the tangent of half the angle CGA.

Example. Let $CA = 462$, $CG = 384$, $AG = 169$; and the radius $= 1$.

384	384
169	169
$553 = s$	$215 = d$
$462 = b$	$462 = b$
$1015 = s + b$	$677 = b + d$
$91 = s - b$	$247 = b - d$

$\frac{(b+d)(b-d)}{(s+b)(s-b)} \times rad.^2 = \frac{677 \times 247}{1015 \times 91} = 1.8104$ nearly, and its square root $= 1.3455$ the *natural tangent* of $53° 23'$.

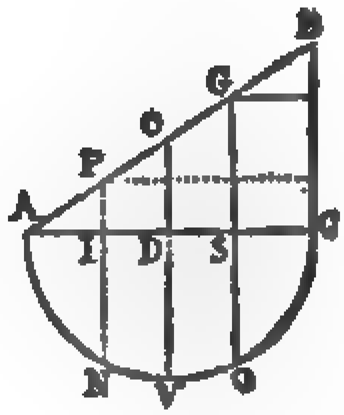
But the operation by logarithms is very short:

1015 *ar. co. log.*	6.993534
91 *ar. co. log.*	8.040959
677 .... *log.*	2.830589
247 .... *log.*	2.392697
rad.^{2} .... *log.*	20.000000
2)	20.257779
$53° 23'$ *tang. log.*	10.128889

Therefore twice $53° 23'$ is $106° 46'$ the required angle CGA.

This rule is preferable to that in vol. 1, art. 229; and should be used when the required angle is very obtuse.

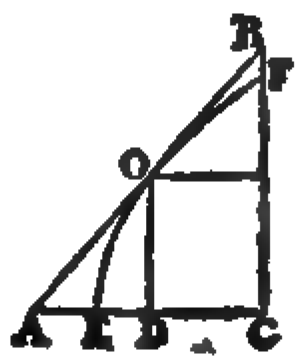
245. *In a right angled triangle* ACB *to inscribe a rectangle* GC *of a given magnitude* (m^2).



Let OD, parallel to BC, bisect AC the base in D. Then by similar triangles, $AD : DO :: AD + DS : \frac{DO}{AD} \times (AD + DS)$ $= SG$, which drawn into SC or $AD - DS$ is the rectangle GC, that is $\frac{DO}{AD} \times (AD + DS)\ (AD - DS) = m^2$, or $DO : AD :: m^2 : (AD + DS)\ (AD - DS)$. Hence the following *construction*:

On AC describe a semicircle, in which, at right angles to AC, apply SQ and IN so that $DO : AD :: m^2 : SQ^2$ or IN^2 (Geom. art. 166); then if SG, IP are perpendicular to AC, either of the rectangles GC, PC is that required. For $(AD + DS)\ (AD - DS) = SQ^2$, by Geom. 97, corol. 1. Therefore $\frac{DO}{AD} \times SQ^2 = m^2$.

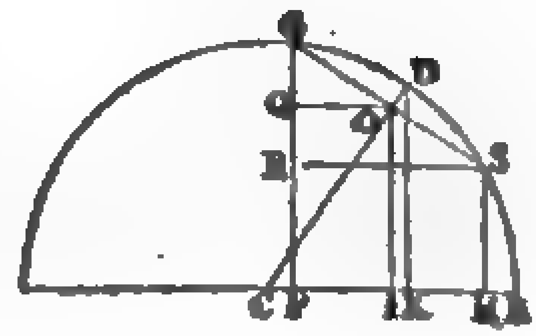
Corol. 1. When $m^2 = AD \times DO$, then $DO : AD :: DO \times AD : AD^2$ (or DV^2), and the points I, S, coincide in D, and DO, DC are the sides of the rectangle, which is a *maximum*, because DV is the greatest possible. And in that case, the problem admits of but one answer.



Corol. 2. Hence we conclude, that if EOF be any curve concave to its axis EC, the greatest inscribed rectangle OC will be when OD bisects the base AC, or the subtangent $DA = \frac{1}{2}$ the base, and the tangent AB is bisected at the point of contact O.

246. *The sines and cosines of two arcs* (BD, DG) *being given, to find the sine and cosine of the sum* (BG), *and difference* (BS) *of those arcs.* (See Trigonom. art. 210).

Draw the sines SH, DK, GP, and the radius CD; also join SG, and let SR and OQ be parallel to the radius CB, and OI perpendicular to CB. Then DK, KC; and GO, OC, are the *sines*, and *cosines* of the arcs BD, DG.



By similar triangles:

CD : CO :: DK : OI, whence $\frac{CO.DK}{CD} = OI = QP$.

CD : CK :: GO : GQ, and $\frac{CK.GO}{CD} = GQ$: therefore

$\frac{CO.DK}{CD} + \frac{CK.GO}{CD} = QP + GQ = PG$ *the sine of the sum*;

and $\frac{CO.DK}{CD} - \frac{CK.GO}{CD} = QP - QG\ (QR) = RP = SH$, *the sine of the diff.*

Again;

CD : CK :: CO : CI, whence $\frac{CK.CO}{CD} = CI$,

CD : DK :: GO : OQ, and $\frac{DK.GO}{CD} = OP = PI = IH$,

whence $\frac{CK.CO}{CD} - \frac{DK.GO}{CD} = CI - IH = CP$, *cosine of the sum*; and

$\frac{CK.CO}{CD} + \frac{DK.GO}{CD} = CI + IH = CH$, *cosine of the difference.*

Hence, if the *sine* and *cosine* of an arc be denoted by S and C, and the *sine* and *cosine* of another arc by s and c, and the *radius* $= r$:

Then, the *sine* of their *sum* $= \frac{Sc + sC}{r}$;

cosine $= \frac{Cc - Ss}{r}$.

sine of their *diff.* $= \frac{Sc - sC}{r}$;

cosine $= \frac{Cc + Ss}{r}$.

When the given arcs are equal, then $S=s$, and $C=c$, and the expressions become $\frac{2sc}{r}$, and $\frac{c^2-s^2}{r}$, or (making the radius $=1$) $2sc$ and c^2-s^2 for the *sine* and *cosine* of *double* the arc whose *sine* and *cosine* are denoted by s and c.

Therefore, if A be the arc whose *sine* is s, and *cosine* c; then the *sine* and *cosine* of $A+A$ (or $2A$) will be $2sc$, and c^2-s^2:

And by the same rule, the *sine* of $(A+A)+A$ (or $3A$) is $2sc \times c + s\,(c^2-s^2)$, or $3s\,c^2-s^3$:

And the *cosine* $(c^2-s^2)\,c - 2sc \times s$, or c^3-3sc. That is; if any *sine*, or *cosine*, be multiplied by $2c$, and the next preceding one by s^2+c^2, and the latter product subtracted from the former, the remainder is the next following *sine*, or *cosine*.

Hence we have

		sines.	*cosines.*
Arc	A	s	c
	$2A$	$2sc$	c^2-s^2
	$3A$	$3sc^2-s^3$	c^3-3s^2c
	$4A$	$4sc^3-4s^3c$	$c^4-6s^2c^2+s^4$
	$5A$	$5sc^4-10s^3c^2+s^5$	$c^5-10s^2c^3+5s^4c$
	$6A$	$6sc^5-20s^3c^3+6s^5c$	$c^6-15s^2c^4+15s^4c^2-s^6.$
		&c.	&c.

Where the law of continuation is manifest, it being such, that all the terms of the *cosine* and *sine* of any multiple arc nA (n being a positive integer) taken in order, are the terms of the binomial $(c+s)^n$. Thus, for example, to find the *sine* and *cosine* of $5A$:

$(c+s)^5=c^5+sc^4+10s^2c^3+10s^3c^2+5s^4c+s^5$, where the first, third, fifth, &c. terms being set down alternately $+$, and $-$, is the *cosine*: and the second, fourth, sixth, &c. connected in like manner, give the *sine*.

247. *The tangents of two arcs being T and t; to find the tangent of the sum of those arcs, and also of their difference.*

If the corresponding *sines* and *cosines* are denoted as in the preceding problem; then by *Trigonom.*

cosine : *sine* :: *radius* : *tang.* (*Geom.* 204)

viz. $Cc-Ss : Sc+sC :: 1$ (*radius*) : $\frac{Sc+sC}{Cc-Ss}$, *tang.* of the sum.

$Cc+Ss : Sc-sC :: 1 : \frac{Sc-sC}{Cc+Ss}$, *tangent* of the difference.

Now suppose m and n to represent the secants of the given arcs, respectively;

Then (*Geom.* 211)

$m : T :: 1$ (*radius*) : $\frac{T}{m} = S$ the *sine*:

$m : 1 :: 1 : \frac{1}{m} = C$ the *cosine*.

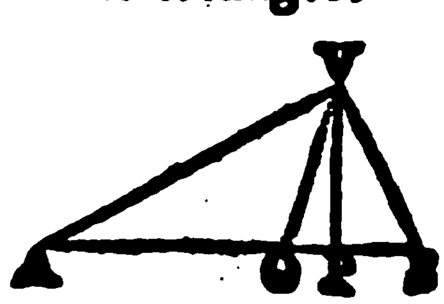
And in like manner we have $\frac{t}{n} = s$, and $\frac{1}{n} = c$: these values being substituted in the foregoing expressions for the tangents,

give $\frac{T+t}{1-Tt}$ the *tangent* of the *sum*.

and $\frac{T-t}{1+Tt}$ the *tangent* of the *difference*.

Remark. If an arc be greater than 90°, its tangent, cosine, &c. are to be used with the negative sign prefixed.

248. *Given the vertical angle* (AVB), *the perpendicular* (VP), *and the rectangle of the segments of the base made by the perpendicular,* (AP × PB); *to determine the triangle.*



Let VO bisect the verticle angle, and put $t =$ the *tangent* of AVO or OVB, and $x =$ the *tangent* of OVP; the *radius* being 1.

Then by the preceding problem $\frac{t+x}{1-tx}$ is the *tang.* of the angle AVP, and $\frac{t-x}{1+tx}$ that of the angle PVB.

Now if the rectangle of the segments be denoted by r, and the perpendicular VP by p, we shall have (by *trigonom.*)

$$1 : \frac{t+x}{1-tx} :: p : \frac{pt+px}{1-tx} = \text{PA}:$$

$$1 : \frac{t-x}{1+tx} :: p : \frac{pt-px}{1+tx} = \text{PB}:$$

Therefore $\frac{pt+px}{1-tx} \times \frac{pt-px}{1+tx} = \frac{p^2t^2-p^2x^2}{1-t^2x^2} = r;$

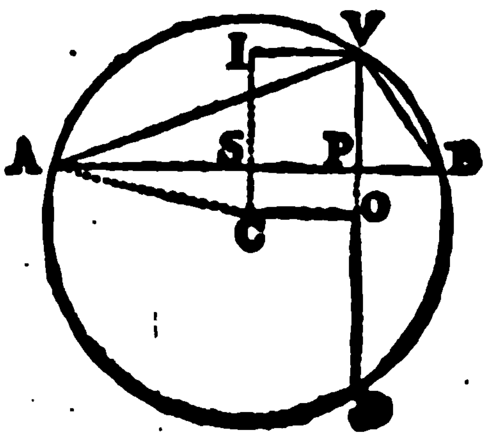
Whence, by reduction, $x = \sqrt{\frac{r-p^2t^2}{rt^2-p^2}}$ the *tang.* of OVP. This added to, and subtracted from $\frac{1}{2}$ the vertical angle, will give the angles AVP, and PVB; and thence the segments of the base, &c. may readily be found by trigonometry.

If p^2t^2 be greater than r, the problem is evidently impossible.

Example. Suppose the vertical angle $= 95°$, the perpendicular $= 60$, and the rectangle of the segments of the base $= 4400$:

Then $t = 1.0913085$ the natural *tang.* of $47\frac{1}{2}°$; and $\sqrt{\frac{r-p^2t^2}{rt^2-p^2}} = .26196$ the natural *tang.* of 14° 41′, nearly, the angle OVP; whence AVP $= 62°.11′$, and PVB $= 32° 49′$. And the sides are found by common trigonometry.

Geometrically.



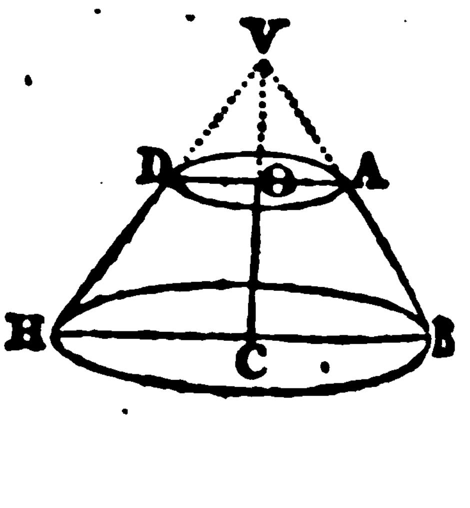
Suppose AVB is the triangle circumscribed by a circle whose center is C. Let the perpendicular VP be produced to D: then since AP × PB is given, and = VP × PD (*Geom.* 97), therefore PD is given, and consequently VD will

also be known. Join CA, and draw CO parallel to AB, and CS perpendicular to AB, then VD and AB are bisected in O and S, respectively (*Geom.* 65), and consequently CS = PO the difference of $\frac{1}{2}$ VD and the perpendicular PV. Also, the angle CAS is the difference of AVB and a right angle (*Geom.* 173). Hence the following *construction* is obvious:

Make the given rectangle AP × PB = m^2 (*Geom.* 184); and we get VP : m :: m : PD; then if a right angled triangle ASC be constructed having the angle SAC = the difference of the given vertical angle and a right one, and the opposite side SC = the difference of VP and $\frac{1}{2}$ VD, the side AS will be $\frac{1}{2}$ the base AB, and AC the radius of the circumscribing circle, which describe, and produce CS till SI = the given perpendicular; then draw IV parallel to the base AB; and the point V in the circle will be the vertex of the required triangle AVB.

Calculation. Taking the *data* as before, we have $\frac{AP \times PB}{VP} = \frac{4400}{60} = 73\frac{1}{3}$ = PD, whence PO or SC = $6\frac{2}{3}$; and the angle CAS = 5°, therefore (by *trigonom.*) as *radius* : *cotang.* 5° :: $6\frac{2}{3}$ (CS) : 76.2 = AS half the base. And, as *sin.* 5° : $6\frac{2}{3}$ :: *rad.* : 76.4914 = CA the radius of the circle, which also is the diagonal of the parallelogram CIVO; and VO being $66\frac{2}{3}$, we get CO = SP = 37.5 nearly, consequently the segment AP = 76.2 + 37.5 = 113.7, and PB = 76.2 − 37.5 = 38.7, nearly; whence AV and VB are readily found.

249. *To investigate the content of the frustum of a Cone or Pyramid.*



Let HDAB be the frustum, and suppose the cone or pyramid to be completed.

Put b^2 = the area of the base HB, t^2 = that of the top DA, h = the height CO, and x = OV, CV being the axis.

Then $b^2 : t^2 :: (h+x)^2 : x^2$ (*Geom.* 132)

or $b : t :: h+x : x$, whence $x = \frac{ht}{b-t} =$ OV, therefore CV $= h + \frac{ht}{b-t} = \frac{hb}{b-t}$.

And $\frac{1}{3}b^2 \times \frac{hb}{b-t} = \frac{\frac{1}{3}hb^3}{b-t}$ the content of HVB, (*Geom.* 133. *cor.* 2)

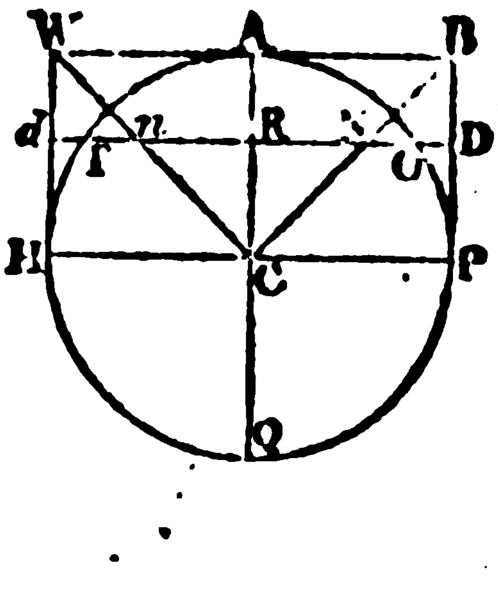
Also $\frac{1}{3}t^2 \times \frac{ht}{b-t} = \frac{\frac{1}{3}ht^3}{b-t}$ the content of DVA:

And the difference, or $\frac{\frac{1}{3}hb^3 - \frac{1}{3}ht^3}{b-t} = \frac{1}{3}h\left(\frac{b^3-t^3}{b-t}\right) = \frac{1}{3}h(b^2+bt+t^2)$.

viz. *The areas of the two ends and the mean proportional between them in one sum, multiplied by $\frac{1}{3}$ of the height*, is the content of the frustum HDAB. The same result as that in *art.* 283 mensuration.

250. *To find the content of a segment of a sphere.*

Let HP be the diameter of a sphere; TAO a segment; and suppose the hemisphere to be circumscribed by a cylinder HWBP. Put RA the height of the segment $= h$; TR or RO the radius of its base $= b$; and $m = .7854$.



Then $\frac{b^2}{h} =$ RQ (*Geom.* 97), and $\frac{b^2}{h} + h = \frac{b^2+h^2}{h} =$ AQ $=$ WB the diam. of the sphere, and $\frac{b^2+h^2}{2h} =$ the radius CA; whence $\frac{b^2+h^2}{2h} - h = \frac{b^2-h^2}{2h} =$ RC $=$ RN $=$ Rn, and $\frac{b^2-h^2}{h} =$ nN.

Now $\left(\frac{b^2+h^2}{h}\right)^2 \times m + \left(\frac{b^2-h^2}{h}\right)^2 \times m + \frac{b^2+h^2}{h} \times \frac{b^2-h^2}{h} \times m \times \frac{1}{3}h$ is the content of the conic frustum nWBN (*by the preceding article*):

Also $\left(\frac{b^2+h^2}{h}\right)^2 \times m \times h$ is the content of the cylinder dWBD:

And (*Geom.* 134. *cor.* 2) the difference of those expressions is the content of the segment TAO;

that is $\left(2\left(\frac{b^2+h^2}{h}\right)^2 - \left(\frac{b^2-h^2}{h}\right)^2 - \frac{b^2+h^2}{h} \times \frac{b^2-h^2}{h}\right) \times \frac{1}{3}mh$,

which, by reduction, becomes $\frac{6b^2+2h^2}{3} \times mh$, or, putting $B = 2b =$ TO, we have $(3B^2 + 4h^2) \times h \times \frac{1}{6}m$ the content of the segment.

In words, *To 3 times the square of the diameter of the base add 4 times the square of the height, then multiply the sum by the height, and that product by the decimal* .1309 (or $\frac{1}{6}$ of .7854), and the result is the content.

251. *A stone being let fall into a well, it was observed, that after being dropped, it was 4 seconds before the sound of the fall at the bottom reached the ear. Hence the depth of the well is required.*

Heavy bodies near the earth's surface descend by their own gravity $16\frac{1}{12}$ *feet* in the first second of time, $48\frac{3}{12}$ in the next, $80\frac{5}{12}$ in the third, and so on, constituting a series of distances in arithmetical progression, the first term being $16\frac{1}{12}$, and common difference $32\frac{2}{12}$. Now let $f = 16\frac{1}{12}$, $d = 32\frac{2}{12}$, $t = 4$ *seconds*, and $x =$ the sum of all the terms in the progression or depth of the well; also put $s = 1100$ *feet* the velocity of sound *per second:*

Then (138) we have $\sqrt{\frac{2x}{d}}$ for the number of terms or time in which the stone is falling to the bottom, ($2f = d$ in this case); and $\frac{x}{s}$ the time of the sound's ascent: therefore both times together must be equal to the whole time t;

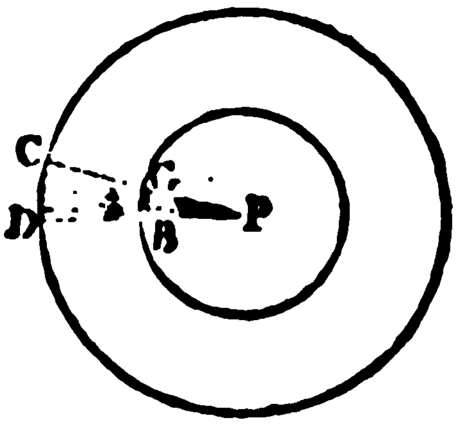
$$\text{viz. } \sqrt{\frac{2x}{d}} + \frac{x}{s} = t, \text{ whence } \sqrt{\frac{2s^2x}{d}} + x = st.$$

Now put $r = \sqrt{\frac{2s^2}{d}}$, and $z^2 = x$;

then $rz + z^2 = st$; whence we have $z = \sqrt{(\frac{1}{4}r^2 + st)} - \frac{1}{2}r$,

or $z^2 = x = (\sqrt{(\frac{1}{4}r^2 + st)} - \frac{1}{2}r)^2 = 231$ *feet*, nearly, by substituting the preceding numeral values of the different letters.

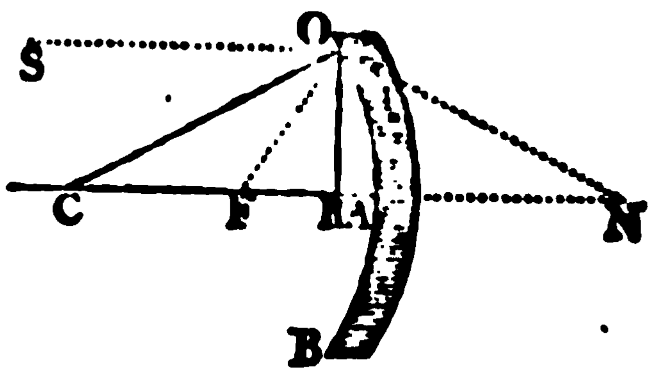
252. *To find the comparative intensity of light at different distances from a lucid point* P.



Let BPG and DPC be similar sectors of two circles described about P as a center.

Now the same light which issues from the point P, and is spread over the sector BPG, will also be diffused through the sector DPC. And it is evident the intensity of the light must grow less, as the space it occupies becomes greater: but the spaces BPG, DPC are as the squares of the radii PB and PD (*Geom.* 105. *corol.*); hence we conclude, that *the intensities are inversely as the squares of the distances*; that is, if the light at B and D are respectively denoted by N and n, then $PB^2 : PD^2 :: n : N$. And the same conclusion is evident in respect of *heat.* Thus if PD is double PB, then the heat at B will be 4 times that at D, supposing P is a point from which heat is emitted.

253. *Let* OAB *be a concave spherical speculum or mirror; to find the point* F *where a ray of light* SO *parallel to the axis* AD *meets the radius* CA *after being reflected from the incident point* O.



Let C be the centre of the spherical surface OAB; CO a radius, and OR perpendicular to the radius CA. Then because CO is perpendicular to the surface of the speculum at O, the angle of reflexion COF is equal to the angle of incidence COS, or the direct and reflected rays SO, OF make equal angles with the perpendicular CO. This is one of the known laws of optics.

Because SO is parallel to CR, the angles SOC, RCO are equal; but the angle COF = COS, therefore the triangle CFO

is isoceles, and FO = FC. Suppose ON = OC, then the triangle CON is isoceles, and RN = RC: moreover, as the angle at C is common to both the isoceles triangles, they must be equiangular or similar; hence CN (2CR) : CO :: CO : CF. Let the radius CO = r, CR = b, and x = CF; then the proportion is, $2b : r :: r : \frac{r^2}{2b} = x =$ CF, whence AF $= r - \frac{r^2}{2b}$ $= \frac{2br - r^2}{2b}$ the distance from the speculum where the reflected ray OF meets the axis.

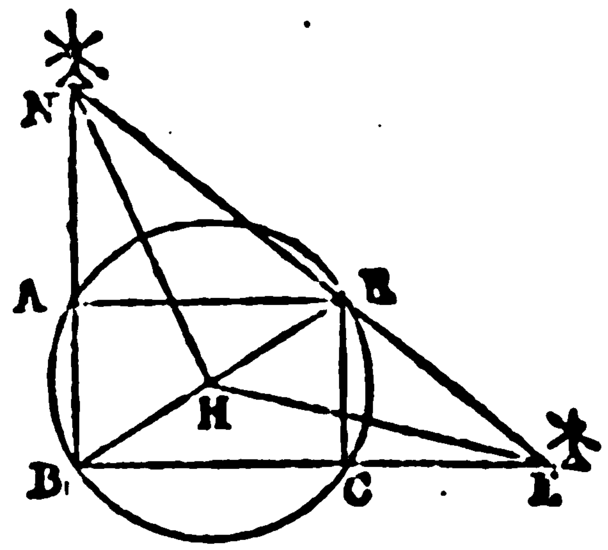
If the incident ray is indefinitely near the axis DA, then b may be taken $= r$, and x becomes $= \frac{1}{2}r$, or FC = FA.

Suppose the radius CO or $r = 6$ *feet* $= 72$ *inches*, and OR $= 20$ *inches*, then CR $= 69.166$ *inches* nearly, $= b$; whence CF $= 37.47$ *inches*. Hence it appears that when a spherical speculum or burning mirror is exposed to the rays of the sun, all the reflected rays are not collected into one point or *focus*, for the extreme rays, or those reflected from O and B, meet the axis at a greater distance from the centre C than those that are incident near the middle of the speculum at A. Thus in the present instance, the point F is 1.47 *inches* from the middle of the radius CA.

The *Geometrical construction* is obvious: For having drawn an incident ray SO (whether it be parallel to the axis CA, or not) join O, C, and make the angle of reflexion COF equal to the angle of incidence COS; then F is the point where the reflected ray meets the axis.

234. *In reconnoitring a country we observed two windmills, one bore* N. *the other* E. *we then proceeded 2 miles in a* NEbE *direction and came to a village that was at an equal distance from those objects; and when we had continued our route 2 miles further in the same direction, we found ourselves upon an height in a line directly between them. Hence the distance from one windmill to the other is required?*

Let N and E be the north and east windmills, B the place where they were first observed, BR the NEbE line, H the village, and R the height in the line NE. About H with the radius HB or HR (2 *miles*) describe a circle, and join RC, RA.



Then, since HE = HN, the points E, N, are equally distant from the circle, and therefore it follows from *Geom.* 98 or 99, that the rectangle (BC + CE) × CE is equal to the rectangle (BA + AN) × AN; whence we have

BC + CE : BA + AN :: AN : CE.

But the triangles ANR, CRE are similar,

whence AR (BC) : CE :: AN : RC (BA);

and by composition, BC + CE : BA + AN :: CE : BA;

therefore by equality (from the first proportion)

CE : BA :: AN : CE; consequently CE is a mean proportional between BA (or RC) and AN.

Also, by similar triangles, RC : CE :: AN : AR (BC); now the 2*d*. term CE being a mean proportional between the 1*st*. and 3*d*. terms RC and AN, the 4 terms RC, CE, AN, BC, are continued proportionals: Hence the question is reduced to that of finding 2 mean proportionals between two given lines (RC, BC), which is a *solid problem*, and consequently will not admit of a geometrical construction by means of right lines and the circle only (232).

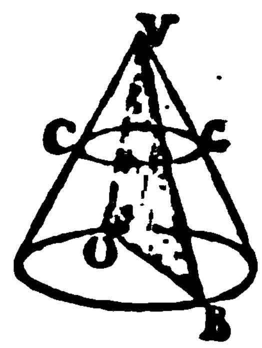
If $d = BR = 4$, $x =$ CE the 2d. term, and s and c are put to denote the *sine* and *cosine* of 33°.45′ the angle RBC (the *radius* being 1); then by *trigonom.* $sd =$ RC, and $cd =$ BC; and the 4 proportionals are sd, x, $\frac{x^2}{sd}$, cd, whence $x \times \frac{x^2}{sd} = scd^2$, which gives $x = \left(cs^2d^3\right)^{\frac{1}{3}} = 2.54196$ *miles*, nearly, = CE; whence by trigonometry, NE is found = 7.7946 *miles*.

CONIC SECTIONS.

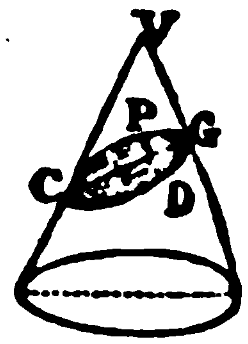
DEFINITIONS.

255. The figures denominated Conic Sections are made by a plane cutting a cone.

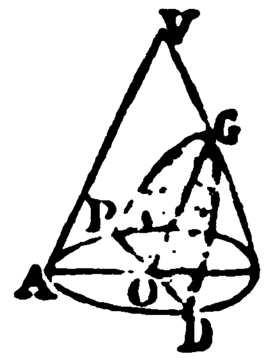
256. If the plane pass through the vertex of the cone, the section will evidently be a triangle. And if it be parallel to the base of the cone, the section is a circle (*Geom.* 121. *corol.*). Thus the section OVB is a triangle; and the section CG a circle. These however, are not called conic sections.



257. When the plane is inclined to the base of the cone, and cuts both sides, the section is an *ellipse*. Thus CG is an ellipse.

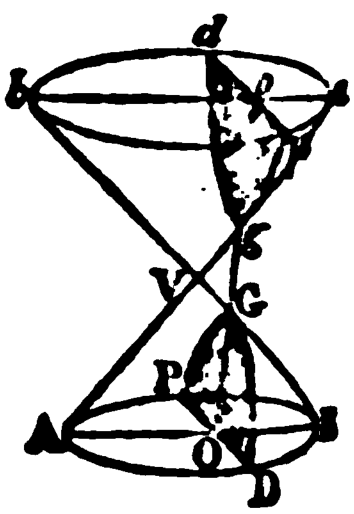


258. If the intersecting plane is parallel to the side, the section is called a *parabola*. Thus if the plane PGD is parallel to VA, the section PGD is a parabola.



259. But if the inclination of the plane be such that it cuts the *opposite cone* when produced, the section is an *hyperbola*.

Thus if V be the vertex common to both cones VAB, V*ab*, and the plane PDG when produced cuts the cone V*ab*, then PDG, *pgd*, are opposite hyperbolas. This section, the parabola, and ellipse, are exclusively called *Conic Sections*.



260. The *vertices* of any section are the points where the intersecting plane meets the opposite sides of the cone.

Thus C and G are the vertices of the ellipse, and G, *g*, those of the opposite hyperbolas. The parabola has only one vertex G.

261. The *axis* or *transverse diameter* is the line connecting the vertices.

Thus CG is the axis of the ellipse; and G*g* that of the hyperbola. But the axis of the parabola is infinite in length; GO being a part of that axis.

262. The *center* of any section is the middle of the axis; consequently that of the parabola is at an infinite distance from the vertex.

263. A *diameter* is any right line passing through the centre and terminated by the curve.

264. An *ordinate* to a diameter is any right line terminated by the curve, and bisected by that diameter.

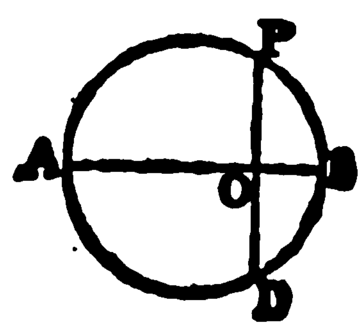
Thus PD is an ordinate to the axis CG in the ellipse, to G*g* in the hyperbola, and to GO in the parabola. The semiordinates OP, or OD, or O*d*, are also called ordinates.

Hence the ordinates to the axis are at right angles to it.

265. An *absciss* or *abscissa* is a part of any diameter contained between its extremities and an ordinate to it.

Thus OG, O*g*, are abscissas.

266. BEFORE we proceed to the properties of the sections, it may be necessary to explain what is understood by the *equation of a curve*.

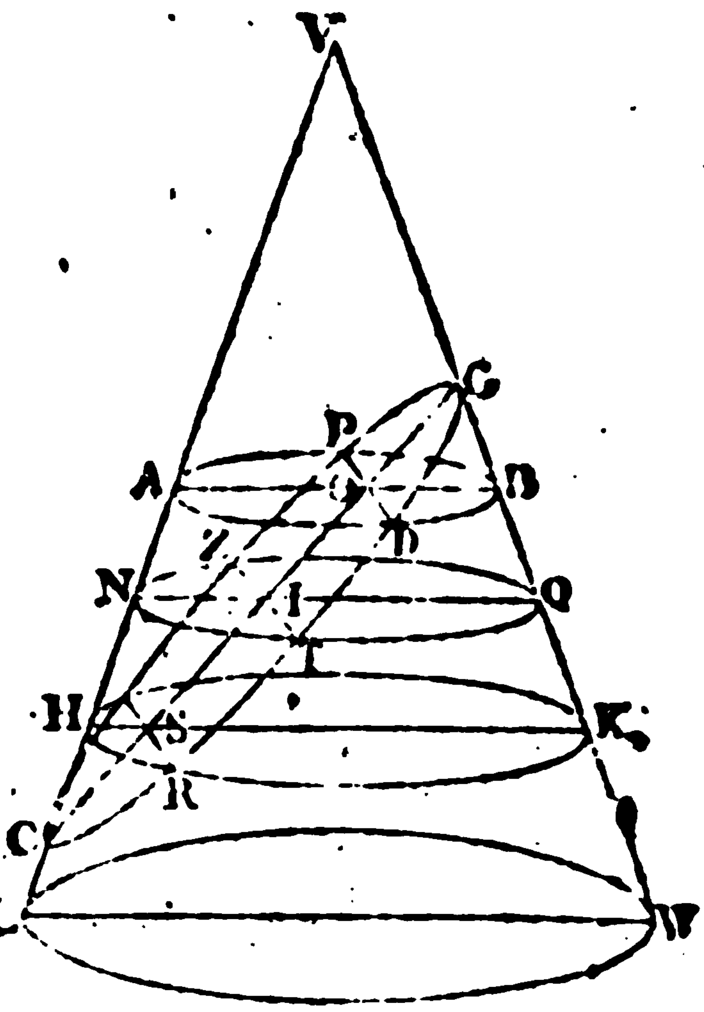


Let AB the diameter of a circle bisect the chord DP; then OD = OP, and AO × OB = OD² (*Geom.* 97. corol. 1), or (AB — OB) × OB = OD²; therefore if the diameter AB = *d*, AO or OB = *x*, and OD = OP = *y*;

then $(d-x)\times x = y^2$, or $dx - x^2 = y^2$ is the *equation of the circle*, and defines the nature of that curve by expressing the relation of the *absciss* x to the corresponding *ordinate* y at any point (O) in the diameter.

To find the Equations of the three Conic Sections.

267. Let the plane CPGDR cut the cone VLW; and suppose $GO = CS$; also let PD and SR be at right angles to GC, and the points P and D at equal distances from the vertex V: then if AB and HK are the diameters of the circular sections through O and S, PD and SR will be in the planes of those circles, and perpendicular to the diameters AB, HK, respectively.



Then the triangles SGK, OGB; OAC, SHC, being respectively similar, we have

$$SK : OB :: SG\ (OC) : OG\ (SC) :: OA : SH,$$

$$\text{whence } SK \cdot SH = OB \cdot OA.$$

And since OD and SR are in the planes of the circular sections whose diameters are AB, and HK, and also perpendicular to those diameters, therefore (Geom. 97. *corol.* 1) $SK \cdot SH = SR^2 = OB \cdot OA = OD^2$, consequently SR and OD are equal; that is, whatever be the curve CPGDR, the ordinates (SR, OD) to the axis CG, are equal at equal distances from the vertices C and G.

Now suppose NQ is the diameter of another circular section of the cone; then IT (in the plane of that circle, and also in

that of the oblique section CPGTR) will be perpendicular to CG and NQ, whence, by similar triangles,

$$SC : SH :: IC : IN,$$
$$SG : SK :: IG : IQ,$$

therefore (96) $SC.SG : SH.SK\ (SR^2) :: IC.IG : IN.IQ\ (IT^2)$;

$$\text{or } SR^2 : SC.SG :: IT^2 : IC.IG :$$

That is, *As the square of any ordinate* (SR^2),
Is to the rectangle of the corresponding abscissas (SC.SG),
So is the square of any other ordinate (IT^2),
To the rectangle of its corresponding abscissas (IC.IG).

If I be the center of the transverse axis CG, then TZ (at right angles to CG) is called the *conjugate* axis.

Let $CG = t$, $TZ = c$, any abscissa (OG or OC) $= x$, and the corresponding ordinate (OD) $= y$. Then the preceding analogy becomes

$$\tfrac{1}{2}t \times \tfrac{1}{2}t : \tfrac{1}{2}c \times \tfrac{1}{2}c :: (t-x)x : y^2,$$
$$\text{or } t^2 : c^2 :: tx - x^2 : y^2;$$

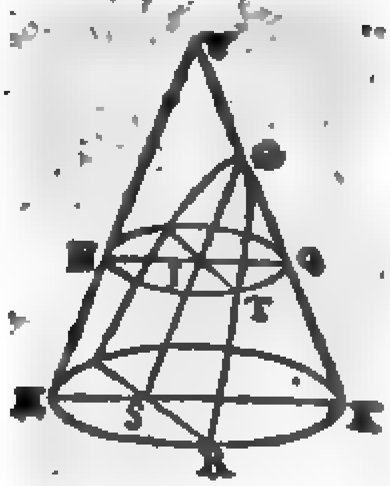
that is $\frac{c^2}{t^2}(tx - x^2) = y^2$. Which is the *equation of an ellipse*, or the *oval* CZPGDTR.

268. When the intersecting plane is parallel to the side of the cone, (PD being at right angles to CG, and its extremities P and D equally distant from the vertex of the cone, as before) then C will be at an infinite distance, and the axis infinite in length, in which case, we conclude that the rectangles SC.SG, IC.IG have the same ratio as SG and IG, consequently the proportion

$$SR^2 : SC.SG :: IT^2 : IC.IG$$

will become $SR^2 : SG :: IT^2 : IG$;

that is, *the abscissas are as the squares of their corresponding ordinates.* But the same conclusion may be obtained thus;



Since $IT^2 = NI \cdot IQ$, $SR^2 = HS \cdot SK$ (by the prop. of the circle), and $NI = HS$, therefore IT^2 and SR^2 will have the same ratio as IQ and SK; but by similar triangles

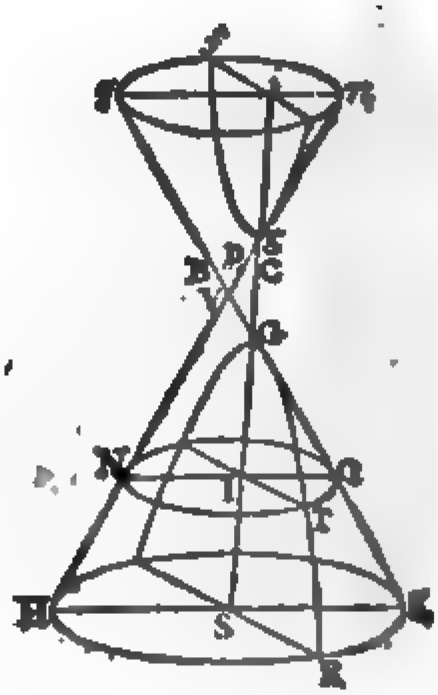
$IG : SG :: IQ : SK :: IT^2 : SR^2$ (by equality);

That is, the squares of the ordinates IT and SR are in the same proportion as the abscissas IG and SG.

Put the absciss $SG = t$, the corresponding ordinate $SR = c$, and x and y for any other absciss and its ordinate:

Then $t : c^2 :: x : y^2$, whence $\frac{c^2}{t} x = y^2$, or, putting $\frac{c^2}{t} = p$, we have $px = y^2$, *the equation of a parabola*, where the constant quantity $\frac{c^2}{t}$, which is a third proportional to the axis and its conjugate, is called the *latus rectum*, or *parameter* of the axis.

269. Let the intersecting plane cut the opposite cones; and suppose $gi = GI$.



By similar triangles,

$IQ : iq :: GI : Gi$,

$IN : in :: gI : gi$,

whence $IQ \cdot IN : iq \cdot in :: GI \cdot gI : Gi \cdot gi$; but these latter rectangles are equal because $Gi = gI$, and $GI = gi$, consequently the two former must also be equal, or $IQ.IN\ (= IT^2) = iq \cdot in = it^2$, hence IT and it are equal, and therefore the opposite hyperbolas are similar and equal.

Again, by similar triangles,

$SK : SG :: IQ : IG,$
$SH : Sg :: IN : Ig,$

hence we have $SK.SH : SG.Sg :: IQ.IN : IG.Ig:$
but $SK.SH = SR^2$, and $IQ.IN = IT^2$ (by prop. of circle),

and therefore $SR^2 : SG.Sg :: IT^2 : IG.Ig,$

that is, *the squares of the ordinates* (SR, IT) *have the same ratio as the rectangles of their corresponding abscissas* (SG, Sg, and IG, Ig), as in the ellipse.

If C be the center of the transverse axis Gg, and CB parallel to NQ or HK; then, by similar triangles,

$IQ : IG :: CB : CG,$
$IN : Ig :: CD : Cg,$

and $IQ.IN : IG.Ig :: CB.CD : CG.Cg:$

Now $IQ.IN = IT^2$, consequently the square of any ordinate (IT^2) and the rectangle of its corresponding abscissas (IG.Ig) have the constant ratio of the rectangle CB.CD to the square of the semi-transverse (CG.Cg), hence a mean proportional between CB and CD will be the semi-conjugate to the transverse Gg. Let this be denoted by $\frac{1}{2}c$, and put $t = Gg$, $x =$ any absciss IG, and $y =$ the ordinate IT; then $t + x = Ig$ the other absciss (instead of $t - x$ as in the ellipse), and we shall have

$y^2 : (t+x)x :: \frac{1}{2}c.\frac{1}{2}c : \frac{1}{2}t.\frac{1}{2}t,$

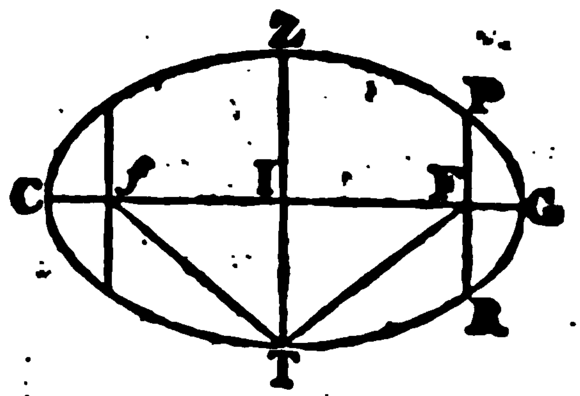
or $y^2 : tx + x^2 :: c^2 : t^2$; whence $\frac{c^2}{t^2}(tx + x^2) = y^2$, *the equation of an hyperbola.*

From the three equations thus obtained, we may derive the other principal properties of the curves by considering the sections in *plano* only, without any farther reference to the solid itself.

Of the Ellipse.

270. Suppose CG *and* ZT *are the transverse and conjugate axes, any absciss* $FG = x$, *and the corresponding ordinate* $FR = y$:

$$\textit{Then } \frac{ZT^2}{CG^2}(CG \times x - x^2) = y^2. \ (267)$$



Let the ordinate PR be a third proportional to the axes CG and ZT, that is $\frac{ZT^2}{CG} = PR$; then $\frac{ZT^2}{2CG} = FR$, and $\frac{ZT^4}{4CG^2} = y^2$;

therefore we have $\frac{ZT^2}{CG^2}(CG \times x - x^2) = \frac{ZT^4}{4CG^2}$,

or $CG \times x - x^2 = \frac{ZT^2}{4}$, this quadratic equation

gives $x = \frac{1}{2}CG \pm \sqrt{\left(\frac{CG^2}{4} - \frac{ZT^2}{4}\right)}$, these two roots or values of x are the abscissas FC and FG;

$$\text{or } FC = IG + \sqrt{(IG^2 - IT^2)},$$
$$\text{and } FG = IG - \sqrt{(IG^2 - IT^2)}.$$

But $FG = IG - IF$, therefore $\sqrt{(IG^2 - IT^2)} = IF$, and consequently $FT = IG$.

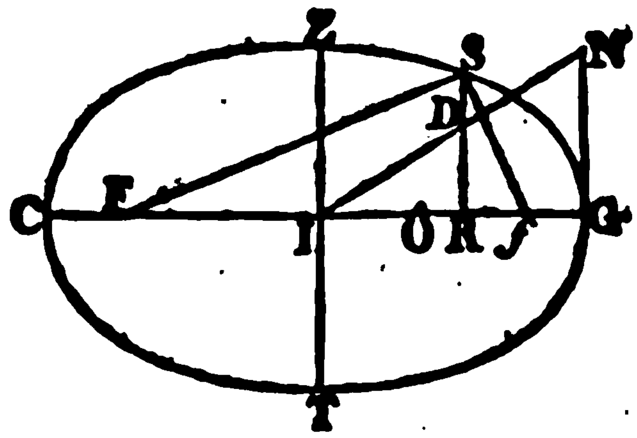
This ordinate PR, which is a third proportional to the axes, is called the *parameter* of the axis CG.

And if $If = IF$ then F and f are the *foci* of the ellipse.

Corol. 1. Hence Ff, the distance of the *foci*, is a mean proportional between the sum and difference of the axes. Or the distance of the *foci* from the center is a mean proportional between the sum and the difference of the semiaxes.

Corol. 2. And when the axes are given, the foci may be found by making TF, T*f*, each = the semitransverse.

271. *If two lines are drawn from the foci to meet at any point in the curve, their sum will be equal to the transverse axis: that is,* $FS + fS = CG$.



Make GN perpendicular to IG, and = IZ the semiconjugate, and join IN; also let RS be an ordinate at right angles to IG.

Then the triangles IRD, IGN being similar, we have

$$IG^2 : GN^2\ (IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IZ^2 :: IG^2 - IR^2 : IZ^2 - RD^2$ (by alternation and division).

And $IG^2 : IZ^2 :: IG^2 - IR^2$ (or $\overline{IC + IR} . \overline{IG - IR}$) $: RS^2$, (267)

$$\text{whence by equality } RS^2 = IZ^2 - RD^2.$$

Now $FR = FI + IR$, and $FR^2 = FI^2 + 2FI \times IR + IR^2$:

but $FS^2 = FR^2 + RS^2$;

whence $FS^2 = FI^2 + 2FI \times IR + IR^2 + IZ^2 - RD^2$ (by addition),

or $FS^2 = FI^2 + IZ^2 + 2FI \times IR + IR^2 - RD^2$;

but $FI^2 + IZ^2 = IG^2$, (270. corol. 1)

whence $FS^2 = IG^2 + 2FI \times IR + IR^2 - RD^2$ (by substitution).

Let IO be a *4th.* proportional to CG, F*f*, and IR, that is, $2IG : 2FI :: IR : IO$; then $2FI \times IR = 2IG \times IO$ this substituted for $2FI \times IR$ in the last equation, and we have

$$FS^2 = IG^2 + 2IG \times IO + IR^2 - RD^2.$$

Again, since $2IG : 2FI :: IR : IO$, or $IG : IR :: FI : IO$, we get $IG^2 : IR^2 :: FI^2$ (or $IG^2 - IZ^2$) $: IO^2$;

But from the similar triangles IRD, IGN,

$$IG^2 : GN^2 \ (= IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IG^2 - IZ^2 :: IR^2 : IR^2 - RD^2$ (by division),

or $IG^2 : IR^2 :: IG^2 - IZ^2 : IR^2 - RD^2$.

Also $IG^2 : IR^2 :: IG^2 - IZ^2 : IO^2$ (from the 4th. proportional), whence by equality, $IR^2 - RD^2 = IO^2$ this substituted for $IR^2 - RD^2$ in the latter of the preceding values of FS^2,

and the result is $FS^2 = IG^2 + 2IG \times IO + IO^2$;

and the square roots give $FS = IG + IO$ or $CI + IO$.

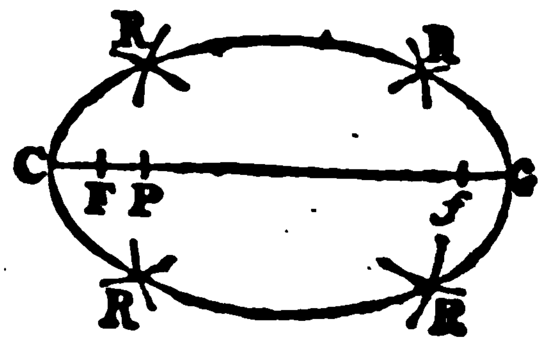
And by proceeding exactly in the same manner with $fR^2 = (fI - IR)^2$ instead of $FR^2 = (FI + IR)^2$, we shall get $fS = IG - IO$,

Therefore FS and fS are respectively equal to OC and OG, the two parts of the transverse diameter.

Corol. Hence is derived the common method of describing this curve with a thread, thus:

Let the thread be equal in length to the transverse CG, and fix its ends at the foci F and f, then move a pencil or pen round by the thread, keeping it always stretched, and the point of the pencil or pen will describe the curve.

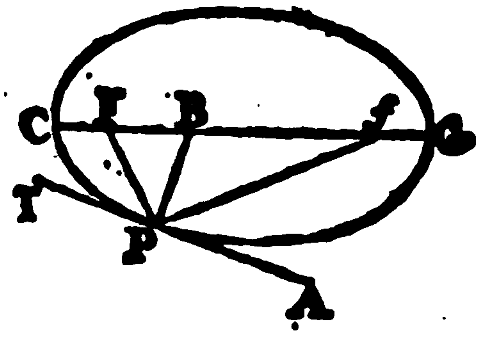
Or the curve may be traced mechanically, thus:



Take any point P in the transverse, and with PC, PG as radii, about the foci F, f, describe arcs intersecting each other in R, R, R, R which will be 4 points in the curve; and the like number may be found by assuming another point in CG, and so on: the curve is then to be drawn through the points of intersection.

272. *To draw a tangent to an ellipse at a given point* P *in the curve.*

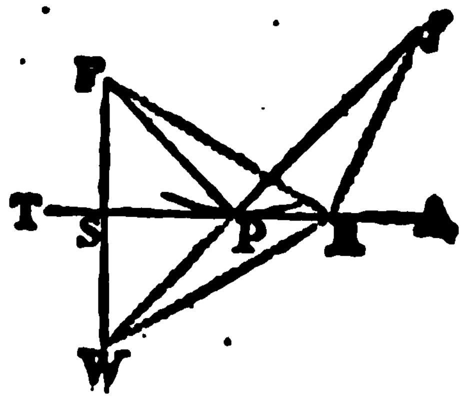
Let the point and the foci F, *f*, be joined, and bisect the angle FP*f* with PB; then TA drawn through P at right angles to PB, or to make the angle FPT $=f$PA, will be the tangent required.



The truth of this construction will be manifest from the following

THEOREM. *If from two given points* F, *f*, *two lines* FP, *f*P *be drawn to meet at a given right line* TA, *and make equal angles* FPT, *f*PA, *the lines* FP, *f*P *taken together will be less than any other two lines* FR, *f*R, *drawn from the same points to meet on the same line* TA.

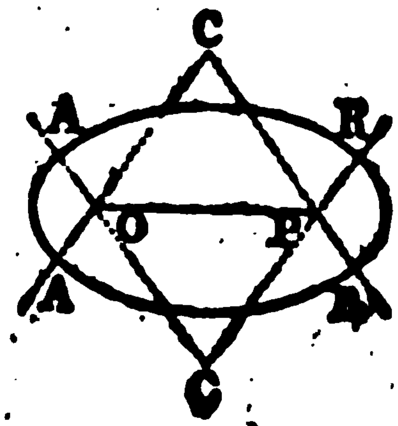
Draw FW at right angles to TA, and produce *f*P to meet it in W; also let WR be joined;



Then since the angle FPS $=f$PR $=$ SPW, the right angled triangles SPF, SPW are similar and equal, and therefore FP $=$ WP, and FP $+ f$P $=$ W*f*; but WR $=$ FR, whence FR $+ f$R $= f$R $+$ RW which is greater than W*f* or its equal FP $+ f$P.

The following method of drawing an Oval is frequently practised by workmen.

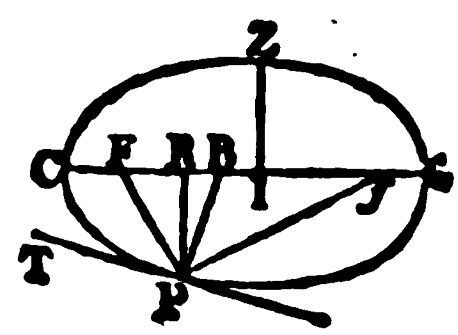
Two equal isoceles triangles OCP, OCP, are constructed on a common base OP, and the sides CO, CP produced; then C, C, are the centers of the circular arcs AR, AR (decribed with a pair of common compasses) and O, P, the centers of the arcs AA and RR.



Hence, if F and f be the foci of an ellipse, and $FP+fP$ equal to the transverse axis, it follows that an elliptical arc described with a thread (as directed in the preceding corollary) will pass through the point P and touch the line TA in that point: for if $FP+fP$ be *increased*, the curve thus described must cut the line in two points, but when *diminished*, it can neither intersect nor touch it, as is evident from the description of the curve.

It has already been observed (253) that opticians find when a ray of light falls on a reflecting surface, the angles of incidence and reflexion are equal; hence, if either focus be a lucid point, and the concavity of the ellipse a polished surface, the rays issuing from that point or focus will be reflected to the other: thus the angle $FPB = fPB$, and the ray FP is reflected in the direction Pf; hence it is, that F, f, are called *foci* or *burning points.*

273. *Let* TP *be a tangent to the ellipse,* F, f, *the foci,* I *the center, and* PR *an ordinate to the axis* CG; *then* $IC^2 : IF^2 :: IR : IB$.



Since PB bisects the angel FPf we have $fP : FP :: fB : FB$ (Geom. 96) whence $fP + FP : fP - FP :: FB + fB : fB - FB$, (by composition and division)

$$\text{or } 2IC : fP - FP :: 2IF : 2IB,$$

$$\text{or } IC : \frac{fP - FP}{2} :: IF : IB,$$

$$\text{and } IC^2 : \frac{fP - FP}{2} \times IC :: IF : IB. \ (92)$$

But in the triangle FPf, I is the middle of the base Ff, PR the perpendicular on that base, and $fP + FP$ (or $2IC$) is the sum of the sides;

whence (245) $Ff : fP + FP$ (or $2IC$) $:: fP - FP : 2IR$,

and $\frac{Ff}{2}$ (or IF) $: IC :: \frac{fP - FP}{2} : IR$,

whence $\frac{fP - FP}{2} \times IC = IF \times IR$:

Therefore $IC^2 : IF \times IR :: IF : IB$ (by substitution),

or $IC^2 : IF \times IR :: IF^2 : IB \times IF$, (92)

and $IC^2 : IF^2 :: IF \times IR : IB \times IF$ (by alternation),

that is $IC^2 : IF^2 :: IR : IB$.

Corol. Let IZ be the semiconjugate axe;

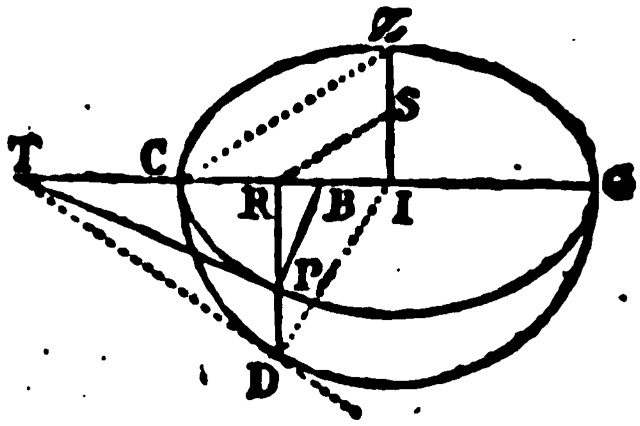
then since $IC^2 : IF^2 :: IR : IB$,

we have $IC^2 : IC^2 - IF^2 :: IR : IR - IB$ (or BR), by division,

that is $IC^2 : IZ^2 :: IR : BR$, because $IC^2 - IF^2 = IZ^2$. (270)

274. *If* TP *be a tangent,* CG *the transverse axis,* IZ *the semiconjugate,* PB *perpendicular to* TP, *and* PR *an ordinate; then* IC *is a geometrical mean between* IR *and* IT.

Let RS be parallel to CZ.



Then $IC^2 : IZ^2 :: IR^2 : IS^2$ (by sim. triang.)

But $IC^2 : IZ^2 : IR : BR$ (273. corol.) $:: IR^2 : IR \times BR$;

whence $IR^2 : IR \times BR :: IR^2 : IS^2$ (by equality),

and therefore $IR \times BR = IS^2$.

Moreover, the right angled triangles TRB, PRB being similar,

we have $TR : RP :: RP : BR$,

whence $TR \times BR = RP^2$.

But $IC^2 : IZ^2 :: IC^2 - IR^2 : IZ^2 - \frac{IR^2 . IZ^2}{IC^2} = RP^2$. (267).

And by sim. triang. $IC^2 : IZ^2 :: IR^2 : \frac{IR^2 . IZ^2}{IC^2} = IS^2$,

whence, by substitution $IZ^2 - IS^2 = RP^2 = TR \times BR$,

or $IZ^2 = IR \times BR + TR \times BR = (IR + TR) BR = BR \times IT$.

And since BR : IR :: BR $\times$ IT : IR $\times$ IT, (92)
that is BR : IR :: IZ^2 : IR $\times$ IT (by substitution):
Also BR : IR :: IZ^2 : IC^2, (273 corol.)
therefore $IC^2 = IR \times IT$; that is, IC is a mean proportional between IR and IT.

Corol. 1. If TD be a tangent to a circle described on the transverse CG, and the points D and I joined, also suppose DR to be perpendicular to CG: then the angle TDI being a right one, we have, by sim. triangles, TI : DI (or IC) :: DI : IR, therefore IC is also a mean proportional between IR and IT in the circle, consequently the point D is in RP produced. Hence it follows, that if any number of ellipses have the same transverse (CG), the tangents drawn from the points (P, &c.) where an ordinate (RD) intersects them, will all meet in the same point (T) in the transverse produced. For IR and IT remain the same, whatever be the length of the conjugate.

Corol. 2. Since $IC^2 : IZ^2 : RC \times RG : RP^2$, (267) in the ellipse;

and $RD^2 = RC \times RG$ in the circle,
it will be $IC^2 : IZ^2 :: RD^2 : RP^2$,
or IC : IZ :: RD : RP;

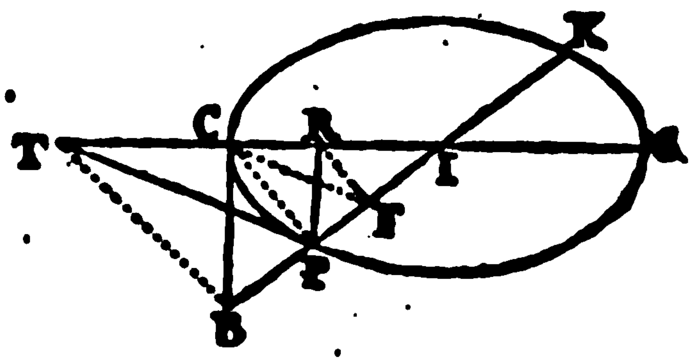
that is, the ordinates RP, RD, have always the same ratio as the semiaxes, or axes of the ellipse:

or *transverse* : *conjugate* :: RD : RP.

Corol. 3. Hence also, *the area of the ellipse will be a geometrical mean between the circles described on the two axes.* For if ordinates (RD, &c.) be drawn from every point in CG, the sum of all the RD's constitutes the area of the semicircle CDG, and the sum of all the RP's that of the semi-ellipse CPG; but the sum of the former to that of the latter is as RD to RP, or as the transverse axe to the conjugate. And any corresponding segments have evidently the same proportion.

Hence, if t be put to denote the transverse, and c the conjugate, then $.7854t^2$ is the area of the circle described on the transverse, and $t : c :: .7854t^2 : .7854tc$ the area of the ellipse, which is a mean proportional between the two areas $.7854t^2$ and $.7854c^2$.

275. *If* CG *be the transverse axis,* TP *and* CB *tangents at* P *and* C, *respectively, and* KPB *a diameter produced; then the triangles* IPT, ICB *will be equal.*



Draw the ordinate PR, and let CF be parallel to TP.

Then the triangles PRI, BCI; CFI, TPI being respectively similar, we have

IF : IP :: IC : IT :: IR : IC (274) :: IP : IB, (by sim. triang.)

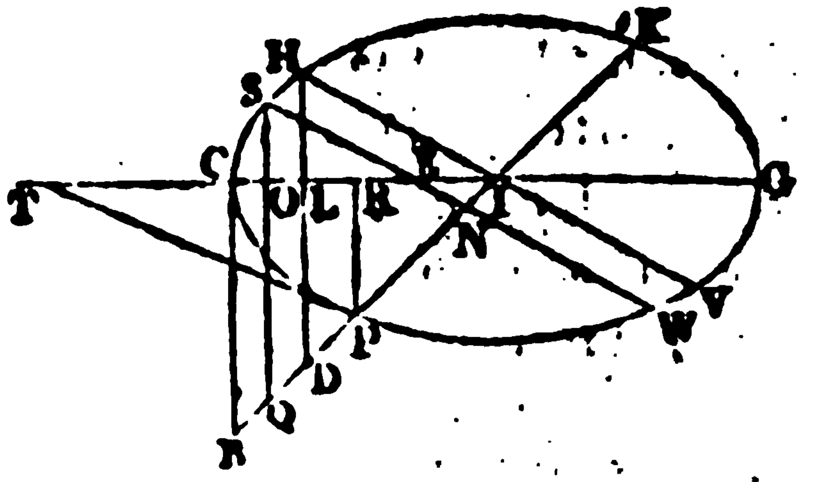
that is, IF : IR :: IP : IC,
and IC : IP :: IT : IB

therefore (Geom. 94. *corol.* 1) RF, CP, TB are parallel to each other.

Now the triangles TCP, BCP on the same base CP, and between the parallels CP, TB, are equal, therefore adding CPI to each we have IPT = ICB.

Corol. The triangle PRT = trapezoid CRPB; this appears by taking the triangle PRI from each of the triangles IPT, ICB.

276. *Let the diameter* HV *be parallel to the tangent* TP, *then* PK *and* HV *are called conjugate diameters. And if* SN *be an ordinate to* PK, *and* HD, SQ *parallel to the tangent* CB, *the triangle* SOE = *trapezoid* COQB.



Let PR be an ordinate to CG:

By similar triangles,

$IC:CB::IR:RP::IO:OQ$, whence by alternation and division,

$IC : CB :: IC+IR : CB+RP :: IC+IO : CB+OQ$,

or $IC+IO : IC+IR :: CB+OQ : CB+RP$ (by alternation).

But

$OC \times OG = OC.(IO+IC)$, and $RC \times RG = RC.(IR+IC)$; whence

$OC \times OG : RC \times RG :: OC.(IO+IC) : RC.(IR+IC)$,

or

$OC \times OG : RC \times RG :: OC.(CB+OQ) : RC.(CB+RP)$ by equality;

$$:: OC.\left(\frac{CB+OQ}{2}\right) : RC.\left(\frac{CB+RP}{2}\right),$$

$:: trapez.\ COQB : trapez.\ CRPB.$

Moreover, the triangles PRT, SOE being similar, we have *triang*. SOE : *triang*. PRT :: $SO^2 : RP^2 :: OC \times OG : RC \times RG$, (267)

or *triang*. SOE : *triang*. PRT :: *trapez*. COQB : *trapez*. CRPB, (by equality):

But (275 corol.) *triang*. PRT = *trapez*. CRPB, therefore, by equality, the *triang*. SOE = *trapez*. COQB. And in the same manner it is proved that the *triangle* HLI = *trapez*. CLDB.

Corol. From the equal triangles IPT, ICB, take the triang. EIN, then the trapez. TPNE = quadrilateral BCEN; from this subtract the trapez. COQB, and add its equal, or the triang. SOE, to the remainder QOEN, and we have the triang. SQN = trapez. TPNE. In like manner, by subtracting CLDB from the triang. ICB, and adding its equal HLI, we get the triang. HDI = triang. ICB = triang. IPT.

277. *If* PK *and* HV *be conjugate diameters, and* SN *an ordinate, as in the preceding Theorem,*

Then $IP^2 : IH^2 :: NP \times NK : NS^2$.

By similar triangles,

triang. IPT : *triang.* EIN :: $IP^2 : IN^2$;

and IPT : IPT − EIN :: $IP^2 : IP^2 - IN^2$ (by division),

that is, *triang.* IPT : *trapez.* TPNE :: $IP^2 : (IP + IN) . (IP - IN)$, or $NK \times NP$.

Moreover, since the triangles HDI, SQN are similar, we have

$IH^2 : NS^2 ::$ *triang.* DHI : *triang.* SQN;

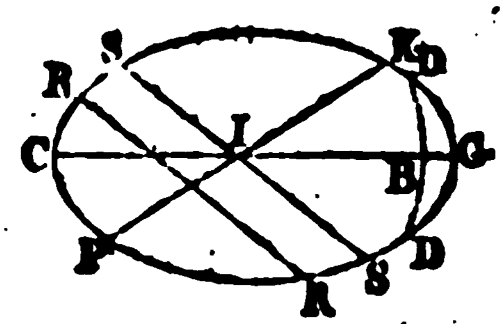
that is $IH^2 : NS^2 ::$ *triang.* IPT : *trapez.* TPNE (by equality) $:: IP^2 : NK \times NP$;

or alternately, $IP^2 : IH^2 :: NK \times NP : NS^2$. And in the same manner it is proved that $IP^2 : IV^2 :: NK \times NP : NW^2$; therefore $NW = NS$.

Corol. 1. Hence any diameter bisects all its double ordinates.

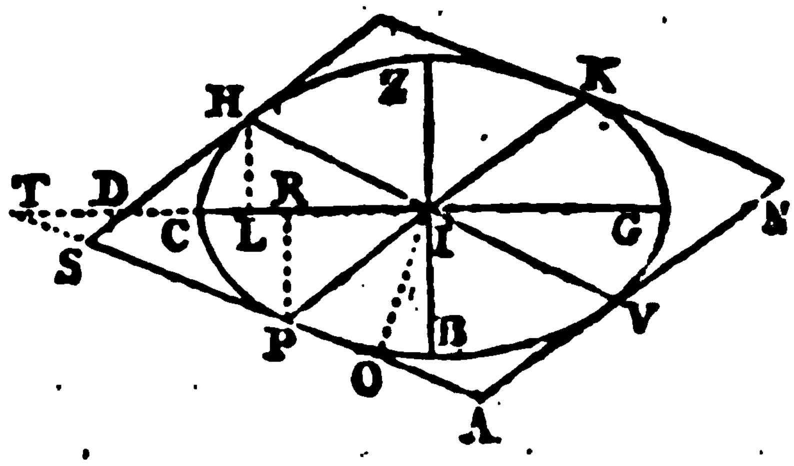
Corol. 2. And the property demonstrated in regard to the transverse axis (Art. 267) is general for any diameter whatever, viz. *the rectangles of the segments of any diameter, are as the squares of their corresponding ordinates.*

Corol. 3. Hence also is derived the method of finding the center and axes of an ellipse; thus,



Draw two parallel ordinates or lines SS, RR, and bisect them with PK which will be a diameter; then about I the center, or middle of PK, describe an arc of a circle meeting the ellipse in two points DD, bisect DD in B, and through B, I, draw GC which will be the transverse. If the arc falls without the ellipse it gives the conjugate.

278. *Every parallelogram circumscribing an ellipse is equal to the rectangle of the two axes.*



Let PK, HV be conjugate diameters, then the tangents through their extremities will form a parallelogram SN. (276).

Draw the axes CG, ZB, and the ordinates PR, HL, also suppose GC, AS are produced till they meet in T, and let IO be perpendicular to SA.

Then (274) $IC^2 = IR.IT$, and $IC^2 = IL.ID$;
therefore $ID : IT :: IR : IL$.

But the triangles IHD, TPI; IHL, TPR are respectively similar,
whence $ID : IT :: IL : TR$,
therefore by equality, $IR : IL :: IL : TR$, or $IL^2 = IR.TR$:
Also, because $IC : IR :: IT : IC$, (274)
we have $IC - IR$ (or RC) $: IR :: IT - IC$ (or CT) $: IC$, by division,
or alternately, $IR : IC :: RC : CT$;
and $IR + IC$ (or RG) $: IR :: RC + CT$ (or RT) $: RC$, by composition,
or alternately, $IR : RG :: RC : RT$, hence $IR.TR = IL^2 = RG.RC$.

Moreover, because $IC^2 : IZ^2 :: RC.RG : RP^2$, (267)
or $IC^2 : IZ^2 :: IL^2 : RP^2$ (by equality),
we have $IC : IZ :: IL : RP$.
In like manner, since $IC^2 : IZ^2 :: LC.LG : LH^2$,
we shall get $IC : IZ :: IR : LH$,
or $IZ : LH :: IC : IR$ (by alternation).
But (274) $IT : IC :: IC : IR$,
therefore by equality $IT : IC :: IZ : LH$, or $IT.LH = IC.IZ$.
But the triangles IOT, ILH being similar, we have
$IH : IT :: LH : IO$, or $IT.LH = IH.IO$;
and therefore the rectang. $IH.IO = IC.IZ$, and $4IH.IO = 2IC.2IZ$.

But the rectang. $IH . IO$ is $=$ the parallelogram PIHS or $\frac{1}{4}$ of the circumscribing parallelogram SN, consequently $2IC . 2IZ$ or the rectang. $CG . ZB =$ the parallelogram SN.

279. *The sum of the squares of any two conjugate diameters, is equal to the sum of the squares of the two axes. That is,* $PK^2 + HV^2 = CG^2 + ZB^2$: (see the preceding fig.)

By the last Theo. $IL^2 = RG . RC = IC^2 - IR^2$ (or $\overline{IC + IR} \times \overline{IC - IR}$),

therefore $IL^2 + IR^2 = IC^2$.

But $IC^2 : IZ^2 :: IL^2 : RP^2$,

and $IC^2 : IZ^2 :: IR^2 : LH^2$,

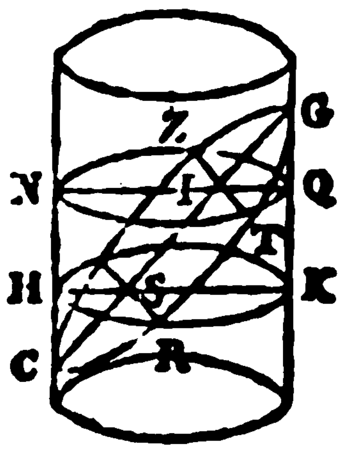
whence $2IC^2 : 2IZ^2 :: IL^2 + IR^2 : RP^2 + LH^2$, (by composition)

that is $2IC^2 : 2IZ^2 :: IC^2 : RP^2 + LH^2$;

therefore by equality $RP^2 + LH^2 = IZ^2$:

Hence the sum of the 4 squares $IL^2 + IR^2 + RP^2 + LH^2 = IC^2 + IZ^2$ the sum of the squares on the semiaxes: but (Geom. 83) the sum of those 4 squares are equal to $IH^2 + IP^2$ the squares on the semiconjugates; therefore $4IH^2 + 4IP^2 = 4IC^2 + 4IZ^2$, or $HV^2 + PK^2 = CG^2 + ZB^2$.

280. *If a cylinder be cut by a plane oblique to its axis, the section is an ellipse.*



Let CZGT be the section, HRK, NTQZ, two sections parallel to the base of the cylinder. Then the equation of the curve is derived exactly as in the cone, (267); thus

By similar triangles,

$SC : SH :: IC : IN$,

$SG : SK :: IG : IQ$,

whence (96) $SC.SG : SH.SK$ (or SR^2) $:: IC.IG : IN.IQ$ (IT^2 or IZ^2),

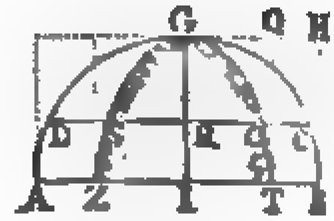
or $SR^2 : SC.SG :: IT^2 : IC.IG$,

That is, *the squares of the ordinates are as the rectangles of the corresponding abscissas.*

The conjugate axis ZT is the diameter of the cylinder, I being the center of the transverse CG.

If CG $= t$, ZT $= c$, SC or SG $= x$, SR $= y$, then the foregoing proportion gives $\frac{c^2}{t^2}(tx - x^2) = y^2$ *the equation of the ellipse.*

281. *The spheroid or solid generated by the revolution of an ellipse about either axis, is $\frac{2}{3}$ of the circumscribing cylinder.*



Let ZTG be a semiellipse, ZT the conjugate axis, IG the semitransverse, and AGB a semicircle described about the center I.

Then if the ellipse and circle revolve about the axis IG, the former will describe a hemispheroid, and the latter an hemisphere.

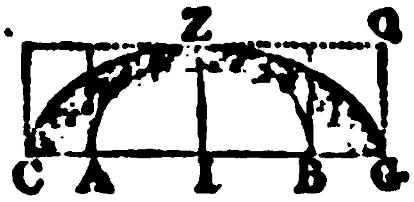
Suppose DC to be a plane parallel to the base AB; then SO and DC will be the diameters of the circular sections of the two solids made by that plane. And because AB : ZT :: DC : SO at every point in IG (274) the surfaces of the corresponding circular sections will be in the constant ratio of AB^2 to ZT^2: if therefore we conceive the two solids to be composed of an infinite number of indefinitely thin elementary circular parallel planes, (Geom. 131) the sum of those in the hemisphere AGB will be to the sum of those in the hemispheroid ZGT, as AB^2 to ZT^2,

that is, $AB^2 : ZT^2$ :: *solid* AGB : *solid* ZGT :

But if AH, ZQ are the cylinders circumscribing the two solids, then $AB^2 : ZT^2$ :: *cylind.* AH : *cylind.* ZQ, (Geom. 131. *corol.* 3)

therefore by equality, *solid* AGB : *solid* ZGT :: *cylind.* AH : *cylind.* ZQ :

But (Geom. 134) the solid or hemisphere AGB $= \frac{2}{3}$ of the cylinder AH, consequently (by the proportion) the solid or hemispheroid ZGT $= \frac{2}{3}$ of the cylinder ZQ.

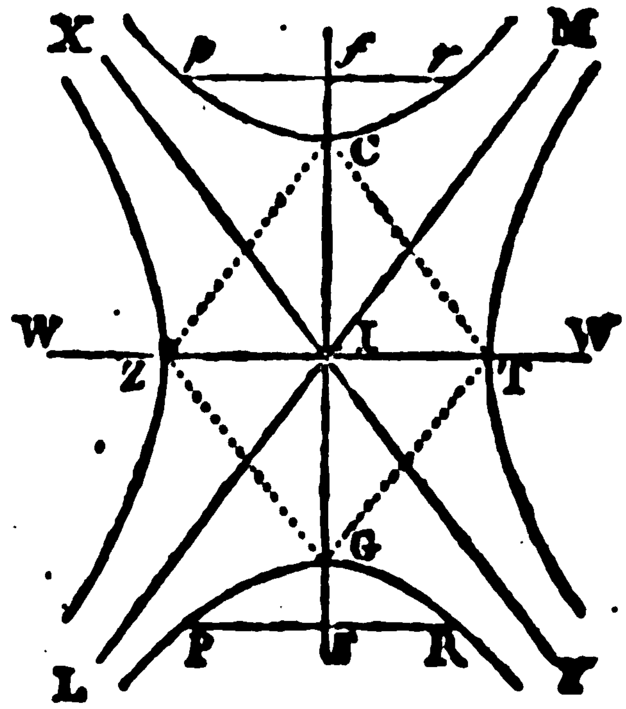


And if the ellipse, and the semicircle described about I with the radius IZ, revolve about the semiconjugate axis IZ, then, in the same manner, it is proved that half the spheroid (or the solid CZG) is equal to ⅔ of the circumscribing cylinder CQ. This is called an *oblate* spheroid, or ellipsoid. But when the transverse diameter of the ellipse is the fixed axis, the solid is a *prolate* spheroid.

Corol. $AB^2 : ZT^2$ (or $DC^2 : SO^2$) :: *spherical segment* DGC : *spheroidal segment* SGO.

Of the HYPERBOLA.

282. The equation of the curve, or fundamental property, is already derived from the cone (269); but in considering the section in *plano*, the following definitions will be necessary.



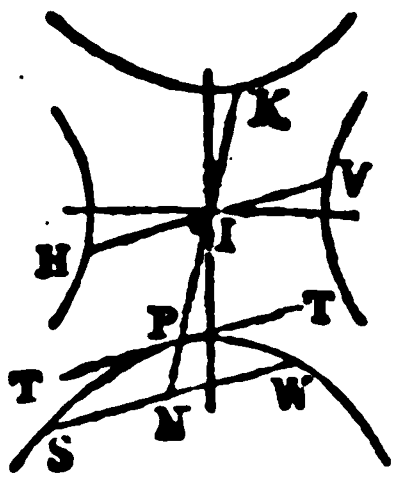
1. If PGR, pCr be the opposite hyperbolas, GC the transverse axis, I its center, F, f, the foci, WIW at right angles to CG, and GZ, GT, CZ, CT, each equal to IF or If, then ZT is the conjugate axis.

2. When the conjugate axis is equal to the transverse, or ZT =CG the curve is called an equilateral hyperbola, or right angled hyperbola.

3. The line PFR at right angles to the axis CF, is called the *parameter*, or *latus rectum*.

4. Two indefinite right lines LM, XY, drawn through the center I parallel to GT, GZ, are called the *asymptotes* of the hyperbola, or of the opposite hyperbolas.

5. If ZT, CG, are made the transverse, and conjugate to two other hyperbolas, whose vertices are Z and T, those are called *conjugate hyperbolas*, with respect to the former.



6. Any right line PK drawn through the center I, and terminated at the opposite sections, is called a *diameter*, and the extremities P, K, its vertices: and HV parallel to TT the tangent at P, is called its *conjugate diameter*.

7. If any diameter KP be continued within the curve, the inner part PN is called the abscissa; and SW parallel to the tangent TT is a double ordinate to the diameter KP.

283. *The square of the distance of the focus from the center is equal to the sum of the squares of the semiaxes: viz.* $If^2 = IG^2 + IZ^2$. (see the following *fig.*)

Let IZ be the semiconjugate, fP the semiparameter or third proportional to the semitransverse IG and semiconjugate IZ:

Then the equation of the curve (269) gives

$IG^2 : IZ^2 :: (If + IG) . (If - IG)$ or $If^2 - IG^2 : fP^2$:

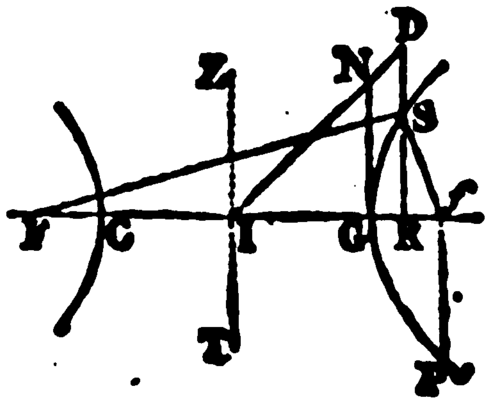
But $IG : IZ :: IZ : fP$ (the semiparam.)

and $IG^2 : IZ^2 :: IZ^2 : fP^2$,

therefore by equality $IZ^2 = If^2 - IG^2$, or $IZ^2 + IG^2 = If^2$; instead of $IG^2 - IZ^2 = If^2$ as in the ellipse, (270 corol. 1).

Corol. Hence the distance from Z to G set on the axis from the center I both ways, give the foci f, F.

284. *If two lines are drawn from the foci to meet at any point in the curve, their difference will be equal to the transverse axis: that is* $FS - fS = CG$.



Make GN perpendicular to IG, and $= IZ$ the semiconjugate, and join IN; also let RS be an ordinate at right angles to IG.

Then RS being produced to D, the triangles IRD, IGN, will be similar, and we shall have

$$IG^2 : GN^2\ (IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IZ^2 :: IR^2 - IG^2 : RD^2 - IZ^2$ (by alternation and division),

And $IG^2 : IZ^2 :: (IR + IG).(IR - IG)$ or $IR^2 - IG^2 : RS^2$, (269)

whence by equality $RS^2 = RD^2 - IZ^2$:

Now $FR = FI + IR$, and $FR^2 = FI^2 + 2FI \times IR + IR^2$,

but $FS^2 = FR^2 + RS^2$,

whence $FS^2 = FI^2 + 2FI \times IR + IR^2 + RD^2 - IZ^2$ (by addition),

or $FS^2 = FI^2 - IZ^2 + 2FI \times IR + IR^2 + RD^2$:

But $IG^2 = FI^2 - IZ^2$, (283)

whence $FS^2 = IG^2 + 2FI \times IR + IR^2 + RD^2$.

Let IO be a fourth proportional to CG, Ff, and IR, that is $2IG : 2FI :: IR : IO$; then $2FI \times IR = 2IG \times IO$, this substituted for $2FI \times IR$ in the last equation, and we have

$$FS^2 = IG^2 + 2IG \times IO + IR^2 + RD^2:$$

Again, since $2IG : 2FI :: IR : IO$, or $IG : IR :: FI : IO$, we get $IG^2 : IR^2 :: FI^2\ (\text{or } IG^2 + IZ^2) : IO^2$.

But from the similar triangles IRD, IGN,

$$IG^2 : GN^2\ (= IZ^2) :: IR^2 : RD^2;$$

whence $IG^2 : IR^2 :: IG^2 + IZ^2 : IR^2 + RD^2$ (by composition);

Also $IG^2 : IR^2 :: IG^2+IZ^2 : IO^2$ (from the 4th. proportional),

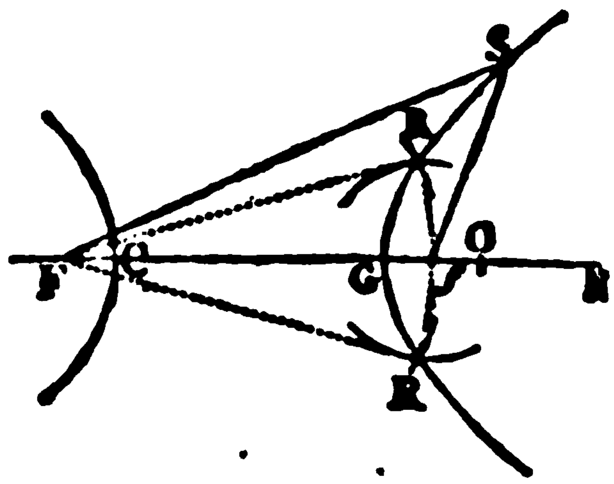
whence, by equality, $IR^2+RD^2=IO^2$ this substituted for IR^2+RD^2 in the latter of the preceding values of FS^2,

and the result is $FS^2=IG^2+2IG\times IO+IO^2$;

and the square roots give $FS=IG+IO$.

And proceeding in the same manner with $fR^2=(fI-IR)^2$ instead of $FR^2=(FI+IR)^2$, we shall get $fS=IO-IG$;

Therefore $FS-fS$ or $IG+IO-(IO-IG)=2IG=CG$ the *difference* of the two lines drawn from the foci to meet in the curve. In the ellipse their *sum* is $=$ the transverse CG. (271)

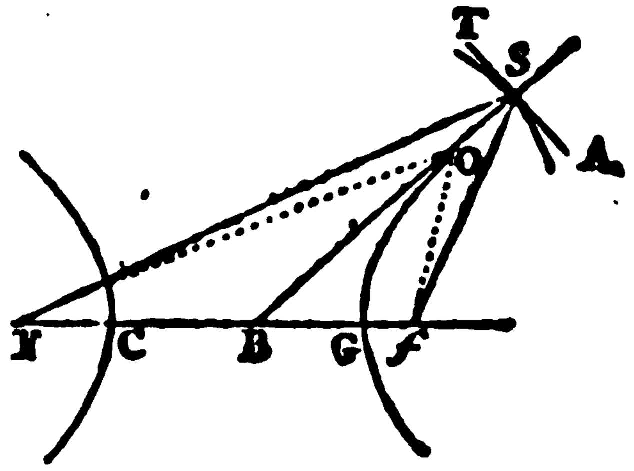


Corol. Hence is derived a method of describing the curve by continued motion when the transverse and foci are given; thus,

Let two threads FS, fS, whose difference in length is $=$ the transverse CG, be fastened at the foci F, f; then if the other ends are tied together (suppose at S) and passed through a small bead or pin S, and the bead or pin be made to move along the threads while they are constantly kept tight, the said bead or pin will, by its motion, describe the curve.

Or the curve may be traced mechanically, thus, Take any point O in the axis fN, then with GO and CO as radii, about the foci f, F, describe arcs of circles intersecting each other in R, R, which will be two points in the curve: and the like number may be found by assuming another point in the axis (fN), and so on. The curve is then to be drawn through the points of intersection.

285. *To draw a tangent to the hyperbola at a given point S in the curve.*



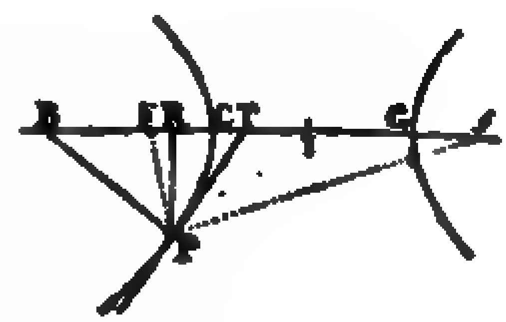
Let SF, Sf be drawn to the foci F, f; then a line SB which bisects the angle FSf is the tangent required.

Suppose TA is drawn through S to make the angles FST, fSA, equal; then if F, f, were the foci of an ellipse, and the length of the threads SF + Sf remained constant, an elliptical arc might be described with that constant length which would touch TA in the point S, and SB would bisect the angle FSf, or stand at right angles to TA (272); but in describing the hyperbola, the threads SF, Sf, are constantly diminished equally in length, and consequently the motion of the point S must, for that reason, be at right angles to the direction of the curve of the ellipse at that point, that is, an indefinitely small part of the hyperbolic curve (SO) will coincide with SB, which therefore must be a tangent to the curve.

Corol. Hence if TA were a reflecting surface perpendicular to the curve, a ray of light FS proceeding from one focus, would be reflected in the direction Sf to the other.

Scholium. In comparing Articles 271 and 280, the reader will perceive that the latter may be considered as a repetition of the former; the difference consisting merely in the signs + and —, which vary in a few steps of the process. This similarity extends to most of the properties of the Ellipse and Hyperbola. We therefore shall only enunciate the theorems in the three following articles, and leave their investigations as exercises for the student, who will find little difficulty in framing the demonstrations when he comprehends what is laid down respecting the ellipse.

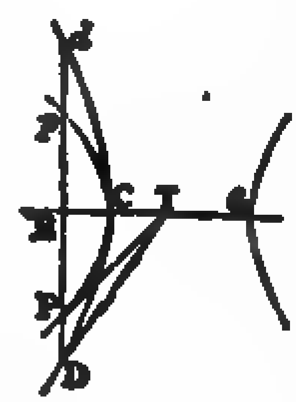
286. Let TP be a tangent to the hyperbola, F, f, the foci, I the center, and PR an ordinate to the axis CG.



Then $IC^2 : IF^2 :: IR : IB$, (PB being perpendicular to the tangent, as in the ellipse, *art.* 273).

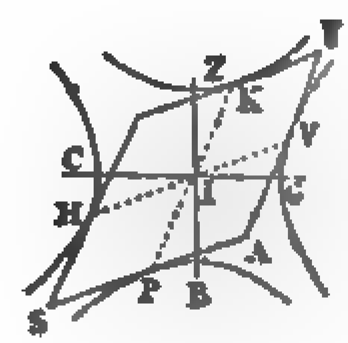
And IC is a geometrical mean between IR and IT; that is, $IR : IC :: IC : IT$.

Hence also, if two or more hyperbolas have the same common axis CG, the tangents at the extremities of the ordinates RP, RD, &c. will all meet in the same point T in the axis, as in the ellipse, (274 *corol.* 1.)



And the ordinates RP, RD have the same ratio as the conjugate axes of the hyperbolas. Whence it follows that the hyperbolic spaces PCp, and DCd, are also proportional to those axes; for each is composed of the like indefinite number of parallel ordinates whose sums are respectively as RP to RD.

287. Every parallelogram inscribed between the four conjugate hyperbolas is equal to the rectangle of the two axes:



That is, the parallelogram $SN = CG \times ZB$.

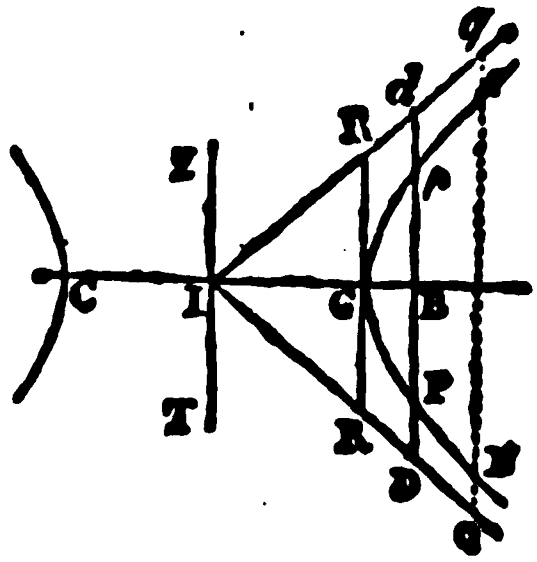
And the opposite sides are bisected at the points of contact H, K, V, P, as in the ellipse (*art.* 278).

288. The difference of the squares of any two conjugate diameters, is equal to the difference of the squares of the two axes:

That is, $HV^2 - PK^2 = CG^2 - ZB^2$. In the ellipse their *sums* are equal, (*art.* 279.)

OF THE HYPERBOLIC ASYMPTOTES.

289. *If* I *be the center of the hyperbola,* CG *the transverse axis,* RR ($=$ ZT) *the conjugate,* ID, I*d the asymptotes,* P*p an ordinate produced to* D *and d; Then* P*d* $\times$ PD $=$ GR^2. (RR *being the tangent at* G).



By similar triangles, $IB^2 : BD^2 :: IG^2 : GR^2$ (or IZ^2):

And (269) $IG^2 : GR^2 :: (CG + GB) \times GB$ or $(IB + IG)\ (IB - IG)$, or $IB^2 - IG^2 : BP^2$;

That is $IB^2 : BD^2 :: IB^2 - IG^2 : BP^2$ (by equality);

or alternately

$$IB^2 : IB^2 - IG^2 :: BD^2 : BP^2;$$

and $IB^2 : IB^2 - (IB^2 - IG^2) :: BD^2 : BD^2 - BP^2$ (by division);

That is $IB^2 : IG^2 :: BD^2 : BD^2 - BP^2$,

or alternately $IB^2 : BD^2 :: IG^2 : BD^2 - BP^2$;

whence $IG^2 : GR^2 :: IG^2 : BD^2 - BP^2$ (by equality);

Therefore $GR^2 = BD^2 - BP^2 = (BD + BP)\ (BD - BP)$ or P*d* $\times$ PD.

Corol. Hence if Q*q* be any other parallel ordinate produced, then N*q* $\times$ NQ $=$ P*d* $\times$ PD: for each is $=$ GR^2.

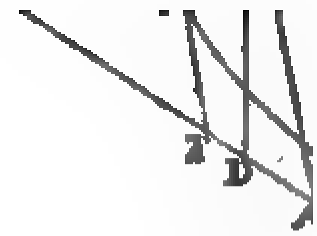
290. *If two parallel lines* A*a,* B*b are drawn through the hyperbola to meet the asymptotes; then p*A $\times$ *pa* $=$ *n*B $\times$ *nb*.

Through *p* and *n* draw ordinates to the axis: Then the triangles *p*DA, *n*QB; *pda*, *nqb*, will be respectively similar;

$pD.pd = nq.nq$;

therefore

$pA.pa = nB.nb$.



291. *If any right line* (B*b*) *be drawn through the hyperbola to meet the asymptotes; then the parts of that line between the curve a… totes will be equal: that is* $OB = nb$.

Let WOS be parallel to qQ or dD. Then th… QnB, SOB, and also bnq, bOW being respectively …

we have $nQ : OS :: nB : OB$,

and $nQ : OW :: nb : Ob$,

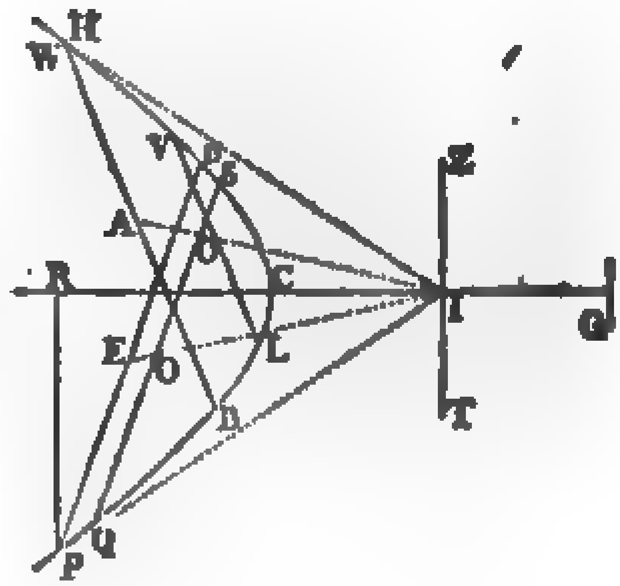
whence $nQ.nQ : OS.OW :: nB.nb : OB.O$… pounding):

But (289 corol.) $nQ.nq = OS.OW$, theref… $= OB.Ob$; whence it follows that $OB = nb$.

Corol. 1. If the tangent TT be parallel to bB, … bisected in the point of contact V; for if Bb be s…

Corol. 2. Hence also it appears, that if the semidiameter IV be produced, it will bisect all the corresponding double ordinates, On, &c. for since TT is bisected in V, it follows from similar triangles, that its parallels aA, bB, &c. are bisected, and because nb = OB, the double ordinates nO, &c. are also bisected by VE.

Corol. 3. Hence when the curve of an hyperbola is given, the axes may be determined thus,



Draw parallel ordinates VL, WB, and SQ, DP, and let them be bisected in O, A, and O, E by the lines AI, EI, then their point of concourse I will be the center. Take any two points P, H, in the curve equally distant from the center I, and bisect the angle PIH with the line ICR; then if IG = IC, CG will be the transverse axe.

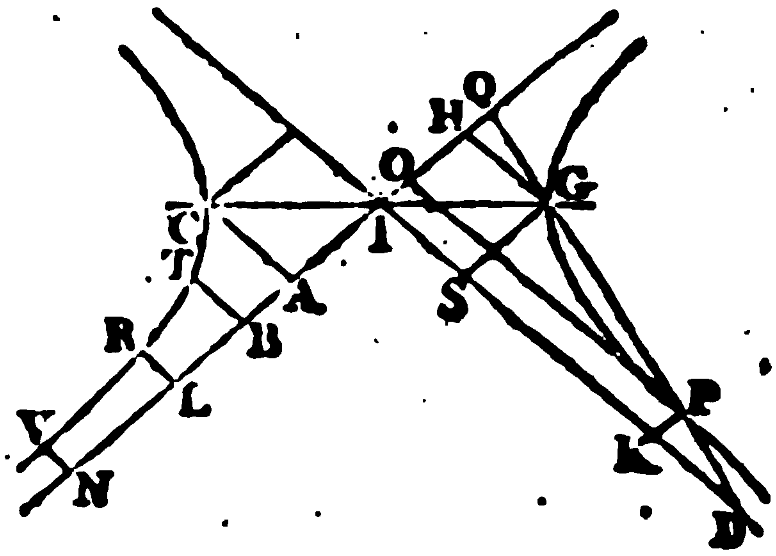
From any point P in the curve, draw an ordinate PR to the axis CG,

$$\text{then (269) } GR \times CR : PR^2 :: CG^2 : (\textit{conjug. axis})^2,$$

$$\text{or } \sqrt{(GR \times CR)} : PR :: CG : \textit{conjug. axis};$$

That is, the conjugate axis ZT is a fourth proportional to the mean proportional between GR and CR, the corresponding ordinate PR, and the transverse axis CG.

292. *If* P *be any point in the curve, and* PK, GS, PO, GH, *parallel to the asymptotes* IQ, ID, *respectively; then the parallelograms* PKIO, GSIH *are equal.*



Through G and P draw QD: Then because PK and KD are parallel to QH and HG, and GQ = PD (291), we shall have HG = KD, and HQ = KP = IO, therefore OQ = IH = HG = GS.

And by sim. triang.

$$HQ : HG :: OQ \text{ (or GS) } : OP;$$

whence $HQ \times OP = HG \times GS$: but $KP = HQ$, therefore $KP \times OP = HG \times GS$.

Now as the rectangles of the sides of the parallelograms SH, KO, about the equal angles KPG, SGH are equal, it follows (from Art. 258 *Mensur.*) that the parallelograms themselves must also be equal.

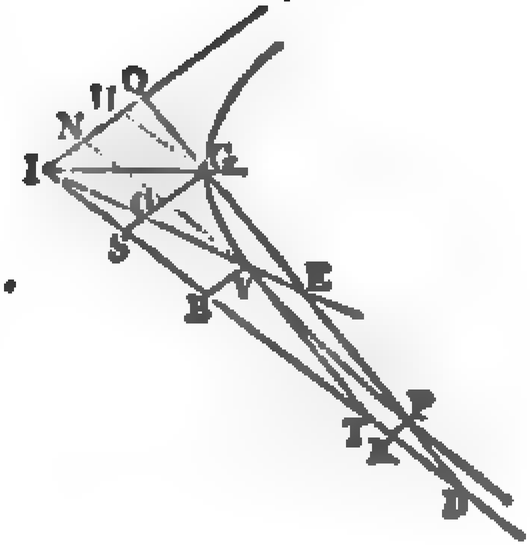
Corol. 1. Hence all the inscribed parallelograms are equal to one another: for each is equal to SGHI.

Corol. 2. It also appears that *the asymptote continually approaches towards the curve, but can never meet it:* for $KP \times OP = HG \times GS$ (*a given magnitude*), consequently $\frac{HG \times GS}{OP} = KP$, which must always be of some length if OP is assignable. Thus suppose HG, GS are each an *inch*, and $OP = 10000$ *miles*, then $KP = \frac{1}{10000 \times 1760 \times 36}$ of an *inch*. The distance of the curve and asymptote therefore diminishes as the latter is increased; on which account, the asymptote is sometimes considered as a tangent to the curve at an infinite distance.

Corol. 3. If IA, IB, IL, IN, &c. are in geometrical progression ascending, then AC, BT, LR, NV, &c. (parallel to the other asymptote) will be a descending progression: for the

rectangles $IA \times AC$, $IB \times BT$, &c. being equal, BT, LR, NV, &c. are reciprocally as IB, IL, IN, &c. Thus if $IA = 1$, $IB = \frac{3}{2}$, $IL = \frac{9}{4}$; $IN = \frac{27}{8}$, &c. then $AC = 1$, $BT = \frac{2}{3}$, $LR = \frac{4}{9}$, $NV = \frac{8}{27}$, &c.

293. *Let* TV *be a tangent at* V, *and* DQ *parallel to that tangent; then if* GS, VB, PK, *are parallel to the asymptote* IQ, VB *will be a geometrical mean between* GS *and* PK; *that is* $VB^2 = PK \times GS$.



The triangles KPD, BVT, SGD being similar, we have

$VB : VT :: PK : PD$,

$VB : VT :: GS : GD$,

whence

$VB^2 : VT^2 :: PK \times GS : PD \times GD$ (by compounding);

But

$VT^2 = PD \times PQ$ or $PD \times GD$, (291 *cor.* 1)

Therefore by equality, $VB^2 = PK \times GS$.

294. *The mixt-lined quadrilinear spaces* GVBS, VPKB *are equal.*

Since VE, the diameter produced, bisects all the double ordinates (291 corol. 2) each of the spaces EVG, EVP is composed of an infinite number of equal ordinates EG, EP, &c. therefore by the *method of indivisibles*, those spaces are equal.

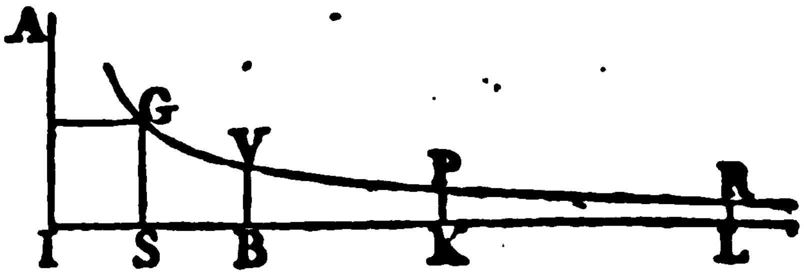
And because the triangles PKD, QHG are similar, and PD = GQ, those triangles are also equal:

Now QD is bisected in E, consequently the triangle EDI = triang. EQI:

From the triang. EDI take EVP + PKD, and from EQI subtract EVG + QHG, and we have the space VPKI = space

VGHI: and if the equal triangles VBI, VNI are respectively subducted from those equal spaces, then the space VPKB = space VGHN:

But (292) the parallelogram BVNI = SGHI; from each of these take the common parallelogram SONI, and we have OVBS = OGHN, and if to each we add the trilinear OGV, there results GVBS = VGHN = VPKB.

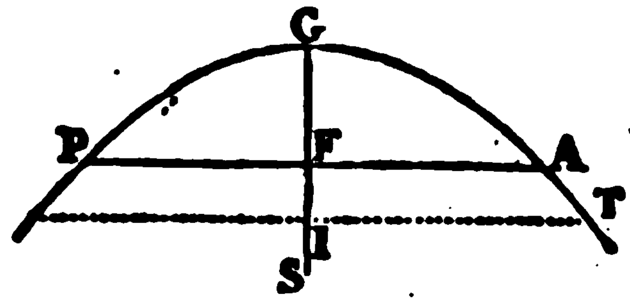


SCHOLIUM. The asymptotic space GVPKS is therefore bisected by the ordinate or line VB which is a geometrical mean between the extremes GS, PK. Hence it appears that when GS, VB, PK, RL, &c. are in geometrical progression, the included spaces GVBS, VPKB, PRLK, &c. are equal; and the spaces GVBS, GPKS, GRLS, &c. proceed in arithmetical progression, while the corresponding distances IB, IK, IL, on the asymptote, are in geometrical progression: the former are therefore analogous to the *logarithms* of the latter. Thus suppose the hyperbola is equilateral or the asymptotes IA, IL are at right angles, and GS = IS = 1, IB = 2, IK = 4, IL = 8, &c. then the area of the space GVBS = 0.693147 the *log.* of 2 or IB; the area GPKS = 1.386294 the *log.* of 4 or IK; the area GRLS = 2.079441 the *log.* of 8 or IL, &c. These logarithms are called *hyperbolic logarithms.*

The system of logarithms however, will vary with the angle made by the asymptotes: Thus if they form an angle of $25° \; 44' \; 25''\frac{1}{2}$, and IS = GS = 1, IB = 2, IK = 4, IL = 8, &c. the area of the rhombus GI will be 0.4342944819; and the asymptotic spaces GVBS, GPKS, GRLS, &c. equal to 0.30103, 0.60206, 0.90309, &c. respectively, which are *Briggs's* logarithms of 2, 4, 8, &c. The area of the rhombus, or which is the same thing, that of any inscribed parallelogram, is called the *modulus* of the system.

Of the PARABOLA.

295. Let G be the vertex, F the focus, GS the axis; then *the ordinate* PA *at right angles to* GS, *is the parameter. And* FG *the distance of the focus from the vertex is* $= \frac{1}{4}$PA.



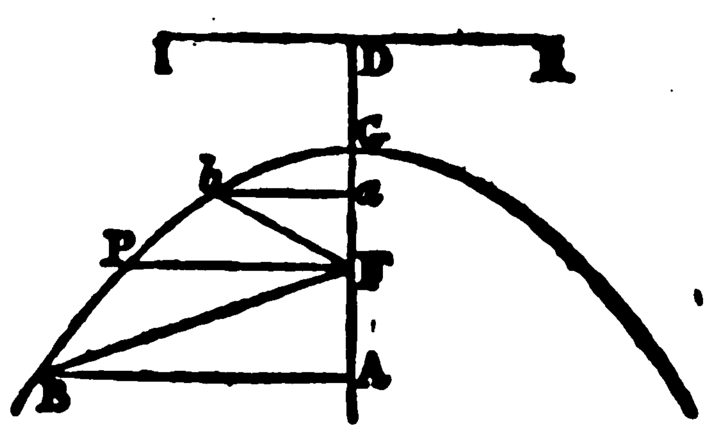
Draw IT parallel to FA: Then from the equation of the curve, (268)

$$GI : IT^2 : GF : \frac{IT^2 . GF}{GI} = FA^2, \text{ or } \frac{4IT^2 . GF}{GI} = PA^2.$$

But $GI : IT :: IT : \frac{IT^2}{GI}$ is the parameter $=$ PA, by hypothesis, (268)

Therefore $\frac{IT^4}{GI^2} = \frac{4IT^2 . GF}{GI}$, or $\frac{IT^2}{GI} = 4GF$, that is, $PA = 4GF$.

296. *Let a line be drawn from the focus to any point* (B) *in the curve, and an ordinate* (BA) *from that point to the axis; also let* GD (*in the axis produced*) *be taken* $=$ GF; *then* $FB = DA$.



Because $FA = GA - GF$,
therefore $FA^2 = GA^2 - 2GA \times GF + GF^2$;

But (268) $BA^2 = p.GA = 4GF \times GA$ (p being the parameter),

whence $FA^2 + BA^2 = GA^2 + 2GF.GA + GF^2$, by addition; and since $FA^2 + BA^2 = FB^2$, we have $FB^2 = GA^2 + 2GF.GA + GF^2$;

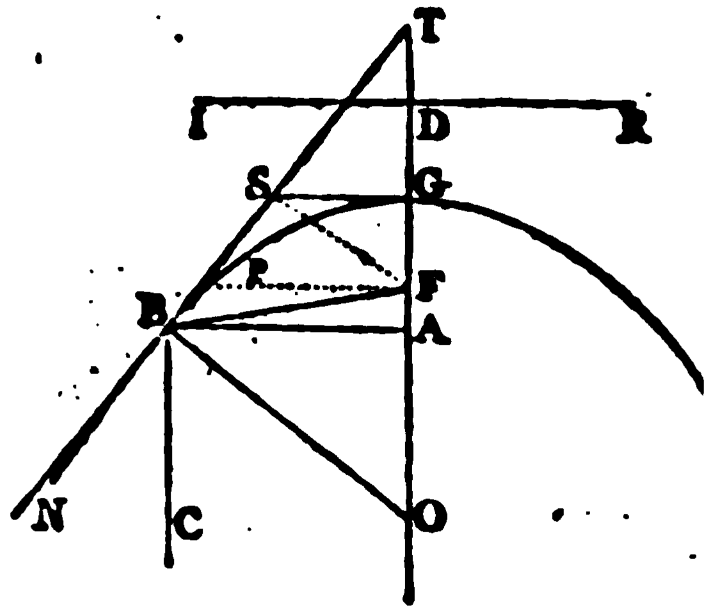
and by extracting the roots, $FB = GA + GF = GA + GD = DA$.

If the point (b) in the curve is above the focus, then $Fa = GF - Ga$, and $Fb = Da$.

Corol. This Theorem affords a ready method of describ the parabola by points, thus: Since the distance of the cu at the extremity of any ordinate from the focus is equal to distance of that ordinate from the point D, if a number of li *ab*, FP, AB, &c. are drawn parallel to DI (at right angles DA) and the distances D*a*, DF, DA, &c. set off from the fo F to meet those lines respectively, the points of concourse be those through which the curve must be drawn.

The line IDR is called the *directrix* of the parabola.

297. *To draw a tangent to the parabola at a given poi in the curve.*



From B draw BF to the focus, and BC parallel to the axis; let BO bisect the angle CBF; then if the angle OBT be made a right one, BT will be the tangent required.

This construction results from considering the parabola as an ellipse whose transverse is infinite in length (268). For a tangent to the ellipse at point is perpendicular to the line which bisects the angle forme the two lines drawn from the foci to meet the curve at that p (272): if therefore the axis is infinite, one of the foci wil at an infinite distance, and the line drawn from that focus n in *that case*, be parallel to the axis.

Corol. 1. Hence, because BO is perpendicular to the c at B, and the angle FBO = OBC, if the concavity of parabola were a polished surface, all rays of light (as CB, falling on that surface parallel to the axis, would be reflect the focus F. (253.)

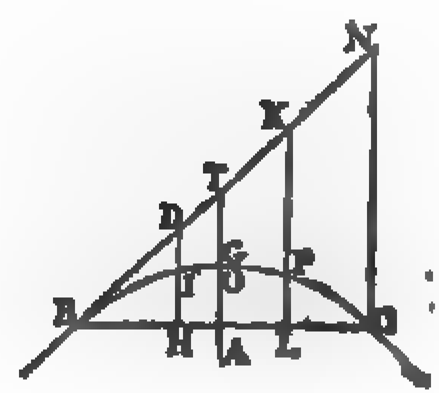
Corol. 2. From this construction and the preceding theorem, it appears that the *subtangent* TA is bisected at the vertex G, that is, $GA = GT$, (BA being an ordinate to the axis). For the angles NBO, TBO being right ones, and the angle CBO $= FBO$, therefore the angle $NBC = TBF$, but the angle NBC $= BTF$, therefore the angles FBT, FTB are equal, and consequently $FT = FB$. But $FB = DA$ (IDR being the directrix), from this take GD, and from FT take its equal GF, and there remains $GT = GA$.

Corol. 3. Hence also, the distance AO is always $= FP$ or half the parameter. For since $FT = FB$, and the angle TBO a right one, F will be the center of a circle passing through T, B, O, and therefore $FO = FT = FB$: but $FB = AF + 2FG$ (or FD) $= AF + AO$, and consequently $AO = 2FG = FP$.

Corol. 4. And the tangent GS is a mean proportional between GF and GA: for BT is bisected by GS, and the angles FST, SGT being right ones, SG is a mean proportional between GF and GT (Geom. 164) or between GF and GA. And FS is a mean proportional between FG and FT.

293. *If* BN *be a tangent at* B, *and the lines* HD, LK, QN, &c. *are parallel to the axis* GA, *these lines will be divided in* I, G, P, &c. *in the same proportion as the double ordinate* BQ *is divided in* H, A, L, &c.

That is, $ID : IH :: HB : HQ$, &c.
$PK : PL :: LB : LQ$, &c.



Draw the ordinate IO, and let the parameter be denoted by p:

Then because $p \times AG = BA^2$, (266)
it will be $p : 2BA$ (or BQ) $:: BA : 2AG$ or AT;

But by sim. triangles, $BH : HD :: BA : AT$,
whence by equality $p : BH :: BQ : HD$;

Moreover, $p \times GO = IO^2$ or HA^2; (268)
and $p \times GA = BA^2$,
whence $p\,(GA - GO) = BA^2 - HA^2$, by subtraction,
or $p \times IH = BA^2 - HA^2$;
therefore, $p : BA + HA :: BA - HA : IH$,
that is, $p : HQ :: BH : IH$;
or alternately, $p : BH :: HQ : IH$;
whence by equality, $BQ : HD :: HQ : IH$,
or alternately, $BQ : HQ :: HD : IH$;
and by division, $BQ - HQ : HQ :: HD - IH : IH$;
That is, $BH : HQ :: ID : IH$.

Corol. Hence the external lines ID, GT, PK, &c. will have the same ratio as the squares of the corresponding tangents BD, BT, BK, &c.

That is $ID : PK :: BD^2 : BK^2$, &c.

For $ID : IH :: BH : HQ$,
and $ID : IH :: BH^2 : BH.HQ$, by equality,
or $ID : BH^2 :: IH : BH.HQ$, by alternation:

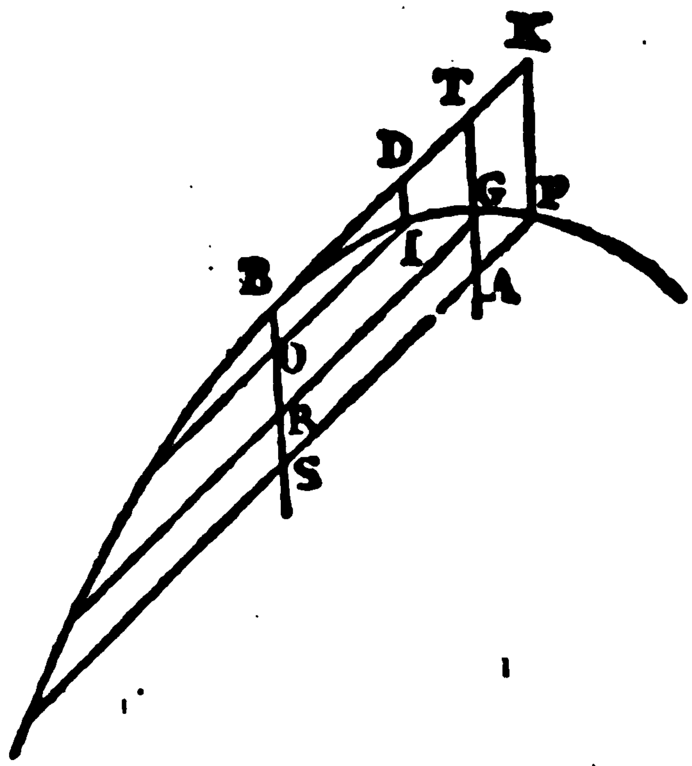
But $\frac{BH.HQ}{p} = IH$, therefore $ID : BH^2 :: \frac{BH.HQ}{p} : BH.HQ$.

In like manner $PK : LB^2 :: \frac{LB.LQ}{p} : LB.LQ$;

And therefore $ID : PK :: BH^2 : LB^2 :: BD^2 : BK^2$, by sim. triangles.

299. *If* BK *be a tangent to the parabola at* B, *then* BS *parallel to the axis* GA, *is a diameter, and* OI, RG, SP, &c. *parallel to the tangent* BK, *are ordinates to that diameter.*

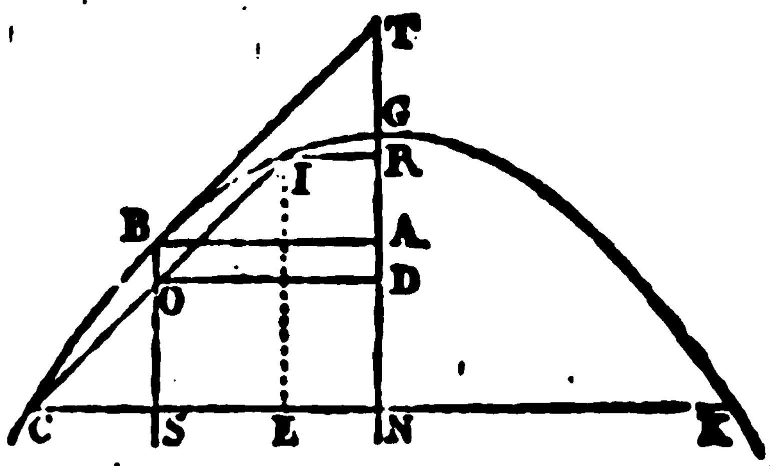
And the abscissas BO, BR, BS, &c. *have the same ratio as the squares of their corresponding ordinates* OI, RG. SP, &c.



Let ID, GT, PK, &c. be parallel to the axis GA or to the diameter BS. Then BOID, BRGT, &c. being parallelograms, the opposite sides will be respectively equal:

And (298 corol.)
$ID : PK :: BD^2 : BK^2$, &c.
that is
$BO : BS :: OI^2 : SP^2$, &c.

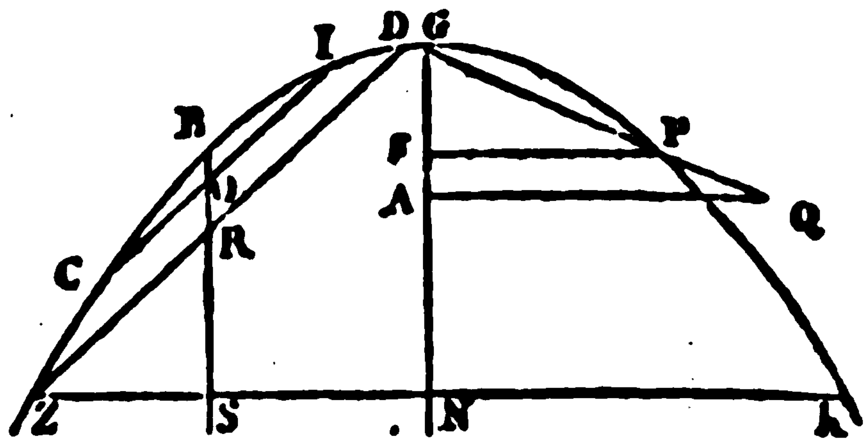
300. *Any diameter* (BS) *bisects all its double ordinates* (IC, &c.) *or lines parallel to the tangent* (BT) *at the vertex* (B) *of that diameter.*



Let IR, BA, CK be ordinates to the axis, and draw IE perpendicular and OD parallel to CN: also suppose p = the parameter.

Then (298) $p \times EI = CE^2 - EN^2 = EK \times EC$,
that is $p : EK :: EC : EI$:
And by sim. triangles $BA : AT$ or $2GA :: EC : EI$,
whence by equality $p : EK :: BA : 2GA$,
or alternately $p : BA :: EK : 2GA$:
But (268) $p : BA :: 2BA : 2GA$,
therefore by equality $EK : 2BA :: 2GA : 2GA$;
consequently EK or $IR + CN = 2BA$. That is, the ordinate BA is an arithmetical mean between the ordinates IR and CN: but $OD = BA$, whence it follows that RN and IC are both bisected by OD.

Corol. Hence when the curve of a parabola is given, the axis and focus are determined by the following construction:

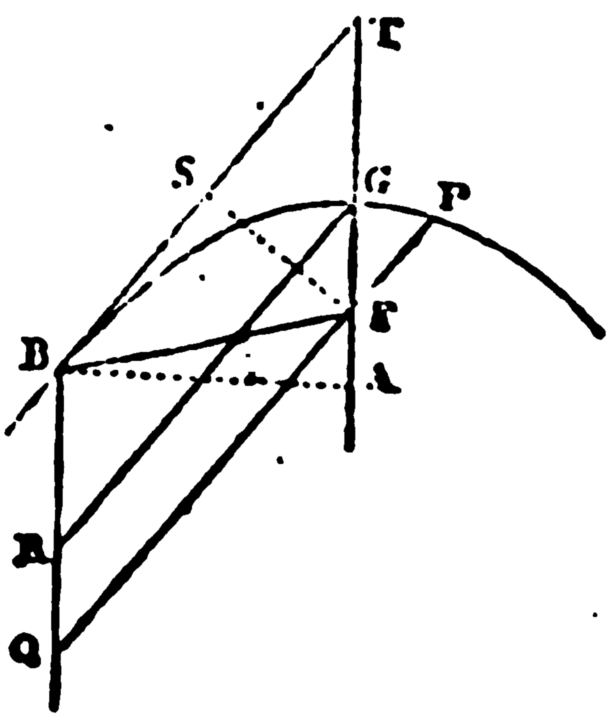


Draw any two parallel lines or ordinates IC, DZ terminated by the curve, and bisect them in O, R with the diameter BS; then, at right angles to BS draw ZK which bisect in N with the perpendicular NG, which will be the *axis*.

To find the *focus*, let AQ be parallel to NK and $= 2AG$, draw QG, and the point P where it intersects the curve, will be the extremity of the parameter of the axis: for by sim. triangles $FP = 2FG$, therefore F is the *focus*.

301. *Let* GA *be the axis, and* F *the focus; then* (295) *the parameter* (p) *of the axis; is equal to* 4FG *or four times the distance of the focus from the vertex* G. *In like manner, if* B *be the vertex of any other diameter, its parameter* (P) *will be 4 times the line drawn from the focus to that vertex. That is,* $P = 4FB$.

Draw GR parallel to the tangent BT meeting the diameter BR in R, also let BA be an ordinate to the axis, and make FS perpendicular to BT.



Then (297. corol. 2) $GA = GT = BR$, therefore the abscissas GA, BR to the ordinates BA, GR are equal:

And (268) $GA = \frac{AB^2}{p}$; also BR (or GA) $= \frac{GR^2}{p}$ (by the definition);

whence $p : P :: AB^2 : GR^2$ or BT^2:

But $FS^2 : FT^2 :: AB^2 : BT^2$ (by sim. triangles),

therefore $p : P :: FS^2 : FT^2$: but $FS^2 = FG . FT$ (297. corol. 4),

consequently $p : P :: FG . FT : FT^2$,

whence $p : P :: FG : FT$ or FB;

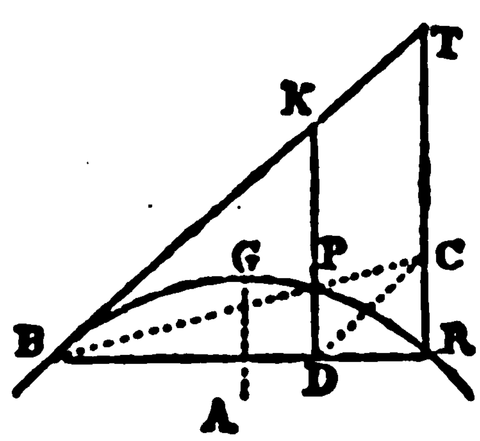
but $p = 4FG$, and therefore $P = 4FB$.

Corol. Let PFQ be parallel to GR. Then because FB $= FT = BQ$, we have $P = 4BQ$ the parameter of the diameter BR; therefore (by the definition) the parameter is the double ordinate drawn through Q, and consequently $2BQ = QP$ the semiparameter. Whence also it appears, that the parameters of all the diameters of a parabola pass through the focus.

• And it may be observed in general, that the properties which have been demonstrated respecting the axis, its abscissas, and ordinates, extend to any other diameter, its abscissas and ordinates.

302. *Let* BR *be any right line terminated by the curve, and* BT *a tangent at* B; *then if* KD *be a line parallel to the axis* GA, *it will be divided by the curve at* P *in the same ratio as* BR *is divided in* D:

That is, PD : KP :: DR : BD.



Draw RT parallel to DK:

Then (299) $KP : TR :: BK^2 : BT^2$.

And $BD^2 : BR^2 :: BK^2 : BT^2$. (by sim. triang.)

therefore by equality

$KP : TR :: BD^2 : BR^2$, whence $KP.BR^2 = TR.BD^2$:

Again, by sim. triang. $KD : TR :: BD : BR$,

and $KD : TR :: BD^2 : BD.BR$, or $KD\,(BD.BR) = TR.BD^2$,

therefore $KP.BR^2 = KD\,(BD.BR)$, or $KP.BR = KD.BD$;

that is $KD : KP :: BR : BD$,

And $PD : KP :: DR : BD$, by division.

Corol. If BC be drawn through the point of intersection P, then DC is parallel to the tangent BT. For the triangles BDP, BRC being similar,

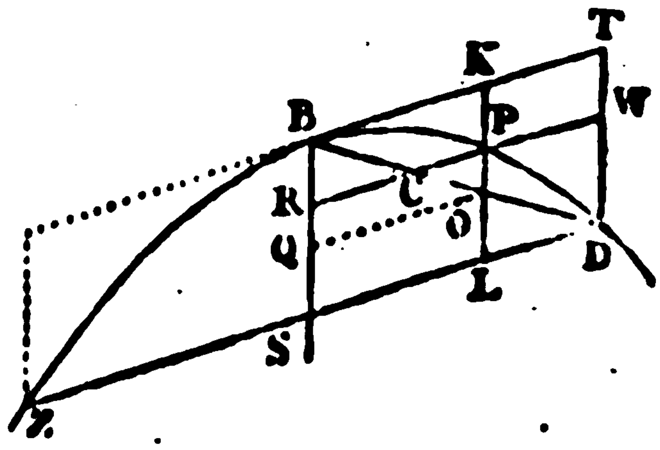
we have BP : PC :: BD : DR,

but KP : PD :: BD : DR,

whence, by equality, BP : KP :: PC : PD, therefore (Geom. 94, corol. 1) the triangles BKP, DCP are equiangular, and DC parallel to BK.

303. *If* BS *be any diameter,* BT *a tangent at* B, *and* ZT *a parallelogram described about the parabola; then if* KL *be a line parallel to* BS, *and* BD *joined,* KO *will be a mean proportional between* KP *and* KL:

That is, KP : KO :: KO : KL.



Draw RPW and QO parallel to BT or SD:

Then (290) BR : BS :: RP^2 or QO^2 : SD^2 (by sim. triang.) :: BQ^2 : BS^2.

And BR . BS : BS^2 :: BQ^2 : BS^2, therefore BR . BS $=$ BQ^2,

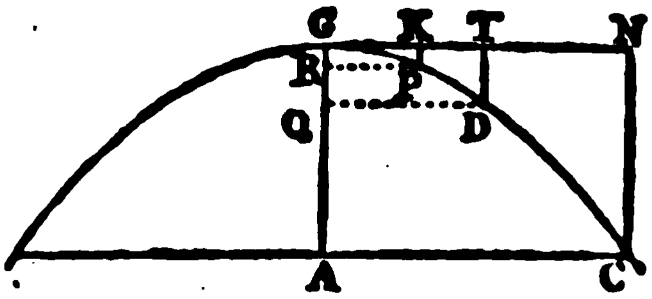
or BR : BQ :: BQ : BS,

That is, KP : KO :: KO : KO.

Corol. Since BR : BQ :: BQ : BS, therefore BQ is a mean proportional between BR and BS: but by sim. triangles, RC, QO (or RP) and SD (or RW) are in the same proportion as BR, BQ, and BS, and consequently RW is divided in C and P so that RC, RP, and RW are also in continued proportion, or RC : RP :: RP : RW.

304. *The area of a parabola is* $\frac{2}{3}$ *of its circumscribing parallelogram:*

That is, the space AGPC $= \frac{2}{3}$ AGNC.



Conceive the surface GPDCN to be composed, or made up of an indefinite number of indefinitely small threads or lines KP, TD, &c. parallel to NC or the axis GA, the longest being, NC, and the shortest at $G = o$:

Then (268) $GA : AC^2 :: GR : RP^2$ or GK^2,

$$\text{therefore } GR \text{ or } KP = \frac{GA}{AC^2} \times GK^2:$$

$$\text{In like manner } TD = \frac{GA}{AC^2} \times GT^2:$$

&c. &c.

Hence $\left(\frac{GA}{AC^2}\right.$ being a constant quantity$\left.\right)$ the sum of all the lines KP, TD, &c. will be $\frac{GA}{AC^2} \times (o^2 + GK^2 + GT^2 + \ldots GN^2)$:

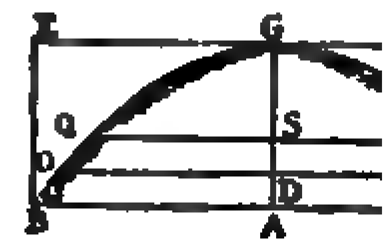
Now if GN is supposed to be divided into an indefinite number of indefinitely small and equal parts, these parts will form an arithmetical progression whose least term is o, and greatest GN: and if n is put to denote the number of terms, the sum of their squares, or the sum of the infinite progression will be $\frac{n^3}{3}$ (179); the first term being o^2, and last n^2. *

* Should the student have any doubts respecting this result from art. 179, the following process will show the truth of the conclusion. Let n or GN be the perpendicular height of a pyramid having a square base whose side is also $=$ GN, and conceive the pyramid to be composed of an infinite number of indefinitely thin square laminæ laid one upon another, the greatest or the base being GN^2, and the least at the vertex $= o^2$; then it is evident the content of the pyramid will be the sum of all the laminæ or series of squares from o^2 to GN^2; but the content of the pyramid is $= GN^2 \times \frac{1}{3}GN = \frac{1}{3}GN^3$, the sum of the series, as above, according to the *Arithmetic of Infinites.*

The summation of such series however, is properly the business of Fluxions, which affords a general method: but the expressions in the article refered to, will answer the purpose in some of the most simple cases.

303. *The content of a paraboloid or solid generat revolution of a parabola* (BGC) *about its axis* (GA *its circumscribing cylinder.*

Suppose the axis of the parabola divided into an infinite, or indefinite number of equal parts, and conceive the paraboloid to be composed of the like or corresponding number of circular sections whose diamete OP, QR, &c. the diameter of the greatest section b and that of the least section (at G) = *o* :



Then the number of sections, or parts into which G posed to be divided, form an arithmetical progressi first term = *o*, last term = GA, and number of t = GA; and the sum of such a series = $(o+\mathrm{GA}) \times$ $\frac{1}{2}\mathrm{GA}^2$.

By Art. 268, we have $\mathrm{GA} : \mathrm{BC}^2 :: \mathrm{GD} : \frac{\mathrm{BC}^2}{\mathrm{GA}} \times \mathrm{G}$

And $\mathrm{GA} : \mathrm{BC}^2 :: \mathrm{GS} : \frac{\mathrm{BC}^2}{\mathrm{GA}} \times \mathrm{G}$

cQR^2 or $\frac{cBC^2}{GA} \times GS =$ that having the diam. QR:

&c. &c.

consequently $\frac{cBC^2}{GA} \times (GA + GD + GS + \ldots\ldots. o)$ will be the sum of all the circular sections, or content of the paraboloid: but the sum $(GA + GD + GS + \&c. \ldots. o) = \frac{1}{2}GA^2$, hence the expression becomes $\frac{cBC^2}{GA} \times \frac{1}{2}GA^2$ or $cBC^2 \times \frac{1}{2}GA$, which is half the content of the circumscribing cylinder BEFC.

Of the Construction of

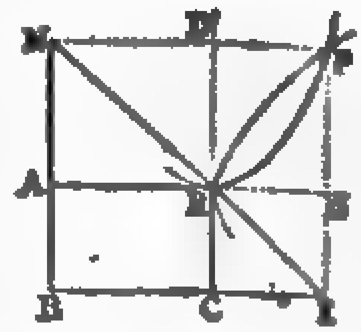
CUBIC and BIQUADRATIC EQUATIONS.

306. We have given the construction of quadratic equations by means of right lines and the circle (233): but either of the conic sections might be substituted for the latter, because their equations are also of two dimensions: the circle however, is prefered on account of the simplicity of its description. A right line can intersect a conic section in two points only, which determine the two roots of a quadratic. But one conic section may cut another in as many points as a cubic, or a biquadratic equation has roots; hence it appears that such equations can be constructed by means of the conic sections, or their roots determined by the intersections of *loci* of two dimensions.

307. *To construct a simple cubic equation $x^3 = a^2b$, or to find two mean proportionals between two right lines denoted by a and b.*

We shall take the example Art. 254, where it is required to find two mean proportionals between the lines RC and BC or RA.

Let the angle ARC be a right one: produce CR and AR, and on the axis RD describe a semiparabola RPD having its parameter $=$ RC, and on the axis RO another RPO whose parameter $=$ RA: then the ordinates PD, PO drawn from the intersection P to the axes, will be the mean proportionals required.

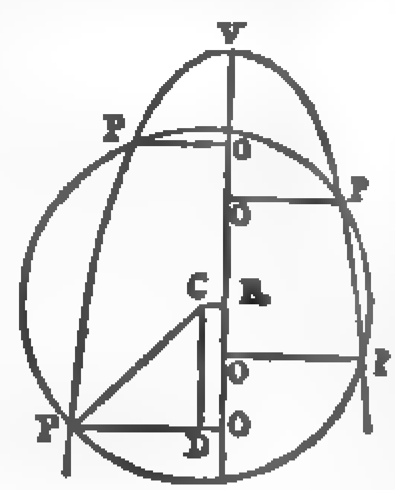


For (268) RD $\times$ *param.* RC $=$ PD^2, and RO $\times$ *param.* RA $=$ PO^2, therefore PD is a mean proportional between RC and RD or PO; and PO is a mean proportional between RO or PD and RA.

This problem is usually constructed by means of the circle and one of the conic sections: the preceding method however, is more simple of explication.

309. *To construct a Biquadratic.* Let the circle whose center is C intersect the parabola PVP in the points P, P, P, P; draw the ordinates PO, PO, PO, PO to the axis VO; also make CD parallel and CR perpendicular to VO, and draw CP.

Put VO $= x$, OP $= y$, VR $= a$, CR $= b$, CP $= r$, and the parameter of the parabola $= p$. Then $px = y^2$, whence $x = \frac{y^2}{p}$. Also PD $=$ PO $-$ DO $= y - b$, and CD $=$ VO $-$ VR $= x - a$: but CP$^2 =$ PD$^2 +$ CD^2, that is $(x-a)^2 + (y-b)^2 = x^2 - 2ax + a^2 + y^2 + b^2 = r^2$, and substituting $\frac{y^2}{p}$ for x, we get



$$y^4 \left.\begin{matrix} -2pa \\ +p^2 \end{matrix}\right\} y^2 - 2p^2by + (a^2 + b^2 - r^2)p^2 = 0,$$ which by

varying the values of the coefficients, may be made to coincide with any proposed biquadratic equation that wants the second term; and then the ordinates on the axis from the points of intersection P, P, P, P, will be the roots of that equation.

For example, suppose the proposed equation to be $y^4 - my^2 + ny - c = 0$.

Let a parabola PVP be described whose parameter $= 1 = p$;

$$\left.\begin{matrix}\text{then } -2pa \\ +\ p^2\end{matrix}\right\} = 1 - 2a = m, \text{ whence } a = \frac{m+1}{2}:$$

$$-2p^2b = -2b = n, \text{ or } b = -\frac{n}{2}:$$

$(a^2 + b^2 - r^2)p^2 = a^2 + b^2 - r^2 = -c$, whence $r = \surd(a^2 + b^2 + c)$:

Now in the axis take $VR = a = \frac{m+1}{2}$, and make RC perpendicular to VR and $= -\frac{n}{2} = b$, then about the center C, with the radius $\surd(a^2 + b^2 + c)$ describe a circle, and the ordinates to the axis from the points of intersection P, P, P, P, will be the four roots of the equation.

When RC represents a negative quantity, the ordinates on that side of the axis are the negative roots, and the contrary.

Corol. 1. If the circle cut the parabola in two points only, the equation has but two real roots, the others being imaginary: and if it touch the parabola, two roots must be equal, because two of the ordinates may be said to coincide.

Corol. 2. Should the circle pass through the vertex V, then $CP^2 = CR^2 + VR^2$, that is, $r^2 = b^2 + a^2$, and the last term of the biquadratic will vanish, if therefore the remainder be divided by y, the result is

$$\left.\begin{matrix}y^3 - 2pa \\ +\ p^2\end{matrix}\right\} y - 2p^2b = 0,$$

which may be made to coincide with any *cubic equation* wanting the second term, and the ordinates will be reduced to three for its roots.

This method of constructing biquadratic and cubic equations which want the second term, is that of Descartes. But the

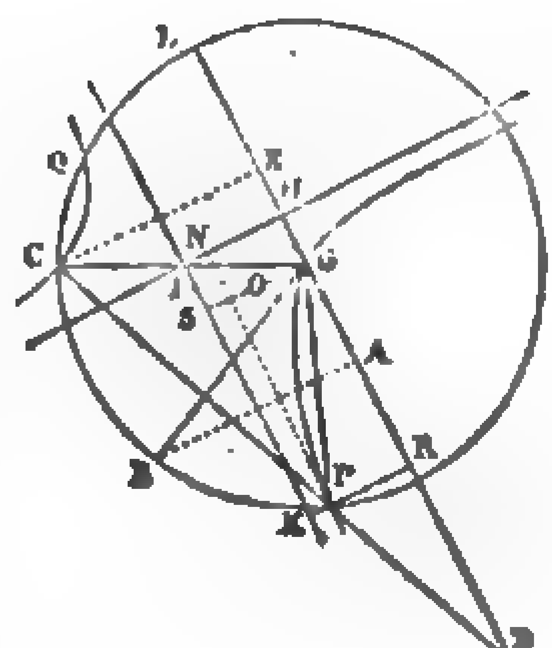
constructions may be made general by refering the lines denoting the roots to a diameter of the parabola that is not the *axis*, as may be seen in Baker's *Geometrical Key*, l'Hospital's. *Conics*, Maclaurin's *Algeb.* &c.

But the same thing may be effected by making use of either of the other conic sections, instead of the parabola, which is usually assumed because its equation consists of two terms only: the ellipse however, is more easy of description.

309. *An angle may be trisected by the construction of a cubic equation.* Thus if s and c denote the *sine* and *cosine* of an arc, the *radius* being 1: then (246) $3sc^2 - s^3$ is the *sine* of three times that arc; but $c^2 = 1 - s^2$ which being substituted for c^2, and we have the cubic equation $3s - 4s^3 = A$, putting $A =$ the *sine* of the angle to be trisected, and consequently the value of s will be the *sine* of $\frac{1}{3}$ of the proposed angle.

The following construction of the problem is by means of the *circle* and *hyperbola*.

Let BGA be the given angle. About G with the radius GB describe a circle; make BA perpendicular to GA, and in AG produced take GH $= \frac{1}{3}$GA; draw HI parallel and equal to $\frac{1}{3}$AB, and let IK be parallel to HA; then between the asymptotes IH, IK describe an hyperbola to pass through the point G, and it will cut the circle in P so that the angle PGA $= \frac{1}{3}$ of the angle BGA.



Through I draw the radius GC, and let CE be perpendicular to GE, also make GS and KPR parallel to IH, and PN to KI.

Then since GH and HI are the halves of GA and AB, the triangles GHI, GAB are similar, and because GC = GB, the triangles GCE, GBA are similar and equal.

And because the parallelograms NK, HS are equal (292) if the common parallelogram NS be subtracted from each, and the parallelogram OR added, the parallelogram SR = parallelogram NR, that is $PR \times RH = KR \times RG$,

$$\begin{array}{lll} \text{or } PR : RG & :: & KR : RH, \\ \text{and} & :: & 2KR : 2RH, \text{ (by doubling)} \\ \text{or} & :: & CE : 2RH, \text{ (because } 2KR = CE); \end{array}$$

But because $HE = HG$, therefore $2RH = ER + RG$, hence we have $PR : RG :: CE : ER + RG$,

or $CE : PR :: ER + RG : RG$.

But $CE : PR :: ED : RD$, by sim. triangles;
whence $ED : RD :: ER + RG : RG$, by equality,
or $ED - RD\ (ER) : RD :: ER : RG$, by division;
therefore $RD = RG$, and consequently $PD = PG$:

Now $GP = GC$, and therefore the angle $GCP = GPC$; but the external angle $CPG = PDG + PGD$, or CPG = twice PGD: in like manner, the external angle $CGE = GCD + GDC$, or CGE = triple PGD, *that is* $BGA\ (= CGE)$ = *triple* PGD.

If the opposite hyperbola be described, it will trisect the supplemental angle BGL, that is, the arc QL is $\frac{1}{3}$ of the arc BCL.

Since all parallelograms inscribed between the asymptotes and curve are equal, the semitransverse or semiconjugate axis of the hyperbola, will be the diagonal of a square whose side is $\sqrt{(GS \cdot GH)}$.

310. OF MECHANICS.

DEFINITIONS.

1. MECHANICS is the science which treats of the motions, velocities, forces, and in general of the actions and effects of moving bodies upon one another. It comprehends *Statics*, on the weight and equilibrium of solid bodies. *Dynamics*, the science of moving powers. *Hydrostatics*, of the gravity, and pressure of fluids. *Hydraulics*, treating of the motion of water, and other fluids, the construction of water-works, &c. &c.

2. *Motion* is a constant and successive change of place. If the body moves equably or passes over equal spaces in equal times, it is called uniform motion. If it increases, or decreases, it is called accelerated, or retarded motion. The motion is also said to be absolute, or relative, according as the body moved is compared with another body at rest, or in motion.

3. *Velocity* or *celerity*, is the quickness or slowness of motion, or the rate at which a body moves. Thus if a body passes uniformly over a space of two feet in half a second of time, it is said to have a velocity of 4 feet per second, or move at the rate of 4 feet in a second.

4. *Quantity of motion* or *momentum*, is the power or force of bodies in motion. This is proportional to the weight or quantity of matter moved drawn into its velocity.

5. *Force* is a power exerted on a body to put it in motion. If it act instantaneously, it is called *impulse* or percussion. If constantly, it is a permanent force like pressure or the force of gravity.

6. *Forces* are also distinguished into *motive*, and *accelerative* or retardive. The motive or moving force relates to the quantity of matter moved as well as the velocity communicated, and is proportional to the momentum or quantity of motion produced in a given time.

7. An accelerating or retarding force is generally understood to be that which affects the celerity only, and therefore it is proportional to the velocity generated in a given time, or to the motive force directly, and the mass or body moved inversely.

Thus if the body or mass B be urged by the moving force F, then $\frac{F}{B}$ will denote the accelerating force; for the magnitude or value of the fraction $\frac{F}{B}$ increases directly as F is increased, but diminishes as B is augmented.

Gravity or the power of gravitation is an accelerating force; for the velocity of a body falling by its own weight, or projected vertically, is continually augmented in the former case, but diminished in the latter, till all its motion in that direction is lost.

8. *Vis inertiæ*, is the innate force of a body by which it resists any endeavour to change its state; this is always proportional to the quantity of matter in the body. Thus if two bodies of the same kind are floating on water, the less or lighter body is more easily moved than the greater, and therefore its vis inertiæ is less.

9. An *elastic* body is that, the position of whose parts being changed by the action of a force, either recovers, or has a tendency to recover its former figure. Thus the strings of a violin are elastic. And a tennis ball rebounds by the force of its elasticity or the force exerted by its parts in recovering their position before impact. Bodies not having this property are called *non-elastic*.

as 1 to 2.

12. *Density* is also the proportion of the quanti in any body to the quantity in another body of the nitude. Thus if a body of any size weigh 6 poun other of equal bulk weigh 4 pounds, the density of to that of the latter is as 3 to 2.

13. *Friction* is a retarding force in machines, the parts rubbing against one another.

311. AXIOMS.

1. Every body perseveres in its state of rest, motion in a right line, unless it be compelled to state by some external force.

2. The alteration or change of motion is always to the force applied, and is made in the direction o line in which it acts.

3. Action and reaction are equal and in contrary

The quantity of matter in a body may be denoted by its weight; therefore

if w = the body or its weight;

m = its magnitude in cubic feet, or any other known measure;

d = its density:

then w is as md, or w is always directly proportional to $m \times d$.

Let $w : m \times d :: a : b$; and suppose the density to be doubled, then the weight must also be double, the magnitude remaining the same,

hence $2w : m \times 2d :: a : b$:

Again, if the magnitude be tripled, it is manifest the weight will also be increased 3 times, and so on:

consequently $2w \times 3 : 3m \times 2d :: a : b$,

That is, the weight or quantity of matter $6w$ is directly proportional to the magnitude $3m$ multiplied by the density $2d$.

Corol. If the magnitude be given, the weight is as the density. And when the density is given, the weight will be as the magnitude.

313. *The momentum or quantity of motion generated by an impulse or momentary force, is as the force that generates it.*

For a double force will manifestly generate a double quantity of motion or momentum; a triple force a triple momentum, and so on. That is, the motion impressed is directly as the percussive or motive force which produces it.

314. *The spaces described in uniform motions, are in the compound ratio of the velocities and the times of their description.*

Thus if the velocity be v feet per second, and the time = t seconds, then the space described in the time t will be $v \times t$ feet: that is, the space is directly as vt. And if s = the space in feet, then $s = vt$.

Corol. 1. Hence if the time be the same, the space described will be as the velocity: but when the velocity is the same, it will vary as the time.

Corol. 2. Since $t = \frac{s}{v}$, and $v = \frac{s}{t}$: therefore in uniform motions, the time is as the space directly, and velocity reciprocally. And the velocity is as the space directly and time reciprocally.

315. *Let m denote the momentum or quantity of motion in a moving body, w its weight or quantity of matter, and v its velocity; then if they are supposed to be variable, m will vary as w × v. That is, the momentum will be in the compound ratio of the mass and velocity.*

If a body be put in motion with any initial velocity by a momentary force, it is manifest that double that force will be necessary to communicate a double velocity, and a triple velocity will require a triple force, and so on: now the momentum being as the generating force (313) it follows, that in the same body, the momentum is as its velocity; but the momentum also increases as the quantity of matter increases, consequently in all bodies it must be as the mass and velocity jointly: or m is directly proportional to wv.

Corol. 1. Hence the ratio of the momenta of two bodies in motion, is compounded of the ratios of their masses and velocities. For let the momentum, weight or mass, and velocity of a body be denoted by M, W, and V, respectively,

$$\text{then } m : w \times v :: M : W \times V$$

$$\text{That is } \frac{m}{M} = \frac{w}{W} \times \frac{v}{V}.$$

Corol. 2. Since $\frac{m}{M} = \frac{w}{W} \times \frac{v}{V}$, we have $\frac{v}{V} = \frac{m}{M} \times \frac{W}{w}$ that is, the ratio of the velocities is compounded of the direct ratio of the momenta, and the reciprocal ratio of the weights or quantities of matter.

SCHOLIUM. To exemplify this proposition in numbers, suppose two cannon shot, one 9*lb.* the other 36*lb.* to strike an obstacle with the respective velocities of 1000 and 800 feet per second; then their momenta or the forces with which they meet the obstacle will be as 9×1000 and 36×800, or as 5 to 16. In this manner the forces of impact or percussion are compared one with another. But it may be observed that such forces cannot be compared with the force of pressure or weight, or bodies at rest, no more than a rectangle can be compared with the line by which it is generated.

316. *If a quiescent body be urged by an uniformly accelerating force during a given time, the velocity generated at the end of that time will be in the compound ratio of the force and time of acting.*

Let t denote the time, and f the constant force; and conceive the time to be divided into innumerable equal particles; then the first impulse will manifestly generate in the body a velocity proportional to the acting force f, which velocity may be considered uniform during the first particle of time, we can therefore denote this velocity by f because it is proportional to that force; now while the body is moving with the velocity f, it receives another impulse equal to the former, which must generate an equal velocity, the body therefore in the second particle of time will move with a celerity proportional to $f+f$ or $2 \times f$; in like manner $3 \times f$ will denote the velocity during the 3*d.* particle of time, and so on; consequently the last velocity or that during the tth. or ultimate particle of time will be represented by $t \times f$.

And in uniformly retarded motions, the diminished velocity will also be in the compound ratio of the retarding force and time.

Corol. 1. Therefore in uniformly accelerated, or retarded motions, the increments, or decrements of velocity, are equal in equal times, because f, $2f$, $3f$, &c. form an arithmetical progression. And hence we can determine the relation between

the time and space described; for it is evident that the space described in the time t with the successive velocities f, $2f$, $3f$, &c. would also be described in the same time with an uniform velocity which is a mean between all the velocities or terms of that series: now the greatest velocity or greatest term of the progression is tf; and as the particles of time are supposed to be indefinitely small, the least term may be taken $= o$; and the number of terms being $= t$, we have $o + f + 2f + 3f + \ldots . tf = (o + tf) \times \frac{1}{2}t$, or $\frac{1}{2}t^2f$ the sum of all the terms or celerities, which being divided by t their number, gives $\frac{1}{2}tf$ the mean velocity, equal to half the greatest (tf); hence it appears, that if the body moved uniformly with half its greatest celerity, it would describe the same space in the same time. Now the space being in the compound ratio of the velocity and time (314) it will therefore be as $\frac{1}{2}tf \times t$ or $\frac{1}{2}t^2f$, that is, as t^2 the square of the time, the force f and body remaining the same. And because the velocities generated, or destroyed, are as the times of description, the space will also be as the square of the velocity. If the body varies, the velocity (with the same force) is inversely as the mass or weight, in which case the space described will be directly as the force and square of the time, and reciprocally as the mass.

Corol. 2. Since in the same body, the momentum is as its velocity, therefore the momentum generated or destroyed by an uniformly accelerating or retarding force acting for any time, is also in the compound ratio of the force and time of acting.

Scholium.

Let $w =$ the weight or mass or quantity of matter in a body,
$f =$ the force constantly acting on it,
$t =$ the time of its acting,
$v =$ the velocity generated in that time,
$s =$ the space described,
$m =$ the momentum at the end of the time t:

Then $\propto$ being the symbol denoting general proportion, we have, from the two last articles, the following relations in uniformly accelerated motions,

$$m \propto wv \propto tf.$$
$$v \propto tf.$$
$$s \propto vt.$$
$$s \propto \frac{t^2 f}{w}.$$
$$\left.\begin{array}{l} s \propto t^2 \\ s \propto v^2 \end{array}\right\} \text{when the force and mass are proportional.}$$

And from these proportions or relations, other comparisons are readily derived. Thus, since equimultiples or submultiples of quantities have the same ratio as the quantities themselves, if (for example) we divide $wv \propto tf$ by w the result is $v \propto \frac{tf}{w}$, that is, the velocity generated or destroyed in any given time, is directly as the force and time, and inversely as its weight or mass when the latter is not given.

Since $s \propto t^2 \times \frac{f}{w}$ is the same as $s \propto \frac{t^2 f}{w}$, if the force ($f$) and mass or weight ($w$) are proportional, then omitting $\frac{f}{w}$, we have $s \propto t^2$: for, by the nature of fractions, when s is as $t^2 \times \frac{f}{w}$, and f as w, s will be as t^2, or $s \propto t^2$, as above. This takes place in bodies acted on by gravity, where the force is proportional to the weight or quantity of matter. But (310 def. 7) if $\frac{f}{w}$ (the accelerating force) $= F$, then $s \propto t^2 F$, whence $t \propto \sqrt{\frac{s}{F}}$. And because $s \propto vt$, we have $t \propto \frac{s}{v}$, therefore by substitution $\frac{s}{v} \propto \sqrt{\frac{s}{F}}$, and $s \propto \frac{v^2}{F}$, whence $v \propto \sqrt{sF}$.

Hence we shall have

$$v \propto \sqrt{sF} \propto Ft.$$
$$t \propto \sqrt{\frac{s}{F}} \propto \frac{v}{F}.$$
$$s \propto t^2 F \propto \frac{v^2}{F}.$$

And given quantities are also to be left out. Thus s varies as vt, or $s \propto vt$; now if v the velocity is given, then $s \propto t$, or the space will vary as the time.

317. To compare the velocities, &c. of two bodies, let W denote any other weight or mass, and F, T, V, S, M, the acting force, time, &c. as above;

Then

$$v : \frac{tf}{w} :: V : \frac{TF}{W}, \text{ whence } \begin{cases} \dfrac{v}{V} = \dfrac{t}{T} \times \dfrac{f}{F} \times \dfrac{W}{w} \\ \dfrac{t}{T} = \dfrac{v}{V} \times \dfrac{F}{f} \times \dfrac{w}{W}, \end{cases}$$

$$s : vt :: S : VT \ldots\ldots\ldots\ldots \frac{s}{S} = \frac{v}{V} \times \frac{t}{T}$$

$$s : \frac{t^2 f}{w} :: S : \frac{T^2 F}{W} \ldots\ldots\ldots\ldots \frac{s}{S} = \frac{t^2}{T^2} \times \frac{f}{F} \times \frac{W}{w},$$

&c. &c.

But numeral results are obtained from quantities denoted by numbers. We shall subjoin an example or two. Let $f =$ the force of gravity, which may be considered as uniform near the earth's surface. Then since it has been found by experiments that a body descends from rest in a perpendicular direction the space of $16\frac{1}{12}$ *feet* in the first second of time, and because an equal space would be described by the body in the same time if it moved uniformly with half its acquired velocity, (316, corol. 1) its velocity therefore at the end of the first second of time will be $16\frac{1}{12} \times 2$, or $32\frac{1}{6}$ *feet* per second; and the celerity generated or destroyed being as the times of description, we have 1 *sec.* : $32\frac{1}{6}$ :: 2 *sec.* : $32\frac{1}{6} \times 2$, or $64\frac{1}{3}$ *feet* per second the velocity at the end of 2 seconds; and therefore $32\frac{1}{6}t$ feet is the velocity per second which bodies acquire in descending perpendicularly from rest, at the end of t seconds. Also since the spaces described are as the squares of the times, or the squares of the generated celerities, we have

$$t^2 : T^2 :: s : S,$$
$$v^2 : V^2 :: s : S,$$

$$\text{whence} \quad S = \frac{sT^2}{t^2} = \frac{sV^2}{v^2}.$$

$$T = t\sqrt{\frac{S}{s}} = \frac{Vt}{v}.$$

$$V = v\sqrt{\frac{S}{s}} = \frac{Tv}{t}.$$

Suppose it is required to find how far a heavy body would descend by the force of gravity or its own weight in 6 seconds of time, and also its velocity at the end of that time. Then $s = 16\frac{1}{12}$, $t = 1$ sec. $T = 12$ sec. and $v = 32\frac{1}{6}$,

and we have $S = \frac{sT^2}{t^2} = 16\frac{1}{12} \times 36 = 579$ *feet*, the distance:

And $V = \frac{Tv}{t} = 32\frac{1}{6} \times 6 = 193$ *feet* per second, the celerity.

Admit a shot to be discharged in a perpendicular direction with an initial velocity of 193 *feet* per second; to what height would it ascend, and what time would elapse before it fell to the ground again?

Here $S = \frac{sV^2}{v^2} = \frac{16\frac{1}{12} \times 193^2}{(32\frac{1}{6})^2} = 579$ *feet*, the height:

And $T = \frac{Vt}{v} = \frac{193 \times 1}{32\frac{1}{6}} = 6$ seconds the time of its ascent, therefore 12 seconds is the time required.

The mass or weight of the body is not considered in these computations, because all bodies would fall equally fast if they were not resisted by the air. The laws of descent therefore suppose that bodies fall in a non-resisting medium. If f and F denote the motive and accelerative forces, respectively, then (310. def. 7) $F \propto \frac{f}{w}$: now if a body descends perpendicular by its own weight or the force of gravity, the weight itself is the motive force, and consequently $\frac{f}{w} = 1$, that is, the accelerative force of gravity F is constant; this is usually expounded by $32\frac{1}{6}$ *feet* the increase of velocity generated by that force in every second of time. Gravity however, strictly speaking, is a variable force, for a body is somewhat heavier near the earth's surface than at any distance above it, because it is more strongly attracted by the earth in the former situation than in the latter.

OF THE

COMPOSITION AND RESOLUTION

318. When the effects of several forces
directions are reduced to that of a single fo
direction only, it is called *composition of f*
versely, the *resolution of forces* consists in
more forces whose joint effect in different d
equivalent to that of a single force in a given

319. *Suppose a body at* B *to be urged in*
and BN *by two forces that would separately*
uniformly along the lines BD *and* BN *in th*
both forces act together, the body, by the
will describe BC *the diagonal of the parallel*
the same time t.

Conceive BD and BN to be two in-
flexible lines or wires in contact with the
body placed between them at the angular
point B; then if the lines begin their mo-
tions together and move parallel to them-
selves in the same plane towards NC and
DC, the body will be carried or urged along
two lines or wires, and constantly move in th
*d*O*n* formed by their intersection, its track
the diagonal BC; for let *bd* and P*n* be any
tions of the moving lines, then because BD
from the position BD to NC in the same tim
uniformly from BN to DC, their velocities a
and BN; and for the same reason, BP and
velocities when the lines are in the positions
the velocities being uniform, the lines BD,
are therefore proportional, consequently (by
intersection O or place of the body will always

of the parallelogram BNCD, and since it is supposed to be always in contact with the moving lines or wires, its situation at the end of the time t is the angular point C.

And the body will also describe the diagonal BC when urged by uniformly accelerating forces, provided they are similar: For let T and t denote the times of describing BD or BN, and BP or Bb, respectively; then the spaces being as the squares of the times (316, corol. 1.) we have

$$\text{BD} : \text{BP} :: T^2 : t^2,$$
$$\text{BN} : \text{B}b :: T^2 : t^2,$$

whence by equality $\text{BD} : \text{BP} :: \text{BN} : \text{B}b$:

That is, the parallelograms BPOb, BDCN are similar, and therefore the angle dOn or situation of the body is always in the diagonal BC as before. The same thing is also manifest in the case of uniformly retarding forces:

Thus, suppose the motion of BD to be $144\frac{1}{4}$ *feet* in the first second of time, $112\frac{7}{12}$ in the next, $80\frac{1}{12}$ in the third, &c. and that of BN 63 *feet* in the first second, 49 in the next, 35 in the third, &c. then, for example, if bd, Pn; NC, DC, are the positions of the lines at the end of the first, and third seconds of time, respectively; we have Bb=$144\frac{1}{4}$, BP=63, BN=$337\frac{1}{4}$, and BD=147 *feet*;

$$\text{and BN} : \text{BD} :: \text{B}b : \text{BP},$$
$$\text{or } 337\tfrac{1}{4} : 147 :: 144\tfrac{1}{4} : 63.$$

Or suppose Bb, BP are described in 2 seconds, then Bb=$257\frac{1}{3}$, and BP=112 *feet*; and $337\frac{1}{4} : 147 :: 257\frac{1}{3} : 112$. Therefore the parallelograms BPOb, BDCN, are similar, as before.

Corol. 1. The velocities at the points P, O, b, and consequently the forces in the directions BP, BO, Bb, are as the lines BP, BO, Bb. And the force in the direction BO is equivalent to, or *compounded of*, the two forces in the directions BP and Bb.

Corol. 2. And since the forces in the directions BP, BO, Bb, may be expounded by those lines, it follows that any single force BO, or BC, can be *resolved into* two other forces acting

BD be uniformly accelerated, like that of a body fal
rest towards the earth; and suppose the line BD t
the direction BN so as to describe each of the equal
lines Bb, bA, An, &c. in the same time t; then if
SG, AG; DC, NC, &c. are the positions of the m
at the end of t, $2t$, $3t$, &c. times, respectively, the la
O, G, C, &c. will be the corresponding places of
and since the lines Bb, BA, BN, &c. (or their equals
DC, &c.) are directly as the times of description, an
tances BP, BS, BD, &c. as the squares of the ti
corol. 1) it will be

$$BP : PO^2 :: BS : SG^2 :: BD : DC^2, \&c.$$

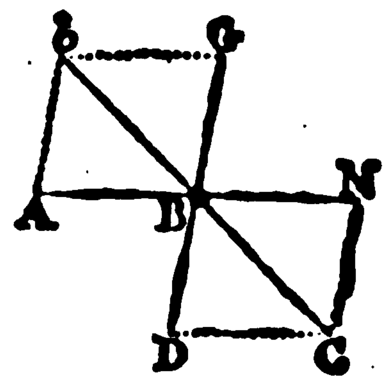
hence the points B, O, G, C, &c. are in the curve of a
And BN is a tangent to the curve at B. (299.)

Corol. 4. Hence also the forces of oblique and direct impact may be compared: Thus, suppose a body to be urged from B in the direction BC by a force denoted by the line BC, then if that force be resolved into two other forces BD and DC (or

BN) the former parallel and the latter perpendicul
obstacle NC, the line DC will represent the force e

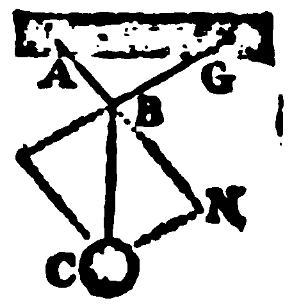
For example, suppose a 48*lb.* shot when moving with a velocity of 1000 *feet* per second should strike an object (NC) in an angle of 50° (NCB), then 48 × 1000 will denote its momentum, and *rad.* : *sine* 50° :: 48 × 1000 : 36770; therefore its force against the obstacle will be less than it would be in a perpendicular direction in the proportion of 36770 to 48000.

320. *If three forces of the same kind* A, G, C, *act together in the same plane against the body* B *in the directions* AB, GB, CB, *and thereby keep it in equilibrio, those forces will be proportional to the sides of a triangle* BDC (*or* BNC) *which are drawn parallel to the directions* AB, GB, *and* CB.



This is manifest from the last proposition corol. 1. for since the force (*b*B) in the direction BC is equivalent to, or compounded of, the two forces in the directions AB or BN and GB or BD, if the former (BC) be exerted in a contrary direction (CB) the effects of the other two will be destroyed, and the body must remain quiescent; the three forces therefore are as BN, NC (or BD), and GB, the sides of the triangle BNC or its equal BA*b*.

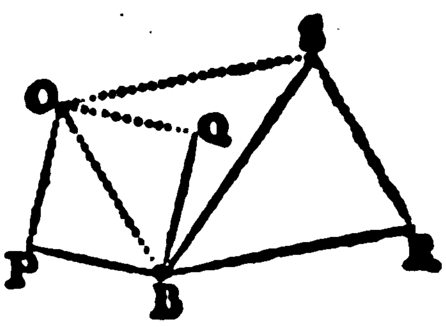
Corol. 1. And because three lines perpendicular to the sides of a triangle will form another similar triangle, the three forces will also be proportional to the sides of that similar triangle.



Corol. 2. Hence if the force in the direction BC be a weight C suspended by three strings or cords AB, GB, BC, the tensions of the cords or the forces by which they are stretched, will be as the sides of the triangle BNC. For example, if BN = 4, NC = 3, BC = 5, the tensions of the cords AB, GB, BC, will be as 4, 3, and 5, respectively.

Corol. 3. The forces in the directions AB, GB may be reduced to a single force (*b*B) acting in a direction contrary to

that of CB, and the body kept in equilibrio by two opposite and equal efforts. But if the body (B) be put in motion by three given forces PB, QB, RB of the same kind, acting in the same plane, then a single force equivalent to all three may be found thus: Complete the parallelogram QBPO, and the diagonal OB will represent a force equal to the two forces PB and QB; and if RS and OS are respectively parallel to BO and BR, the two forces OB and RB will be reduced to the diagonal SB or the single force SB, which therefore is the force equivalent to the three given forces; that is, the single force SB acting in the direction SB would have the same effect on the body B as the three given forces acting together in the directions PB, QB, and RB.



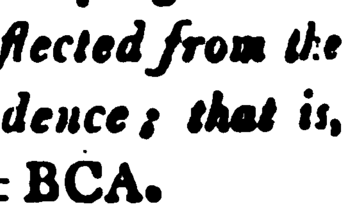
Hence it appears that any single force may be resolved into three or more forces acting in different directions.

SCHOLIUM. What is advanced in this last article will hold true in all kinds of forces whatever, whether of impulse or percussion, pushing, or drawing, or whether instantaneous, or continual, provided they are similar.

ON THE COLLISION OF BODIES.

321. *If a perfectly elastic spherical body* B *impinge on an immovable plane* AG, *it will rebound or be reflected from the surface in an angle equal to the angle of incidence; that is, if* C *be the point of impact, the angle* DCG = BCA.

Let BC denote the force of the body in that direction, which suppose to be resolved into two other forces BP and BA, the former parallel, and the latter perpendicular to the plane AG; then if we conceive the body to be urged or carried along

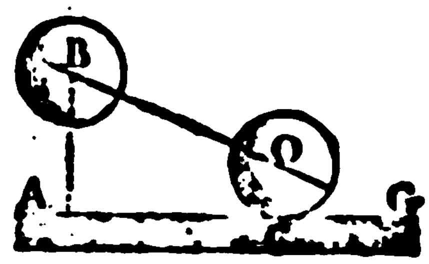


the diagonal BC by those forces or lines moving parallel to themselves, it must meet the plane in the point C with a force equal to PC; and since there is no resistance in the direction of the surface AG, the force (BA or PC) in that direction will not be retarded by the stroke, the body therefore, after impact, is urged by two forces respectively equal to the two former, namely, one in the direction of the surface CG as before, the other in that of CP, this latter is the re-acting or restoring force (*def.* 9) which, if the body be perfectly elastic, is equal and contrary to the compressing force PC (*ax.* 2 and 3); hence, by composition, CD the track of the body after impact, must be inclined to the reflecting surface in the same angle as before.

Corol. 1. The velocity with which the body quits the reflecting surface is equal to that at the time of impact, because the generating forces are equal and in similar directions.

Corol. 2. Since the times of compression and restitution are not instantaneous, the body is moved in the direction CG during those times by the force AC, and consequently the point of incidence and that of reflection cannot *accurately* be the same if the body is elastic.

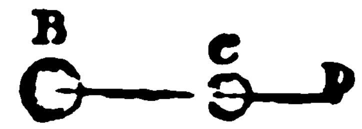
Corol. 3. When the surface AG is not smooth, the body will be reflected with a whirling motion: For let O be its center, and C the point of impact; then while the body is retarded in the direction AG by the friction at C, the force in the direction BO must produce a motion by which it endeavours to roll. This is confirmed by experience, for spherical bodies are seen to acquire a rotatory motion when reflected obliquely. This motion may affect the direction of the body when it quits the plane: And if the body is not perfectly elastic, the restoring force will be less than the compressing one: on these accounts, it is probable that the angles of incidence and reflection are always different.



Corol. 4. If the body be non-elastic, it will not acquire or generate a restoring force by impulse, (*def.* 9) it must therefore after the impact, be carried along the surface CG by the force acting in that direction.

Remark. If AG be a polished surface, and BC a ray of light proceeding from the lucid point B, the ray will be reflected in the direction CD, that is, the angles of incidence and reflection are equal in that case. This is a fundamental law of Optics, founded in nature according to some writers, because 'tis said *nature always acts by the most expeditious methods;* for the sum of the lines BC and DC are *less* than the sum of any other two lines that can be drawn from the points B and D to meet in the surface AG (*Theorem*, art. 272). Sir I. Newton however, has shewn that the reflection of light is not effected by its particles striking against bodies, but by some repelling power that extends beyond their surfaces. But if a particle of light moving along BC be struck in a direction (CP) perpendicular to the surface AG, either at C, or before it reaches that point, so that its velocity is not changed by the impulse, it will be reflected in an angle equal to that of incidence (*corol.* 1.)

322. *Suppose* B *and* C *are two equal non-elastic bodies, and let the body* B *strike the quiescent body* C *in the direction of their centers with a velocity of V feet per second, then after the impact they will proceed together as one body in the direction* CD *with a velocity equal to half V.*



For both bodies being non-elastic, they cannot generate any force that will cause them to recede from one another; and since a double quantity of matter is moved by the same force, (that of B at the impact) the velocity must be diminished in the same proportion (315), that is, the velocity is reciprocally as the augmented weight or mass;

$$B : V :: B + C \text{ (inversely)} : \frac{V \times B}{B + C}, \text{ or } \tfrac{1}{2} V \text{ when } C = B.$$

Corol. The momentum of both bodies moving together, will be the same as that of B before the stroke; for let B and C denote the weights or quantities of matter in the bodies B and C, whether equal or unequal: then $V \times B$ will represent the momentum of B, and $\frac{V \times B}{B+C} \times (B+C)$ or $V \times B$ that of both after the impact.

Suppose $B = 12$lb. $c = 4$lb. and $V = 20$ *feet*, then $\frac{V \times B}{B+C} = \frac{20 \times 12}{12+4}$ $= 15$ *feet* the velocity per second of both together after their congress.

323. *Let the body B moving with a velocity $= V$ overtake and strike the body C whose velocity in the same direction is $= v$; then if the bodies are non-elastic, they will proceed together as one mass with a velocity* $= \frac{V \times B + v \times C}{B+C}$.

For the force lost in B by the stroke is communicated to C, because action and re-action are equal; and therefore the force or momentum of both moving together is equal to the sum of the separate momenta, that is $V \times B + v \times C$; and this divided by the mass $B+C$ gives $\frac{V \times B + v \times C}{B+C}$ the velocity.

Let $B = 16$lb. $C = 4$lb. $V = 10$, and $v = 5$;

Then $\frac{V \times B + v \times C}{B+C} = \frac{10 \times 16 + 5 \times 4}{16+4} = 9$ *feet*, the velocity per second.

Corol. 1. If C be quiescent, $v = o$; and $\frac{V \times B}{B+C}$ is the velocity with which they proceed together after impact.

Corol. 2. If C moves in a contrary direction, or towards B, then $\frac{V \times B - v \times C}{B+C}$ will denote the velocity after the stroke. And when $V \times B - v \times C = o$, all motion is destroyed by the concourse. But if $V \times B - v \times C$ be negative, both bodies will move together towards B.

Suppose B $=4$, C $=16$, $V=5$, and $v=10$;

Then $\frac{V\times B-v\times C}{B+C}=\frac{5\times 4-10\times 16}{4+16}=-7$ the velocity, v

therefore is towards B after impact.

Corol. 3. The velocity lost by B is $V-\frac{VB+vC}{B+C}=\frac{VC-}{B+}$

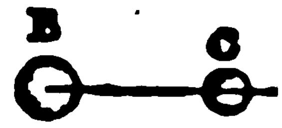
that is, $B+C : C :: V-v :$ velocity lost by B.

In like manner, $B+C : B :: V-v :$ velocity gained by C

If the bodies move in contrary directions, that body prevail whose momentum is greatest, but its velocity wil diminished, consequently $V-\frac{VB-vC}{B+C}$, or $\frac{VC+vC}{B+C}$ is velocity lost,

That is, $B+C : C :: V+v :$ velosity lost by B, supposi the most powerful of the two.

324. *If a non-elastic body* B *impinge directly on a* *but perfectly elastic body* C *with a given velocity, it will bound with the same velocity; and the whole force exerte* C *against the striking body* B, *is double the force of im when both bodies are non-elastic.*



For if both were non-elastic, the motion or force of B would only be destroyed by the impact, or the bodies would adhere; but when C is perfectly elastic, it not only destroys all that m or force but exerts another force equal and contrary to it i action of recovering its figure before the stroke; consequen will recede with its former velocity: and as the elastic bod first destroys and then restores the same force, its effect is do that of a non-elastic body. And if both bodies are perf elastic, the effect is the same, for the whole force of restitu must be equal to that of compression.

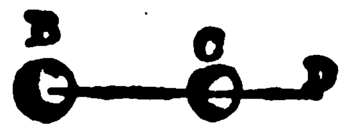
Corol. 1. If C be moveable, the velocity lost by B, communicated to C by the stroke will be double what

would be were the bodies non-elastic; for the restoring force acts just as much in the direction of B's motion as against it, consequently while the bodies recede, B is retarded and C urged by additional forces equal to that of impact; that is, the velocities lost, or communicated by collision, are twice as great in elastic as in non-elastic bodies.

Corol. 2. Since the force lost in one body is gained by the other, if B and C are equal, and both perfectly elastic, C being moveable, the striking body B will rest after collision, and the other C move with a velocity equal to that of B before the impact.

Corol. 3. Hence it appears that the velocities are relatively the same before and after the impulse, that is, the bodies will be equally distant from one another at equal times before and after the impact.

325. *If the body* C *moving towards* D *with a celerity* $= v$ *be struck by the body* B *whose celerity in the same direction is* $= V$; *to find their velocities after the impulse, supposing both are perfectly elastic.*



It follows from *art.* 323 corol. 3, and *art.* 324 corol. 1, that the celerity lost by B after the impact, is $\frac{VC - vC}{B + C} \times 2$, and therefore $V - \frac{VC - vC}{B + C} \times 2$, or $\frac{V(B - C) + 2vC}{B + C}$ is its velocity in the direction BD or DB, according as the expression is positive or negative.

And (by the same corollaries) $v + \frac{VB - vB}{B + C} \times 2$ or $\frac{v(C - B) + 2VB}{B + C}$ is the velocity of C in the direction CD.

But if C be moving in a contrary direction, or towards B, then by making v negative, the same expressions become

$$\frac{V(B-C)-2vC}{B+C} \text{ the velocity of B;}$$

$$\frac{v(B-C)+2VB}{B+C} \text{ the velocity of C,}$$

towards D when the expressions are positive, but in the opposite direction if they are negative.

Let B = 12*lb.* $V = 5$ *feet* per second; C = 8*lb.* and $v = 60$ *feet* per second; and suppose the bodies move in contrary directions,

then $\frac{V(B-C)-2vC}{B+C} = \frac{5(12-8)-2\times 60\times 8}{12+8} = -47$, velocity of B,

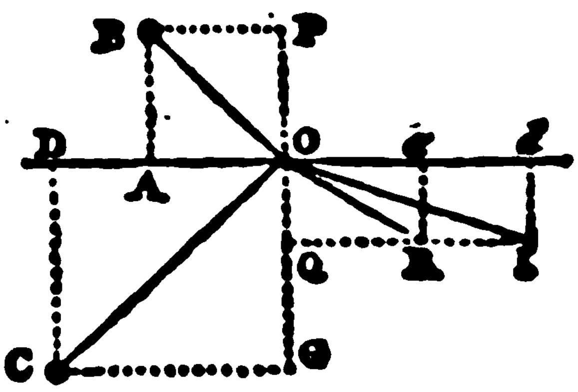
$\frac{v(B-C)+2VB}{B+C} = \frac{60(12-8)+2\times 5\times 12}{12+8} = 18$, velocity of C;

Therefore after collision, the bodies will move again in contrary directions, with velocities of 47, and 18 *feet* per second, respectively.

Corol. If C be at rest, then by making $v = 0$, we shall get the velocities in that case.

The preceding method of investigation will answer when one of the bodies is non-elastic; or when both are imperfectly elastic, provided the forces of elasticity are known.

326. *If the non-elastic bodies* B *and* C *move in the same plane, and strike one another obliquely at the point* O *with given velocities; to determine their directions and velocities after collision.*



Let BO and CO be taken in the ratio of the respective velocities; and suppose D*d* is drawn to touch the bodies at their point of contact O. Complete the rectangles BAOP, CDOG; then the celerity BO is resolved into two others AO and PO, and the celerity CO into DO and GO (319, corol. 2): now as the efforts of the bodies against each other are made in the line joining their centers, those forces are not affected by

the velocities (AO, DO) in the direction Dd; consequently the velocities, in the line (PG) in which they act against one another, are denoted by PO and GO: and $\frac{B \times PO - C \times GO}{B + C}$ is the celerity with which they would proceed together after direct impact with the velocities PO and GO (323, corol. 1): If B be the most powerful, let OQ (in OG) be made $= \frac{B \times PO - C \times GO}{B + C}$, and take $Oa = OA$, and $Od = OD$, and complete the rectangles $OQRa$, $OQId$; then the diagonals OR and OI will be the directions and velocities of B and C, respectively.

If the bodies are elastic, they will be reflected after impact; but the construction is no ways different: for having found the velocities in the line PG by art. 325, the result will point out whether they must be set off on the same, or on contrary sides of O.

N. B. The bodies are supposed to move along OR and OI after impact; strictly speaking however, their centers do not describe those diagonals, but lines parallel to them.

SCHOLIUM.

The preceding conclusions respecting the collision of bodies are confirmed by experiment, abstracting from the imperfection of materials; for it is probable there is no surface perfectly smooth, nor any hard bodies either perfectly elastic or non-elastic. Some experiments however, made with a view to ascertain the force of bodies in motion, seem to have misled several eminent mathematicians of the last century. Thus because it is found that a hard body impinging on soft and yielding substances of uniform consistence will penetrate to depths proportional to the squares of the velocities of impact, it has been infered that the momentum or force of bodies in motion, instead of being compounded of its velocity and mass (def. 4) is as the *square* of the velocity into the mass: this erroneous conclusion

results from ascribing a whole effect to part of its cause: for the whole effect (or depth to which the body penetrates) is not produced by the motion or force of the body at the moment of impact, but by its successive efforts during the time of penetration, each effort being as the body drawn into the velocity with which it is moving. So bodies when projected vertically rise to heights proportional to the squares of the initial velocities (316, corol. 1) and during the time of ascent act against gravity, which, like the soft and yielding substances, is an uniformly retarding force; but to infer from this, that the force of the ascending body at any point of time is as the square of its velocity into the mass, would be contrary to theory and experiment.

Balls discharged from guns would penetrate wood, banks of earth, &c. to depths proportional to the squares of the velocities of impact, provided the resistances were uniform. And Mr. Robins found that musket bullets of equal size when shot against a block of elm with velocities of 1700, 730, and 400 *feet* per second, penetrated to the depths 5, 1, and $\frac{1}{4}$ inches, respectively: these numbers are not exactly as the squares of the velocities; but "a greater coincidence cannot be expected when the unequal texture of the same piece of wood, and the change of the form of the bullet by the stroke are considered." (Gunnery, Chap. 2. Prop. 8). These experiments however, have been objected to as inconclusive *.

In estimating the force of a pile engine, the velocity of the weight or ram is easily determined: but if the pile be heavy, its momentum should be taken into consideration, because the ram and pile proceed as one body after the impact: and if the ground resist uniformly, the pile will sink to depths proportional to the squares of the velocities with which it begins to move.

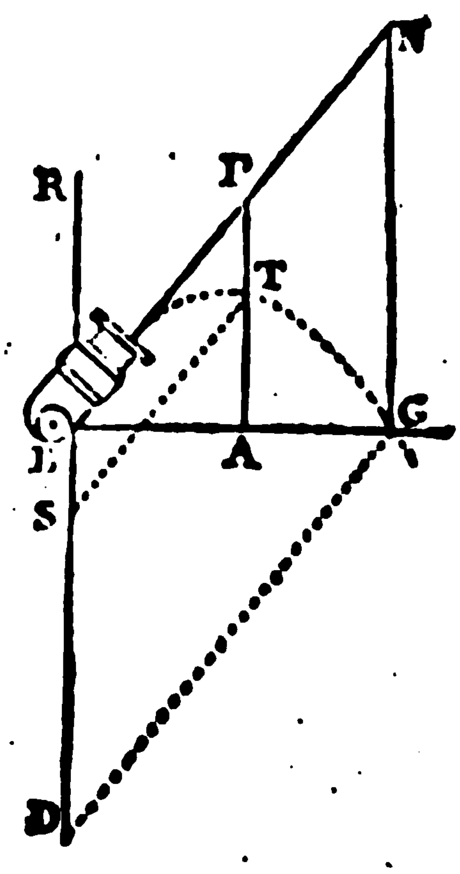
Bodies impinging with equal momentums may have different effects. Thus a 48*lb.* shot with a velocity of 1000 *feet* per

* Hutton's Math. and Philos. Dictionary, art. GUNNERY.

second, and a battering ram whose weight is 12000*lb.* moving with a velocity of 4 *feet* per second would have equal momentums, for 48 × 1000 = 12000 × 4 : but the former when discharged against a wall (for example) might pass through it without any other effect than that of driving out a few bricks or stones; whereas an impulse of the ram would probably cause a large breach: for that part of the wall upon which the ball impinges is separated and driven out before it can communicate much motion to the adjacent parts; but the shake is extended to a considerable distance by the slow movement of the battering ram, because the parts struck adhere together for a longer time.

Of PROJECTILE MOTION.

327. *Let the equal lines* BD *and* CN *be perpendicular to the plane of the horizon represented by* BC; *and suppose a shell is discharged from the mortar* B *in the direction* BN *with a velocity that would carry it uniformly from* B *to* N *in the same time that a heavy body would descend by its gravity from* B *to* D; *then if the motion of the shell is not affected by the resistance of the air, it will describe the parabolic curve* BTC. (Art. 319, corol. 3).



BC is the horizontal range or amplitude.

CBN is the angle of elevation.

The velocity with which the shell quits the mortar, is the initial or projectile velocity.

And if the perpendicular BR be made equal to the height to which the body would ascend if projected vertically, it will represent the *impetus.*

TA is the altitude of the projection, T being the highest point in the curve.

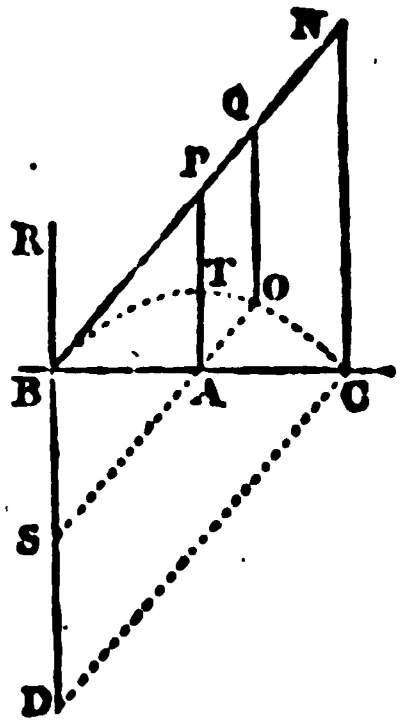
Thus suppose the angle of elevation CBN $= 45°$, the time of flight or that in which the shell describes the curve $= 12$ seconds $= t$, and $d = 16\frac{1}{12}$ *feet*; then $dt^2 = 2310$ *feet* $=$ BD or CN (317) $=$ BC the range in this case; and $dt^2\sqrt{2} =$ the tangent BN, which divided by t (the number of seconds) gives $dt\sqrt{2}$ *feet* the projectile velocity per second, hence (317) the vertical height to which it would ascend in t seconds is $\frac{d \times (dt\sqrt{2})^2}{4d^2}$ $= \frac{1}{2}dt^2$ the impetus BR; which therefore is $=$ half the range at an elevation of 45°.

Corol. 1. Let PA be perpendicular to the horizontal line BC, and TS parallel to PB; then the velocity of the projectile in the direction of gravitation at any point T, is to the projectile velocity in the direction BN, as 2BS or 2PT to PB. For BP and BS are described in the same time; but a body descending from rest through BS would acquire a velocity that would carry it uniformly through 2BS in the same time (317); and as the spaces described with uniform motions are as the velocities, therefore 2BS or 2PT is to BP, as the perpendicular velocity at T, to the projectile velocity in the direction BP.

Corol. 2. The horizontal celerity of the projectile is uniform; for the celerity along BN is uniform, and BA is directly as BP, by similar triangles. Hence also, because the velocity in the direction BC is constant, the celerity in the direction of the curve at any point (B) is as the secant of the angle of elevation; for BP is the secant to the radius BA. Therefore if T be the vertex of the parabola, the motion in direction of the curve will be slowest at that point; and the projectile will move with equal celerities at equal distances from it.

Corol. 3. Let SO, parallel to the tangent BN, bisect BD; then, as the velocity acquired in descending through BD is 2BD or twice the velocity of the projectile at B, therefore 2BS the velocity acquired at S, which is half that at D, will be equal to

the projectile velocity; and by the first corol. as *velocity* in direction of gravity : *velocity* in direction BQ :: 2BS or 2QO : BQ; therefore 2BS or 2QO = BQ, because the velocities or two first terms of the proportion are equal. Whence (301) SO is the semi-parameter to the diameter BS: and when the elevation is 45°, A will be the focus of the parabola; and the height AT = $\frac{1}{4}$ of the range BC.



Corol. 4. Because, when S is the point where the celerity acquired by a body falling freely by the force of gravity from B, would be equal to the projectile celerity at B, the impetus BR is = BS, consequently $2BS = 2BR = 2QO = BQ = SO$, and $BQ^2 = 4BR^2 = 4QO^2 = BR \times 4QO$. But (299) PT, QO, NC, &c. are as BP^2, BQ^2, BN^2, &c. or $PT : QO :: BP^2 : BQ^2$,

$$\text{whence } BQ^2 = \frac{BP^2 \times QO}{PT} = BR \times 4QO, \text{ or } \frac{BP^2}{4PT} = BR,$$

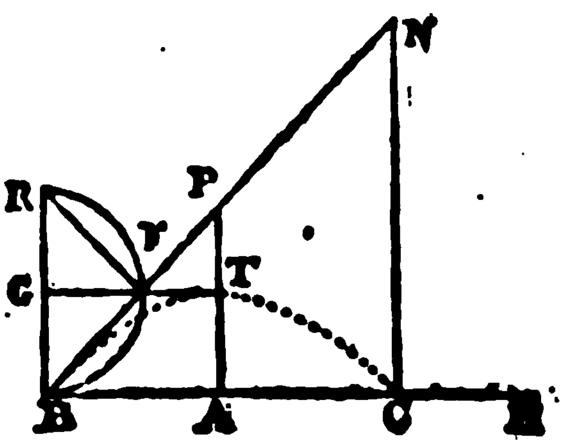
that is, $BR : BP :: BP : 4PT$,

Also, $BR : BQ :: BQ : 4QO$,

$BR : BN :: BN : 4NC$, &c.

329. *Having the impetus, and elevation, to determine the random or horizontal range, and the greatest height to which the projectile will rise.*

If BH be the horizontal line, BR the impetus, and NBC the angle of elevation; then by the last of the preceding corollaries, we have to construct the right angled triangle BCN so, that BN is a mean proportional between BR and 4NC,



On BR describe a semicircle, and take BN = 4BF; let fall the perpendicular NC; and BC is the horizontal range.

Draw RF; then the triangles BCN, BFR being similar, we have

$$\mathrm{BR} : \mathrm{BF}\ (\text{or } \tfrac{1}{4}\mathrm{BN}) :: \mathrm{BN} : \mathrm{NC},$$
$$\text{and } \mathrm{BR} : \tfrac{1}{4}\mathrm{BN} :: 4\mathrm{BN} : 4\mathrm{NC},$$
$$\text{that is, } \mathrm{BR} \times 4\mathrm{NC} = \mathrm{BN}^2,$$

Therefore BN is a mean proportional between BR and 4NC.

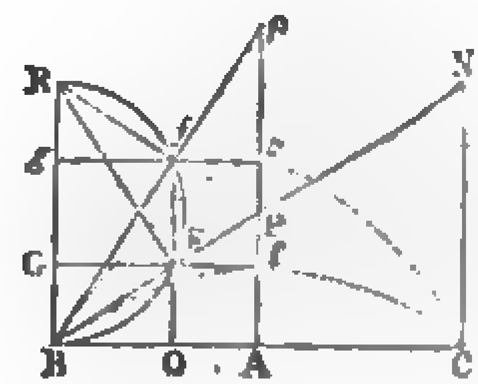
If BC be bisected by the perpendicular AP, and GFT drawn parallel to BC or perpendicular to BR, T will be the vertex of the parabola; and BG or AT its height above the horizon BC. For PB = PN, and FP = FB, and the triangles FBG, FPT being similar, BG is = PT = AT, therefore T is the vertex. (297, corol. 2)

And because BN = 4BF, the range BC = 4GF.

329. *Having the projectile velocity, and the distance* BC *of an object* C *on the horizontal plane; to find the angle of elevation of the mortar or cannon at* B, *so as to hit that object.*

If v = the projectile velocity, and $d = 16\tfrac{1}{12}$ *feet*, then $\frac{d \times v^2}{(2d)^2}$ or $\frac{v^2}{4d}$ is the impetus BR. (327)

On BR describe a semicircle; take BO = $\tfrac{1}{4}$BC, and erect Of perpendicular to BC; then through F and f draw BN, Bp, and either of the angles CBN, CBp, is the elevation required.



For let the perpendicular Ap bisect BC, and draw gft parallel to BA; then it is proved that t is the vertex of the parabola BtC, in the same manner as T is found to be that of the parabola BTC.

Corol. 1. Hence there are two elevations which give the same range with the same velocity; one being as much above 45° as the other is below it.

Corol. 2. When BO or $\frac{1}{4}$ of the range BC, is $=$ the radius of the circle or $\frac{1}{4}$ the impetus BR, then O*f* will touch the circle, and the points F, *f*, coincide, in which case the elevation becomes 45°. The range therefore at 45° elevation is the greatest because its fourth BO will be a maximum.

330. Let s and c denote the *sine* and *cosine* of the angle of elevation.

- r the horizontal range or amplitude BC.
- h the greatest height AT or A*t*.
- m the impetus BR.
- v the projectile velocity or the number of *feet* per second the projected body would describe with its first or greatest velocity.
- t the time of flight.
- $d = 16\frac{1}{12}$ *feet*.

Then from the similar triangles BFR, BCN, we have

$$rad. : BR :: sin. \text{ angle } BRF : BF,$$

That is, $1 : m :: s : sm = BF$, and $4sm = BN$.

And in the triangle BCN

$$rad. : 4sm \ (BN) :: c : 4csm = r = BC \text{ the range:}$$

But $2cs$ is the *sine* of double the angle whose *sine* is s (246), therefore $4cs$ is twice the *sine* of double the elevation; consequently if $a =$ the *sine* of twice the elevation, $2am$ is the horizontal range, or $2am = r$. Hence the ranges with the same impetus, are as the sines of double the elevations: for let A denote the sine of twice any elevation, and R the corresponding range, then $2Am = R$, and $m = \frac{R}{2A}$; also $2am = r$, whence $m = \frac{r}{2a} = \frac{R}{2A}$, or $\frac{r}{a} = \frac{R}{A}$, that is $a : r :: A : R$.

331. If the elevation be the same, but the velocities different, the horizontal ranges are as the squares of the velocities. For let M be the impetus, R the corresponding horizontal range, and a the sine of double the angle of elevation, as above; then $2aM = R$, whence $a = \frac{R}{2M}$; also $2am = r$, and $a = \frac{r}{2m} = \frac{R}{2M}$, that is, $m : M :: r : R$; but if V be the velocity corresponding to the impetus M, then m being $= \frac{v^2}{4d}$, and $M = \frac{V^2}{4d}$ (327) we have $v^2 : V^2 :: r : R$.

332. If both elevations, and also the velocities, are different, the ranges are in the compound ratio of the squares of the velocities and the sines of double the angles of elevation. Thus, let A denote the sine of double any angle of elevation, M, V, and R the corresponding impetus, velocity, and range; then since $2AM = R$, and $2am = r$, we have $\frac{2AM}{2am} = \frac{R}{r}$, that is, $AM : am :: R : r$; but $M = \frac{V^2}{4d}$, and $m = \frac{v^2}{4d}$, whence by substitution $AV^2 : av^2 :: R : r$.

333. To determine the height, AT for example, we have $BF = sm$ (330), whence

$rad. : sm :: s : s^2m = OF = AT = h$ the height; s being the sine of the elevation OBF to $rad.$ 1. But if the time t be given, then $\frac{1}{4}dt^2 = m$, and the height $h = \frac{1}{4}dt^2s^2$.

334. From the preceding articles, we collect the following expressions, namely

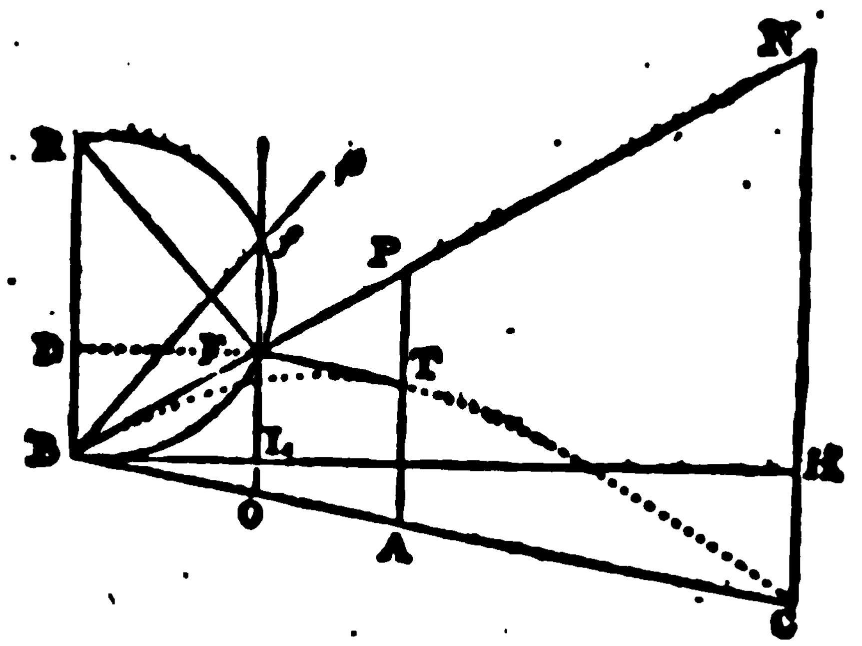
$$m = \frac{v^2}{4d},\ r = 4csm = 2am,\ h = s^2m = \tfrac{1}{4}dt^2s^2;$$

whence $v = \sqrt{4md}$, $m = \frac{r}{2a} = \frac{h}{s^2} = \frac{1}{4}dt^2$, $a = \frac{r}{2m}$,

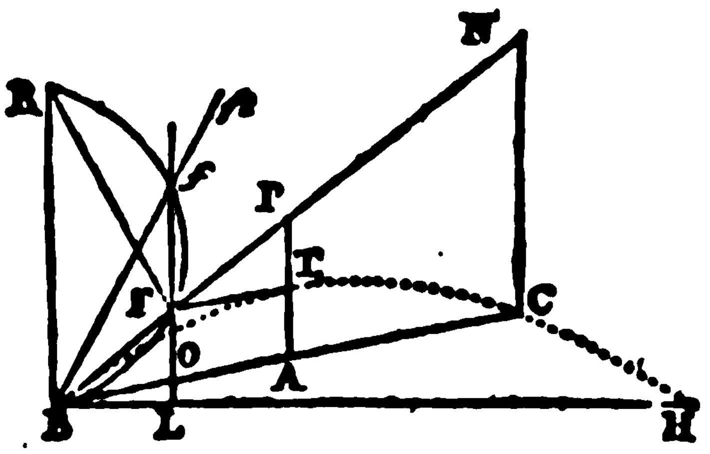
$$s = \frac{r}{4cm} = \sqrt{\frac{h}{m}},\ t = \sqrt{\frac{2h}{ds^2}} = \sqrt{\frac{2m}{d}}.$$

And by substitution, a variety of theorems may be found for the different cases on horizontal planes.

335. *Having the velocity, or impetus, and the angle of direction, to find the range on a plane inclined to the horizon.*



Let BH represent the horizontal line; BR (perpendicular to BH) the impetus; BC the oblique plane; and BN the direction of the projectile.



On the impetus BR describe the segment of a circle to contain an angle BFR equal to the supplement of RBC; take BN = 4BF, and draw NC perpendicular to BH; then BC is the range on the plane BC.

Join RF, and draw FO parallel to NC. Then since the angles BFR, BOF are equal, and the angle RBF equal to BFO, the angles FBO, FRB are therefore equal, and consequently the triangles BFR, BOF are similar: whence BR : BF ($\frac{1}{4}$BN) :: BN : NC, and we have BR $\times$ 4NC = BN^2 as in Art. 328; therefore BN being a tangent to the parabola at B, the curve will pass through the point C. And by sim. triang. BC = 4BO.

Corol. 1. If the impetus BR and range BC are given, the direction of the projectile is found thus: Let the circle be described as above; take BO = $\frac{1}{4}$BC, and draw OF*f* perpendicular to BH, then through F, *f*, draw BN, B*p*, and either of those directions is that required, as in horizontal ranges.

Corol. 2. But if Of touch the circle, the points F, f, will coincide, and the direction bisects the angle RBC between the plane and impetus. And because in that case, BO is a maximum, therefore when the direction of the projectile is equally distant from the vertical BR and plane BC, the range BC will be the greatest possible, as in horizontal ranges.

336. Let S = the *sine* of FRB or FBO the angle of elevation above the plane.

C = the *sine* of BFR or BOF the *cosine* of the plane's inclination to the horizon.

c = the *sine* of RBF or BFO the *cosine* of the elevation above the horizon.

m = the impetus.

r = the range.

t = time of flight.

v = the velocity.

h = AT the greatest vertical height above the plane.

Then $C : \text{BR} :: S : \frac{S \times \text{BR}}{C} = \text{BF}$,

$C : \text{BF}\ \left(\frac{S \times \text{BR}}{C}\right) :: c : \frac{Sc \times \text{BR}}{C^2} = \text{BO}$, and $\frac{4Sc \times \text{BR}}{C^2}$ the range BC, or $\frac{4Scm}{C^2} = r$.

337. Let AP, parallel to CN, bisect BC; then since TA is parallel to the axis of the parabola (which is perpendicular to BH), BC is a double ordinate to the diameter TA, therefore (298) AP is bisected in T, and FT (parallel to OA) a tangent to the curve at T, and consequently the triangles BFO, FPT are similar and equal;

hence, *sin.* BOF : $\frac{S \times \text{BR}}{C}$ (BF) :: *sin.* FBO : FO;

That is $C : \frac{Sm}{C} :: S : \frac{S^2 m}{C^2} = \text{FO} = \text{AT} = h$ the greatest vertical height.

338. The time of describing the curve BTC is equal to the time that a body would be falling freely through NC or 4FO or $\frac{4S^2m}{C^2}$, (319 corol. 3); but if P be any space descended, then (317 schol.) $\sqrt{\frac{P}{16\frac{1}{12}}}$ is the time; therefore, putting $d = 16\frac{1}{12}$ *feet*, the time of flight will be $\sqrt{\frac{4S^2m}{dC^2}}$, or $\frac{2S}{C}\sqrt{\frac{m}{d}} = t$ seconds.

N. B. If S be taken for the *sine* of fBO the highest elevation, the computations refer to the upper parabolas; these however, are omitted in both figures.

339. The expressions $\frac{4Scm}{C^2} = r$, and $\frac{2S}{C}\sqrt{\frac{m}{d}} = t$

$$\text{give } m = \frac{C^2r}{4Sc} = \frac{dC^2t^2}{4S^2} = (\text{by art. } 331)\ \frac{v^2}{4d};$$

$$C = \sqrt{\frac{4Scm}{r}} = \frac{2S}{t}\sqrt{\frac{m}{d}};\ c = \frac{C^2r}{4Sm};$$

And by substitution we get the following theorems for the range, elevation, time, and velocity, on oblique planes:

$$r = \frac{Scv^2}{dC^2} = \frac{4Scm}{C^2} = \frac{dct^2}{S} = \frac{ct^2v^2}{4Sm}.$$

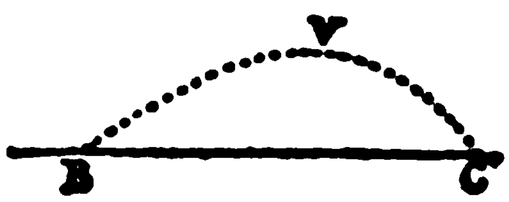
$$S = \frac{dtC}{v} = \frac{Ct}{2}\sqrt{\frac{d}{m}}.$$

$$t = \frac{2S}{C}\sqrt{\frac{m}{d}} = \frac{vS}{dC} = \frac{vrC}{4dcm} = \sqrt{\frac{Sr}{dc}}.$$

$$v = 2\sqrt{dm} = C\sqrt{\frac{dr}{Sc}} = \frac{dtC}{S} = \frac{4dtcm}{rC}.$$

340. The preceding deductions from the properties of the parabola however, are of little use in the practice of Artillery, on account of the very great resistance of the air, which in swift motions, is sometimes more than 20 times the weight of the projected body. And in consequence, the horizontal ranges are often less than $\frac{1}{5}$ of what they would be were the projections

made in vacuo. For example, if a cannon shot be discharged with an initial velocity of 1600 *feet* per second at 45° elevation, then $\frac{1600^2}{4 \times 16\frac{1}{12}} = 39793$ *feet* the impetus, and $2 \times 39793 = 79586$ *feet*, or upwards of 15 miles would be the horizontal range according to the parabolic theory; whereas in actual practice, it is found to be less than 3 miles. And the curve described is not at all similar to a parabola; its vertex or highest point, instead of being vertical to the middle of the range, is nearer the farther extremity, where the curve meets the horizon in a greater angle than that in which the body was projected. See the adjacent figure, where BC represents the horizontal line, and BVC the track described in the flight from B to C.



Many attempts have been made to investigate the nature of this curve, but from hypothetical data; and hence no theory has yet been found to agree with practice; but this will not be considered as extraordinary, since it is known by experiment that the same weight of shot, length of barrel, and quantity of powder frequently give different ranges.

Another great irregularity in the firing of shot is the deflection of the ball to the right or left of the mark. A deviation of this kind is likely to take place when there is considerable windage; for if the ball in its passage along the bore should touch one side, it will be reflected to the other, and again rebound to the opposite side, and so on, and thus acquire a kind of zig-zag motion; in which case the ball must quit the piece in a direction inclined to the axis of the bore. And the friction on that side of the mouth of the cannon touched by the shot when it quits it, will give the ball a whirling motion; the side of the ball therefore which moves foremost will be unequally resisted by the air in consequence of this rotatory motion; which is another cause of deflection to the right or left, except

the axis of rotation be at right angles to the vertical plane in which the projection is made. Bullets discharged from a rifled barrel have the axis of rotation in the direction of the piece, and consequently that side of the bullet which moves foremost is equally resisted by the air in all its parts.

341. But the parabolic theory may sometimes be useful in slow motions if we employ data derived from good experiments, and proceed by comparison in circumstances not very dissimilar. Dr. Hutton found by experiments made at Woolwich that "shot which are of different weights and impelled by the firing of different quantities of powder acquire velocities which are directly as the square roots of the quantities of powder, and inversely as the square roots of the weights of the shot, nearly:"

That is, if $p =$ the *lbs.* of powder,
$w =$ the weight of the shot in *lbs.*
$v =$ its initial velocity:

then if P, W, and V denote any other weight of powder, shot, and velocity, we have

$\frac{\sqrt{p}}{\sqrt{w}} : v :: \frac{\sqrt{P}}{\sqrt{W}} : V$, or $\frac{p}{w} : v^2 :: \frac{P}{W} : V^2$, which, when the balls are equal, or $W = w$, becomes $p : v^2 :: P : V^2$, that is, the squares of the velocities are as the quantities of powder, nearly. Which conclusion agrees with the experiments of Mr. Robins. Very small charges however, and such as exceed those that give the greatest velocity, are excepted.

342. Here follow some Examples in numbers.

1. If a ball of 1*lb.* acquire a velocity of 1600 *feet* per second when fired with 8 ounces of powder, what will be the velocity of a 13 inch shell weighing 196*lb.* when fired with 9*lb.* of powder?

Here $p = \frac{1}{2}$, $w = 1$, $v = 1600$, $P = 9$, $W = 196$;

and $\frac{p}{w} : v^2 :: \frac{P}{W} : \frac{wv^2P}{pW} = \frac{2304000}{98} = V^2$, and the square root $= 485$ *feet*, the velocity required.

2. If the horizontal range of a shell be a mile when discharged at 45° elevation, how far will it range when the elevation is 32°. 20′, the charge of powder being the same?

Here a = *sin.* of twice 45°, r = 1760 yards, A = *sin.* of twice 32°.20′; and $\frac{rA}{a}$ = the range, (art. 330);

r = 1760............ *log.* 3.2455
A = *sin.* 64° 40′..... *log.* 9.9561
The *range* .... = 1591 yards *log.* 3.2016

Remark. Here it is supposed that the greatest range is at 45° elevation, but this will not be the case, except in very slow motions with great weight of shell or ball, for small shot discharged with considerable velocities are found to range the farthest when projected at about 30° elevation.

3. The horizontal range being 1760 yards at 45° elevation, then what must be the elevation with the same charge of powder to strike an object at the distance of 1591 yards?

In this example r = 1760, R = 1591, a = *sine* of twice 45°, or the *sine* of 90°; and since $\frac{rA}{a} = R$ (in the preceding examp.) we have $A = \frac{aR}{r}$ the *sine* of double the required elevation:

a = *sin.* 90°.................. *log.* 10.0000
R = 1591..................... *log.* 3.2017
r = 1760....................... 6.7545 ar. comp.
sin. 60°.40′ or 115° 20′......... *log.* 9.9562

and the halves of 64° 40′ and 115° 20′ are 32° 20′ and 57° 40′ the required *elevations.* (329 corol. 1)

4. With what impetus, velocity, and charge of powder must a 13 inch shell be fired at an elevation of 32°.12′ to strike an object at the horizontal distance of 3250 feet?

If r = 3250, a = *sin.* of twice 32° 12′, m = the impetus, v = the velocity, and $d = 16\frac{1}{12}$ *feet*:

Then (334) $m = \frac{r}{2a}$, and $v = \sqrt{4md}$.

r = 3250................ *log.* 3.5119
a = *sin.* 64° 24′......... *log.* 0.0449 ar. comp.
2 .... 9.6990 ar. comp.
Impetus 1803 *log.* 3.2558

4	*log.*	0.6021
impetus	*log.*	3.2558
$d = 16\frac{1}{12}$	*log.*	1.2064
	2)	5.0643
Velocity 341	*log.*	2.5321

The charge is determined by comparing the velocity 341 with that of the shell when fired with a different quantity of powder: thus in examp. 1. the velocity with 9*lb.* is 485 feet,

Hence $485^2 : 341^2 :: 9lb. : 4.44lb.$ nearly, the *charge* required.

5. The horizontal range of a shell at 22° of elevation being 1400 yards, then how far will it range at an elevation of $29\frac{1}{2}°$ with the same charge of powder?

Here $a =$ *sin.* of twice 22°, $r = 1400$, and $A =$ *sine* of twice $29\frac{1}{2}°$: and $\frac{rA}{a} =$ the range, (art. 330);

$r = 1400$	*log.*	3.1461	
$A =$ *sin.* 59°	*log.*	9.9331	
$a =$ *sin.* 44°		0.1582	ar. comp.
Range = 1728 yards	*log.*	3.2374	

6. If the horizontal range of a shell be 1300 yards with 7*lb.* of powder, what charge will throw it 1000 yards, the elevation being 45° in both cases?

The squares of the velocities being nearly as the quantities of powder, we have (art. 331),

$1300 : 7lb. :: 1000 : 5\frac{5}{13}lb.$ the *answer.*

7. If the range of a shell when fired with 5*lb.* of powder at an elevation of 25° be 1600 yards on an horizontal plane, then how far will it range at an elevation of 30° when the charge is 4*lb.*

Since the velocities are nearly as the square roots of the quantities of powder, the squares of the velocities may be represented by 5 and 4:

Let $r = 1600$, $a =$ *sine* of twice 25°, $v^2 = 5$, $A =$ *sine* of twice 30°, $V^2 = 4$, and $R =$ the required range:

Then (332) $a v^2 : AV^2 :: r : R$, or $\frac{rAV^2}{av^2} = R$.

$r = 1600$	...... log.	3.2041	
$A = sin.\ 100°$	...... log.	9.9933	
$V^2 = 4$	...... log.	0.6021	
$a = sin.\ 50°$	......	0.1157	ar. comp.
$v^2 = 5$	......	9.3010	ar. comp.
Range = 1645 yards	log.	3.2162	

8. If the horizontal range of a shell at 34° elevation be 1100 yards, what is the time of flight?

Let m = the impetus, $a = sin.\ 68°$ (twice the elev.), $r = 3300$ *feet* the range, $d = 16\frac{1}{12}$ *feet*, and t = the time of flight:

Then (331) $m = \frac{r}{2a}$, and $t = \sqrt{\frac{2m}{d}}$; and substituting $\frac{r}{2a}$ for m, we get $t = \sqrt{\frac{r}{ad}}$:

$r = 3300$	...... log.	3.5185	
$a = sin.\ 68°$	......	0.0328	ar. comp.
$d = 16\frac{1}{12}$	......	8.7936	ar. comp.
	2)	2.3449	
Time nearly = 14.9 seconds	log.	1.1724	

When the elevation is 45°, then $a = sin.$ twice $45° = 1$, and the expression $\sqrt{\frac{r}{ad}}$ becomes $\sqrt{\frac{r}{d}}$, and if d be taken = 16 (neglecting the fraction $\frac{1}{12}$) we shall have $\frac{1}{4}\sqrt{r}$ for the *time of flight*, nearly, in that case.

9. What will be the range of a shot on a plane which ascends 10° 20′, and on another which descends 10° 20′, the impetus being 2500 feet, and the elevation of the piece 34° above the horizon?

$$\left.\begin{array}{l} 34° - 10° \ 20' = 23° \ 40' \\ 34° + 10° \ 20' = 44° \ 20' \end{array}\right\} \text{elevations above the planes.}$$

If $S = sin.\ 23°\ 40'$, $C = cos.\ 10°\ 20'$, $c = cos.\ 34°$, $m = 2500$, and r = the range:

Then (339) $\frac{4Scm}{C^2} = r$:

4 .... log. 0.6021		C .... log. 9.9929	
s .... log. 9.6036			2
c .... log. 9.9186		C^2......... log. 9.9858	
m .... log. 3.3979			
C^2............ 0.0142 ar. comp.			

log. 3.5364 .... 3439 feet, *range* on the *ascending* plane.

And when $S =$ *sin.* 44°. 20′, the same expression gives $r = 5987$ feet, the *range* on the *descending* plane.

10. What quantity of powder will throw a 13 inch shell 3439 feet on a plane which ascends 10° 20′, the mortar being elevated 34° above the horizon?

Here $r = 3439$, $d = 16\frac{1}{12}$; and $S =$ *sin.* 23° 40′, $C =$ *cos.* 10° 20′, $c =$ *cos.* 34° as in the last example:

And (339) $C\sqrt{\frac{dr}{Sc}} = v$ the velocity, or $\frac{drC^2}{Sc} = v^2$:

d .... *log.* 1.2064
r .... *log.* 3.5364
C^2............ *log.* 9.9858
S .... 0.3964 ar. comp.
c .... 0.0814 ar. comp.
2) 5.2064
Velocity = 401 feet, *log.* 2.6032

Then, as in examp. 4.
$485^2 : 401^2 :: 9$*lb.* : 6.15*lb.* nearly, the required charge.

11. In what time will a shell strike a plane which descends 7°, the impetus being 2000 feet, and the mortar elevated 45° above the horizon?

Let $S =$ *sin.* 45° + 7°, $C =$ *cos.* 7°, $m = 2000$, $d = 16\frac{1}{12}$ *feet*:

Then (339) $\frac{2S}{C}\sqrt{\frac{m}{d}} = t$ the time.

m .... *log.* 3.3010
d .... *log.* 1.2064
2) 2.0946
1.0473
2 .... *log.* 0.3010
S = *sin.* 52° *log.* 9.8965
C .... 0.0033 ar. comp.
Time = 17.7 *seconds* *log.* 1.2481

12. The time of flight of a shell was observed to be 14.4 seconds on a plane which ascends 8½°; what was the elevation of the mortar, the impetus being 2304 feet?

Here $m = 2304$, $C = cos.\ 8° 30'$, $t = 14.4$, $d = 16\frac{1}{12}$:

And (339) $\frac{Ct}{2}\sqrt{\frac{d}{m}} = S$, or $\frac{C}{2}\sqrt{\frac{t^2d}{m}} = S$, the sine of the elevation above the plane.

t .... *log.*	1.1584	
	2	
t^2 .... *log.*	2.3168	
d .... *log.*	1.2064	
m	6.6375	ar. comp.
2)	0.1607	
	0.0803	
C .... *log.*	9.9952	
2	9.6990	ar. comp.
Elevation above plane 36° 30′ *log. sine*	9.7745	
8 30		
Elev. above horizon 45 0		

13. What must be the elevation of a mortar to throw a shell 6745 feet on a plane which descends 8° 15′, the impetus being 3000 feet?

Let $m = 3000$, $r = 6745$, $C = cos.\ 8° 15'$ or $sin.\ 81° 45'$, $T = tang.\ 8° 15'$.

Then $m : C :: \frac{1}{2}rC - mT :$ *cosine* of an angle, half of which added to, and subtracted from half the supplement of 81° 45′, give two directions that will answer the question.

$\frac{1}{2}r = 3372\frac{1}{2}$	*log.* 3.5280		m	*log.* 3.4771
C	*log.* 9.9955		T	*log.* 9.1613
$\frac{1}{2}rC =$ 3337	*log.* 3.5235		$mT = 435$,	*log.* 2.6384
$mT =$ 435				
diff. 2902				

m	6.5229	ar. comp.
C .... *log.*	9.9955	
2902 .... *log.*	3.4627	
cosine 16° 46′ .... *log.*	9.9811	
half 8 23		
49 8	half the supplement of 81° 45′	
sum 57 41		
diff. 40 45		

sum 57° 41′, diff. 40° 45′ — the two required *elevations* above the plane, or 49° 26′, and 32° 30′ above the horizon.

For the construction of this case, see *art.* 335, corol. 1. But the proportion is investigated as follows:

Draw FD parallel to LB or perpendicular to BR (see the *fig.* art. 335); then since BO $= \frac{1}{2}$BC $= \frac{1}{2}r$, and $C =$ *sin.* angle BOL or BFR, therefore BL or FD $= \frac{1}{2}rC$. Hence in the triangle BFR we have given BR (the impetus), the opposite angle BFR, and the perpendicular FD, to determine the angles FRB, FBR.

Let the segment *RfFB* contain the given angle BFR, and suppose G the center, and draw *Vf* parallel to BR, and WI parallel to DF: then since the angle WBG is the difference between the angle BFR and a right one, it is equal to the inclination of the plane and horizon, and C is the *sine* of the angle BGW, and T the *tangent* of WBG; therefore $\frac{1}{2}m$ being $=$ WB, we have $\frac{1}{2}mT =$ GW; but DF $=$ WI, consequently GI $= \frac{1}{2}rC - \frac{1}{2}mT$; and by trigonometry,

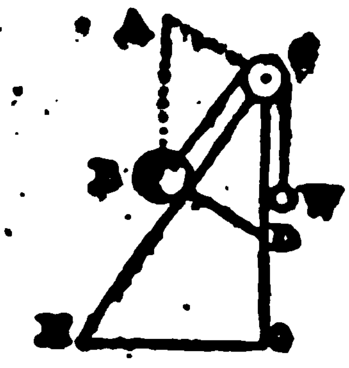
WB : *sin.* WGB :: BG : *radius*,
GI : *sin.* GFI :: GF : *radius*, but BG $=$ GF, therefore
by equality WB : *sin.* WGB :: GI : *sin.* GFI,
that is $\frac{1}{2}m$: C :: $\frac{1}{2}rC - \frac{1}{2}mT$:
or m : C :: $\frac{1}{2}rC - mT$: *cosine* IGF, which angle being at the center, is $= f$BF the difference of the required angles FRB, FBR whose sum is the supplement of BFR.

When the plane is ascending, the center G will fall on the other side of BR, and GI the third term of the proportion will be $\frac{1}{2}rC + \frac{1}{2}mT$.

We have made use of logarithms because the computations are much shorter than by natural sines.

OF THE FORCE AND DESCENT OF BODIES ON INCLINED PLANES. —— MOTION OF PENDULUMS.

343. *Let* CO *be perpendicular to the horizon* HO, *and* CH *an oblique plane; and suppose the body* B *is sustained on the plane by the string* BC (*parallel to* CH) *fastened at* C; *then if* BD *be perpendicular to* CH, *the triangles* CBD, COH, *are similar*,



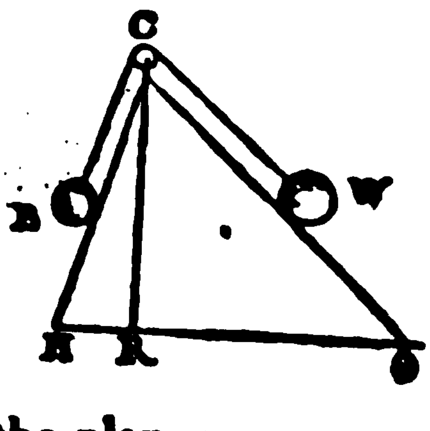
and the weight of the body B,
the tension of the string CB *or the force of the body in the direction* CB,
the pressure against the plane HC,
are respectively as CD, CB, BD; *or* CH, CO, HO *the sides of the triangle* COH.

Let BA be parallel and equal to DC; then the force of gravity or the whole weight of the body acts in the direction AB or CD, the sustaining power in the direction HB or BC, and the opposing force of the plane in the direction DB; and since these three forces keep the body in equilibrio, they are as the sides of the triangle CBD (320) or COH.

Or because the sides of the triangle COH are as the sines of the opposite angles, the weight, the power in the direction BC, and the pressure on the plane, are as the radius, sine, and cosine of the plane's elevation above the horizon.

Suppose HC = 5, CO = 4, HO = 3, and the weight B = 15*lb.* then the sustaining force or power in the direction BC = 12*lb.* and the pressure against the plane = 9*lb.*

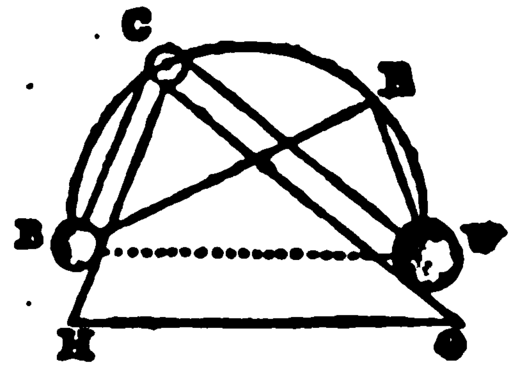
Corol. 1. Hence if the weight B = 15*lb.* be connected with another weight W = 12*lb.* by a flexible line BCW (considered as having no weight) that moves freely over a pin or pulley at C, the weight W acting in a perpendicular direction, will just prevent the other from descending along the plane, or the two weights will be in equilibrio.



Corol. 2. Hence also, if the two planes HC, OC are of an equal height above the horizon HO, and the weights B and W, connected by the line BCW moveable over the pulley C, are in the same proportion as the lengths HC and OC, the weights will mutually sustain each other on the planes.

Let HC = 5, OC = 7, the weight B = 15*lb*. then 5 : 15 :: 7 : 21*lb*. the weight W. And if the height CR = 4, then HR = 3, and OR = $\sqrt{33}$, and the pressure of B against the plane HC = 9*lb*. and that of W against the plane OC = $\frac{21}{7}\sqrt{33}$ = 17.23*lb*. nearly.

Corol. 3. If a circle be described through B, C, and W, the pulley, instead of being at C, may be at any point R in the arc BCW, and the weights B and W will remain in equilibrio when the connecting line is brought into the position BRW and its length = BR + RW.

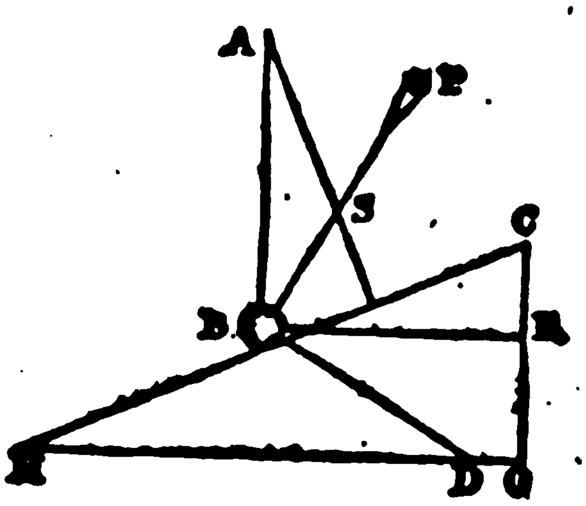


For the sustaining forces in the directions CB, CW being equal, and the angle CBR equal to CWR, it follows from the resolution of forces, that the weights are equally sustained in the directions BR and WR.

The angle CBR or CWR is called the angle of *traction*. And because BR is below the plane BC, the pressure of B against that plane is augmented, and that of W against CO diminished.

344. *Suppose the body or weight* B *is sustained on the plane* HC *by a string* BP *fastened at* P; *then if* BD *be perpendicular to* BP, *the weight* B, *sustaining force, and the pressure on the plane, will be respectively as* HD, BD, *and* HB.

Let BA be perpendicular to the horizon HO, and AS to the plane HC; then (320) the three forces acting on B, namely, that of gravity in the direction AB, the sustaining force in the direction BP, and the opposing force of the plane in the direction SA, will be as the sides of the triangle BAS. But because AB and AS are perpendicular to HO and HC, and AB and BS to HO and BD, re-



spectively, the angle BAS = BHD, and A
quently the triangles HBD, ASB are simil
proportional; that is, the forces are as the
drawn perpendicular to the directions of th

Suppose HD = 5, HB = 4, BD = 2, and th
Then 5 : 20 :: 4 : 16*lb*. the pressure on the pl
5 : 20 :: 2 : 8*lb*. the sustaining force in

Corol. 1. The sustaining power (BP)
parallel to the plane BC; but greatest in t
in that case it is equal to the weight B.

Corol. 2. But when the sustaining pow
to the horizon, the weight of the body B, t
and the pressure on the plane, are respecti
perpendicular OC, and length of the plane

For the sides of the triangle BCR or HC
to the directions of the three forces in equil

Let HO = 4, HC = 5, OC = 3, and the we
Then 4 : 20 :: 3 : 15*lb*. the sustaining pow r.
4 : 20 :: 5 : 25*lb*. the force against the p

345. *Let the body* B *upon an inclined p*
librio with another W *hanging freely; th*
motion, their perpendicular velocities wi
their weights, or the weights multiplie
velocities are equal.

Let the weight B be made to descend
a small distance on the plane from B
to H. Draw OD, BA perpendicular
to PH, and GR, BS perpendicular to
the horizon HO; then BS will be the
perpendicular descent of B; and be-
cause BH is supposed to be a very small
considered as the difference of the lengths

positions of the connecting line), which difference is the perpendicular ascent of the weight W. And since the triangles GRO, HBO are similar, and also the quadrilaterals HABS, HDGR, we have

GR : DH :: BS : AH

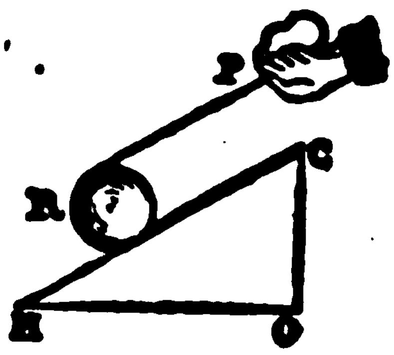
GR : DH :: GO : HO :: *weight* W : *weight* B (344)

whence by equality

BS : AH :: *weight* W : *weight* B, that is, the perpendicular velocities into the weights are equal.

Corol. Hence if two weights are in equilibrio on two inclined planes, their perpendicular velocities will be reciprocally as the masses when they are put in motion.

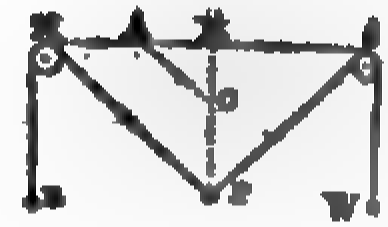
346. *If a cylinder* R *is sustained on the inclined plane* HC *by a power at* P *drawing one end of the rope* CRP *parallel to the plane, the other end being fixed at* C; *this power is to the weight of the cylinder, as half the height* CO, *is to the length of the plane* HC.



For (343) CH : CO :: *weight of cylinder* : *sustaining power in the direction of the plane*; but the sustaining power is equally divided between the parts of the rope RC and RP; therefore CH : ½CO :: *weight of cylinder* : *sustaining power at* P.

SCHOLIUM. Hence it appears, that when the inclined plane is of sufficient length, a great weight may be raised to a given height by a small (comparative) force.

347. *Let* HO *be an horizontal line,* H *and* O *two pins or pullies,* B *and* W *two equal weights connected by a small flexible line* BHPOW *that moves freely over* H *and* O, *and* P *another weight at the middle of* HPO; *to find the length of the perpendicular* RP *when the three weights rest in equilibrio.*



Since the force of P sustains both the equal weights B and W, the force exerted by ½P in the direction HP must be equal to the weight B; hence, by the resolution of forces, HP : RP :: force of ½P in the direction HP : its force in the direction RP, therefore HP and PR have the same ratio as the weight B and ½P; whence the following construction is obvious.

From any point C in the perpendicular RP draw CA so that ½P : B :: CR : CA; make HP parallel to AC, and P is the place of the weight.

Since $AR = \sqrt{(AC^2 - RC^2)}$, we have, by sim. triangles,

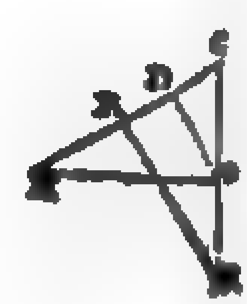
$$\sqrt{(AC^2 - RC^2)} : RC :: HR\ (= \tfrac{1}{2}HO) : \frac{RC \times HR}{\sqrt{(AC^2 - RC^2)}}$$

$= RP$, where AC and RC may be any quantities in the proportion of B and ½P; therefore substituting B and ½P for AC and RC, we get $\dfrac{P}{\sqrt{(4B^2 - P^2)}} \times \tfrac{1}{2}HO = RP$.

Suppose each of the equal weights = 5*lb.* P = 6*lb.* and HO = 16 *feet* then RP = 6 *feet.*

Corol. If the weight P be equal to, or greater than both the other weights together, it will constantly descend, and consequently there can be no equilibrium.

348. *If a body descend from rest along the inclined plane* CH, *the space it describes, is to the space it would describe in the same time when falling perpendicularly, as the height of the plane* OC, *to its length* CH.

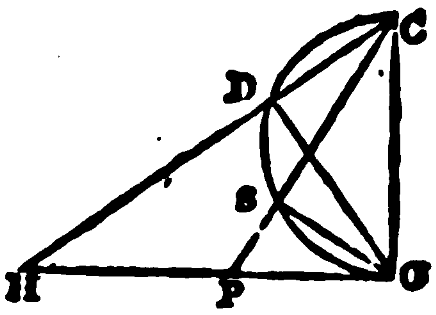


The force with which the body endeavours to descend along the plane CH, is to the force by which it is urged in a perpendicular direction, as OC to CH (343): and since those forces are uniformly accelerating, the velocities acquired will be as the forces because the times are equal

(316), that is the velocities generated in the same time are as OC to CH; but (316 corol. 1) double the whole spaces described are as the velocities, and therefore the spaces are also as the velocities, or as OC to CH.

Corol. 1. Hence if CR = CO, and CB = CH, the triangles COH, CRB, are similar and equal, and therefore a body would descend from C to R in the same time that it would fall from C to B. Let OD be parallel to BR or perpendicular to HC, then by similar triangles, the times of descent through CD and CO are also equal.

Corol. 2. And if CP be another plane, and OS perpendicular to CP, then the time of descending from C to S is the same as that through the perpendicular CO; consequently the times along CD and CS are equal: but the *locus* of the right angles CDO, CSO, &c. is a semicircle described upon CO (Geom. 72); therefore the times of descent through all chords (CD, CS, DO, &c.) drawn from the extremities of the vertical diameter of a circle are equal, and equal to the time of the perpendicular descent through that diameter.



Corol. 3. The velocity acquired upon an inclined plane (CH) is to the velocity acquired in the same time by falling perpendicularly, as the height of the plane (CO) to its length (CH): Or as the sine of the plane's elevation to the radius.

349. *The time of descent along the plane* CH, *is to the time of falling through the perpendicular* CO, *as the length of the plane* CH, *to its height* CO. (see the last *fig.*)

Let T = the time of descent along CH, and t = the time in descending from C to D or from C to O. Then since the body is urged down the plane by an uniformly accelerating force, we have (316 corol. 1)

$$T^2 : t^2 :: CH : C$$
$$\text{and } CH : CO :: CO : \frac{C}{C}$$
$$\text{whence } T^2 : t^2 :: CH : \frac{C}{C}$$
$$\text{or } T^2 : t^2 :: CH^2 : C$$
$$\text{That is } T : t :: CH : C$$

Corol. Hence the times of desc the same height, are as their length

350. *A body acquires the same v an inclined plane* CH, *as by falli* CO *the height of that plane.*

Let OD and DL be perpendicul and CO, respectively. Then since mean proportional between CO an will be CO : CL :: CD² : CL² (Ge or $\sqrt{CO} : \sqrt{CL} :: CD : CL$.

Now CL being the height of th corol. 3)

veloc. at D : *veloc.* at O :: C

whence *veloc.* at D = $\frac{C}{\ }$

Also because the velocities acqu square roots of the heights CO and

veloc. at O : *veloc.* at L :: $\sqrt{\ }$

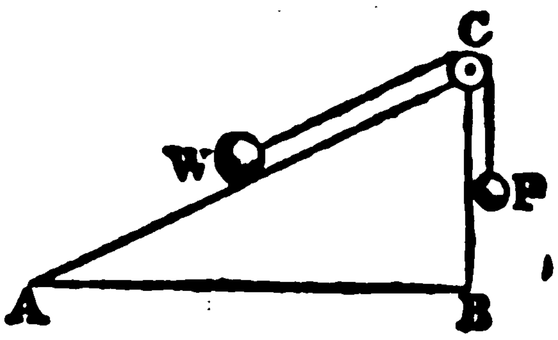
therefore *veloc.* at L = $\frac{C}{\ }$

velocity acquired at L in the perpe the velocity acquired at D on the acquired velocities at H and O are

Corol. 1. Hence the velocities a from the same height, down any pl right line, are equal to one another.

Corol. 2. And the velocities acquired by descending down any planes, are as the square roots of the heights.

351. In Emerson's Mechanics, Prop. 31, we find the following Scholium, "If it be required to find the position of the plane AC, whose height BC is given; so that the given weight W may be raised through the length of the plane AC, in the least time possible, by any given power P, acting in direction WC. Make $AC = \frac{2W}{P} \times BC$, and you have your desire." It may be worth while to show how this construction is derived.



Let the power P, denoted by a weight, descend perpendicularly and draw the weight W in the direction AC by means of a string passing over a pulley at C.

Put the height $BC = h$, $x = AC$ the required length, and $t =$ the time:

Then (343) $AC : BC :: W : \frac{BC \times W}{AC}$, or $\frac{hW}{x} =$ the power or relative weight which urges the weight W down the plane AC; and since P is the power or force in the direction CP, the difference $P - \frac{hW}{x}$, or $\frac{Px - hW}{x}$ will be the force which accelerates the bodies in motion. Now (317, corol.) the square of the time being as the space divided by the accelerating force, we have $t^2 \propto x \div \frac{Px - hW}{x}$, or $t^2 \propto \frac{x^2}{Px - hW}$; therefore the expression $\frac{x^2}{Px - hW}$ is to be the least possible or a *minimum*; consequently $\frac{x^2}{x - \frac{hW}{P}}$ must be the least possible, because in that case, any multiple or submultiple must also be the least possible.

Let $\frac{hW}{P} = g$; then $\frac{x^2}{x - g}$ must be a minimum.

Suppose $\frac{x^2}{x - g} = m$; then $x^2 = mx - mg$, and $x^2 - mx = -mg$, a quadratic equation, which completed is $x^2 - mx + \frac{1}{4}m^2 = \frac{1}{4}m^2 - mg$;

Whence $x = \frac{1}{2}m \pm \sqrt{(\frac{1}{4}m^2 - mg)}$, where if $\frac{1}{4}m^2$ be less than mg, the expression $\sqrt{(\frac{1}{4}m^2 - mg)}$ becomes impossible; therefore when the value of $\frac{1}{4}m^2$, or m is the least possible, $\frac{1}{4}m^2$ must be $= mg$, and $\sqrt{(\frac{1}{4}m^2 - mg)}$

$=o$, whence $\frac{1}{2}m=g$, and $\frac{1}{2}m=2g$, and therefo
$\sqrt{(\frac{1}{4}m^2-mg)}=2g\pm o$; that is $x=2g=\frac{2hW}{P}$

Here the pulley is supposed to move without fric
no sensible weight.

Corol. Hence it appears that the momentum of
that of the weight W.

352. *If* C *be a point in the vertical lin
direction of the plane* CD *along which a
from rest, to meet a plane or right line* AB
in a given time.

Upon CO equal to the perpendicular di
which the body would descend in the given
let a semicircle be described; draw CD an
to the intersections D and P; and either of
directions is that required, as is manifest fro
348 corol. 2.

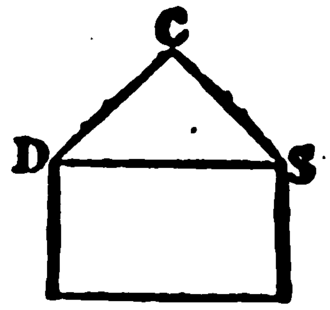
The distance CB and angle ABC are su
hence if the perpendicular RS be let fall fro
AB, the lines BP, BD, and angles PCB,
found by trigonometry.

353. If the circle should touch, instead
intersecting the line AB, then CD drawn
the point of contact D, will be the direct
sought; and the time of descent from C to A
is the least possible, because the radius, or
diameter of the circle is the least possible.
construct this case: Upon AB let fall the p
pendicular CP, bisect the angle PCB with C
parallel to PC; then R is the center, and R
circle.

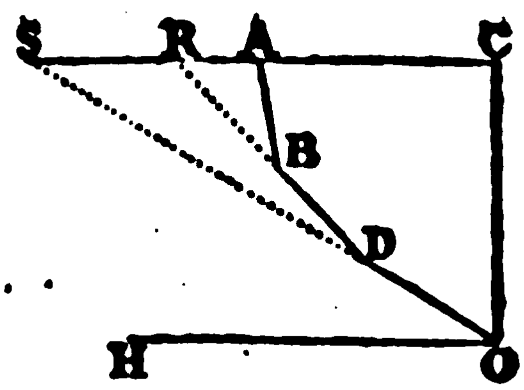
For the angle RDB is a right one, and
tangent to the circle; and because the a
$=$ DCR, the triangle DRC is isoceles, con
the radius.

Corol. 1. Hence when the time of descent is the least possible, the direction of the plane CD with the vertical CB, is equal to half the complement of the angle ABC which the plane or line AB makes with the vertical.

Corol. 2. And therefore if we would construct the roof of a house that the rain may run off in the least time, make the angle of the ridge DCS a right one, or each of the angles CDS, CSD equal to 45°.



354. *If a body descend from* A *to* O *down any number of contiguous planes* AB, BD, DO, *it will acquire the same velocity at* O *as a body falling perpendicularly the same height* CO, *provided the velocity is not altered by the different directions at* B, D, &c.



Draw CS parallel to the horizon HO, and produce the planes DB, OD, to meet CS. Then the velocities acquired in descending down the planes AB, RB, will be equal, because their heights are equal (350, corol. 1), and therefore the velocities acquired in the descent from R to D along the continued plane RD, is the same as that acquired in descending through both planes AB and BD; in like manner, the velocities generated in the descent along SO, and down the planes RD and DO, are also equal, and equal to that acquired in the perpendicular descent from C to O (350, corol. 1).

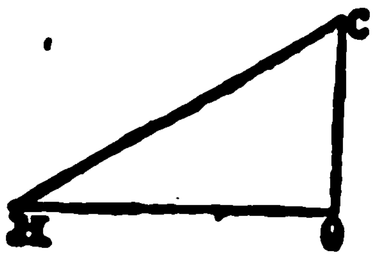
Corol. 1. If the lines AB, BD, DO, &c. are supposed to be diminished indefinitely in length, we may consider them as ultimately forming a curve line; and hence it is infered, that a body acquires the same velocity in descending along any curve, as in its descent through the same perpendicular height.

Corol. 2. Hence it also appears, that bodies acquire the same velocity in descending through any curves and planes of

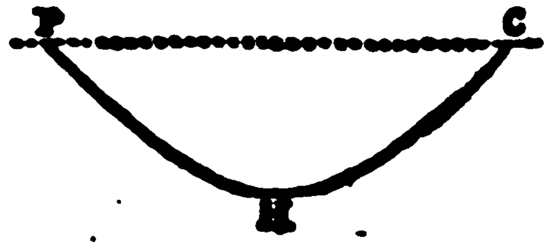
the same perpendicular height. And therefore when their velocities are equal at any particular height, they will be equal at all other equal heights.

Corol. 3. The velocities acquired by descending through any planes or curves, are as the square roots of the perpendicular heights (350, corol. 2).

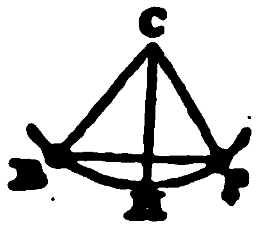
SCHOLIUM. In the preceding articles it is supposed that planes are perfectly smooth, or that bodies are not in the least retarded by friction in oblique descents. And since those motions are uniformly accelerated by the constant force of gravity, the descent on inclined planes is subjected to the laws already laid down for accelerated motion in a perpendicular direction (*art.* 316, 317). Thus, the velocities acquired in descending from rest along CH are as the times. And the spaces described as the squares of the times, or the squares of the acquired velocities. Also if the body were projected along the plane from H towards C with the velocity it acquired in the descent from C to H, the velocities and times of ascent would be equal to those of descent at equal distances from H.



Consequently, if a body descend freely along the curve CH, the velocity or force acquired at the lowest point H, would be sufficient to make it ascend up the curve HP, to the same perpendicular height, in the same time.

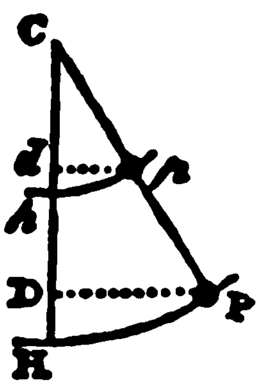


Hence if P be a body attached to one end of a small non-elastic string CP, the other end being fixed at C, and the body left to descend freely, it will describe the circular arc PHB; for the force acquired at the lowest point H would make it ascend to B in the horizontal line PB; and its motion being lost at that point, it



will descend back again, and rise to the same horizontal line BP in the same time: in this manner it would continue to oscillate or vibrate through the same arc BP by the force of gravity only, if all causes of resistance were removed. But experience proves that the air, and the friction, &c. at the center of motion or suspension C, act as retarding forces, by which means the vibrations are successively shortened till the body loses all motion at the lowest point H. A body moving in this manner is called a PENDULUM. And in theory, when all the matter of the pendulum or vibrating body is supposed to be in a single point (P), and the sustaining line CP without gravity, it is called a simple Pendulum. Pendulums that regulate the going of clocks continue their motion by the force of a weight, or spring, &c. called the maintaining power.

355. *If two pendulums* Cp, CP, *vibrate in similar arcs* ph, PH, *the times of vibration will be as the square roots of their lengths.*



Draw pd, PD, perpendicular to the vertical CH. Then (354, corol. 3) the velocities acquired at the lowest points h, H, in describing the arcs ph, PH, are as the square roots of the vertical heights hd, HD; but the sectors hCp, HCP, are similar, therefore the radii hC, HC, and also the arcs ph, PH, have respectively the same ratio as the homologous lines hd, HD; that is, the velocities acquired in descending from rest, are constantly as the square roots of ph, PH, the spaces described; and therefore the times of description are as $\sqrt{ph}$ and $\sqrt{PH}$, or as $\sqrt{Cp}$ and $\sqrt{CP}$.

Corol. Hence if T and t denote the respective times in which the pendulums CP, Cp, perform a vibration; then $\sqrt{CP} : \sqrt{Cp} :: T : t$, or $CP : Cp :: T^2 : t^2$, whence $Cp \times T^2 = CP \times t^2$, that is, the lengths are as the squares of the times of a vibration.

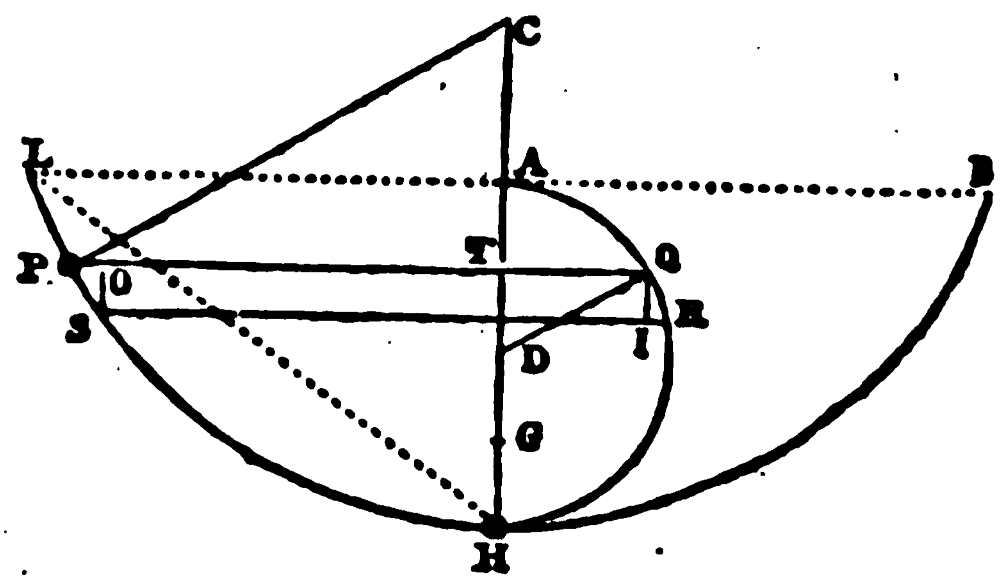
It is found by experiment that the length of a simple pendulum which vibrates once in a second of time in the latitude of London, is $39\frac{1}{8}$ *inches*, nearly; hence the length of a pendulum that vibrates once in $\frac{1}{2}$ a second is required?

The preceding expression gives $Cp = \frac{CP \times t^2}{T^2} = \frac{39\frac{1}{8} \times (\frac{1}{2})^2}{1^2} = 39\frac{1}{8} \times \frac{1}{4}$ $= 9.8$ *inches*, nearly.

If a musket ball be suspended by a very fine thread or wire, and the distance from the point of suspension to the center of the ball is 50 *inches*; how many times will it oscillate in a minute?

Here $t^2 = \frac{Cp \times T^2}{CP}$, and $t = \sqrt{\frac{Cp \times T^2}{CP}} = \sqrt{\frac{50}{39\frac{1}{8}}}$ seconds the time of one vibration, and $\sqrt{\frac{39\frac{1}{8} \times 60^2}{50}} = 53$ nearly, the number of vibrations or oscillations in 60 seconds.

356. But the vibrations in circular arcs of different lengths are not isochronal, that is, they are not performed in the same time. When however, the arcs are very small, the differences are found to be almost insensible. And in that case, *the time of one vibration, is to the time in which a body would descend perpendicularly through half the length of the pendulum, as the circumference of a circle, to its diameter.* (Emerson's Mechan.)



Let CH or CP be the pendulum vibrating in the arc LHB, C the point of suspension, and CH perpendicular to the horizon. On AH describe a a semicircle about the center D, and let PS be an indefinitely small part of the arc LPH; draw PQ, SR parallel to the horizontal line LB, and make SO, QI perpendicular to PQ, and join DQ.

Let t denote the time in which a body would descend perpendicularly through $2CH$ or along the chord LH. Then the velocities acquired in the descent through $2CH$, and by the pendulum in falling through the arc LP, will be as the square roots of the vertical heights, or as $\sqrt{2CH}$ and $\sqrt{AT}$; and the space that would be described in the time t with the velocity acquired in falling through $2CH$ is $4CH$, (316 corol. 1) which therefore may represent the velocity. And since the arc PS is indefinitely small, we may conceive it to be described with the uniform velocity denoted by $\sqrt{AT}$. But in uniform motions, the times of description are as the spaces divided by the velocities (314 corol. 1): hence $\frac{4CH}{\sqrt{2CH}} : t :: \frac{PS}{\sqrt{AT}} : \frac{t.PS}{2\sqrt{(2CH.AT)}}$ the time of describing PS.

If the arcs PS, QR, on account of their smallness, are considered as right lines, the triangles CPT, PSO, and DQT, QRT will be similar, respectively; whence $CP : PT :: PS : \frac{PT.PS}{CP}$, or $\frac{PT.PS}{CH} = OS = QI$; and $DQ : TQ :: QR : \frac{TQ.QR}{DQ}$, or $\frac{TQ.QR}{DH} = QI$, therefore $\frac{PT.TS}{CH} = \frac{TQ.QR}{DH}$; whence $PS = \frac{TQ.CH.QR}{PT.DH}$, which being substituted for PS in the expression $\frac{t.PS}{2\sqrt{(2CH.AT)}}$, and the time of describing PS is $= \frac{t.TQ.CH.QR}{2\sqrt{(2CH.AT)}\,(PT.DH)}$ or $\frac{t.TQ.CH.QR}{\sqrt{(2CH.AT)}\,(PT.2DH)}$.

But by prop. of the circle, $TQ = \sqrt{(AT.TH)}$. And $PT = \sqrt{[(CH+CT)TH]}$, these values of TQ, and PT being substituted, and we have the time of describing PS $= \frac{t\sqrt{(AT.TH)}.CH.QR}{\sqrt{(2CH.AT)}\sqrt{[(CH+CT)TH]}2DH}$, which reduced becomes $\frac{t\sqrt{2CH} \times QR}{\sqrt{(CH+CT)}4DH}$.

But $4DH = 2AH$. And $\sqrt{(CH + CT)} = \sqrt{(2CH - TH)}$; therefore by substitution, the time of describing PS $= \frac{t\sqrt{2CH}}{\sqrt{(2CH - TH)\,2AH}} \times QR$.

But if we suppose QR to be the arithmetical mean of all the QR's in the semicircle, the corresponding mean of the TH's will be DH or half AH, because the least TH is $= o$, and the greatest $= AH$; hence the time of describing PS $= \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times QR$. Now all the QR's is the semicircle ARH, and all the PS's the arc LPH; therefore the time of describing the arc $LPH = \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times 2ARH$. And the time of one vibration along the arc $LHB = \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times 2ARH$. But when the arc LHB is indefinitely small, DH may be taken $= o$, and the expression becomes $\frac{t\sqrt{2CH}}{\sqrt{2CH \times 2AH}} \times 2ARH$, or $\frac{\frac{1}{2}t \times 2ARH}{AH}$. But when t is the time of perpendicular descent through 2CH, $\frac{1}{2}t$ is the time of descent through $\frac{1}{2}$ CH;

Therefore, as the diameter AH,

is to the circumference 2ARH,

so is $\frac{1}{2}t$ (the time of descent through half the length of the pendulum CH),

to $\frac{\frac{1}{2}t \times 2ARH}{AH}$ the time of one vibration in a very small arc.

Corol. 1. Hence the time of descent through a small arc (LPH), is to the time of descent along its chord (LH), as the diameter of a circle, to $\frac{1}{4}$ of its circumference.

For $\frac{\frac{1}{2}t\,.\,ARH}{AH}$ is the time of descent through the arc, and t the time along the chord,

And $AH : \frac{1}{2}ARH :: t : \frac{\frac{1}{2}t\,.\,ARH}{AH}$.

Corol. 2. If DH be bisected in G, and $T = \frac{t . ARH}{AH}$ the time of one vibration in a very small arc; then the time of vibration in any arc, will be $T + \frac{DG}{CH + CG} \times T$, nearly.

For we found the time of vibration in LHB

$$= \frac{t . \sqrt{2CH . 2ARH}}{\sqrt{(2CH - DH)2AH}} = \frac{t . ARH}{AH} \sqrt{\frac{2CH}{2CH - DH}} = T\sqrt{\frac{2CH}{CH + CD}};$$

and the lines 2CH, CH + CG, and CH + CD, are in arithmetical progression, the common difference being GH; but if DH = o, they become equal, and therefore since DH is very small, they are nearly in geometrical progression; hence

$$\sqrt{\frac{2CH}{CH + CG}} = \frac{CH + CG}{CH + CD},$$

therefore the time of vibration

$$= T \times \frac{CH + CG}{CH + CD} = T \times \frac{CH + CD + DG}{CH + CD} = T \times \frac{CH + CD}{CH + CD} + \frac{DG}{CH + CD} \times T$$

$$= T + \frac{DG}{CH + CD} \times T.$$

Corol. 3. Hence also, we can determine the perpendicular descent of a body near the earth's surface in a second of time.

For it has been found by experiment that a simple pendulum $39\frac{1}{8}$ or rather 39.13 inches long, vibrates once in a second of time in the latitude of London. Therefore, putting $c = 3.1416$ the circumference of a circle whose diameter is 1, $p = 39.13$, $t =$ the time of perpendicular descent through $\frac{1}{2}p$, and $d =$ the distance required;

Then, 1 *sec.* : t :: c : 1, whence $t = \frac{1}{c}$ the time of descent through $\frac{1}{2}p$.

And the spaces described being as the squares of the times of description (316 corol. 1), we have $\frac{1}{c^2}$: $\frac{1}{2}p$ :: (1 *sec.*)^2 : d, or $\frac{1}{2}pc^2 = d = \frac{39.13}{2} \times 3.1416^2 = 193.1$ *inches*, nearly, $= 16\frac{1}{12}$ *feet*, for the descent of gravity in 1 second of time. Which agrees with experiment.

Remark. Pendulums of the same length vibrate quicker in remote latitudes than near the equator. For, by reason of the spheroidal figure of the earth, the distance from its center increases

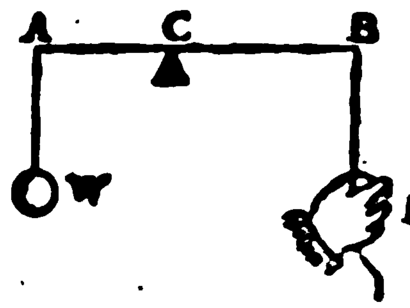
and therefore gravity decreases, as we approach the equator. But the greatest diminution arises from the rotation of the earth on its axis; for the parts near the equator move quicker, and thereby have a greater tendency to recede from the axis of motion, than bodies at the surface in other latitudes. Gravity, therefore, is greatest at the poles, and least at the equator.

OF THE MECHANICAL POWERS.

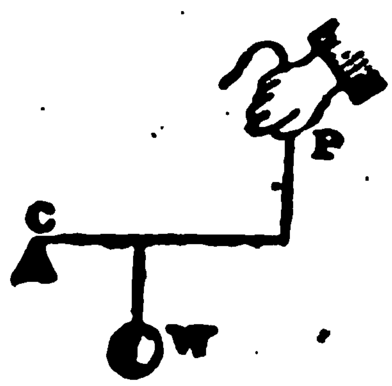
357. THESE are certain machines or instruments contrived for moving great weights with small force. They are commonly reckoned six in number; namely, the Lever, the Wheel and Axle, the Pulley, the Wedge, the Screw, and the Inclined Plane.

THE LEVER.

358. A LEVER is a rod or bar of wood or metal, as a handspike, a crow-bar, the beam of a pair of scales, &c. It is usually represented by an inflexible line without weight to render the demonstrations more simple. There are four kinds of levers.



A lever of the first kind has the fulcrum or prop between the weight and the power. Thus if AB be a rod or bar, W a weight attached to the end A; P a power acting at the other end B, and C the fulcrum or prop that supports AB. Then C is the center of motion, and AC, and BC are the arms of the lever AB. Of this kind are balances, scales, scissars, pincers, &c.



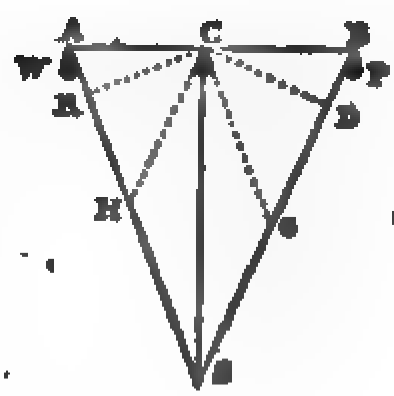
A lever of the second kind has the weight W between the power P and the fulcrum C. As oars and rudders, bellows, cutting knives fixed at one end, &c.

A lever of the third kind has the power P between the weight W and the fulcrum C. Such as tongs, sheep-shears, a man raising a ladder, the bones and muscles of animals, (Borelli de Motu Animalium).

A bended lever is the fourth kind. As a claw-hammer drawing a nail. This however, is only a species of the first kind, because the fulcrum is between the power and the body to be moved.

259. *Let the weights* W *and* P *be attached to the ends of the inflexible line or lever* AB, *and suppose* C *is the fulcrum or prop supporting the weights; then if they are in equilibrio, the distances* AC *and* CB *will be reciprocally as the weights; that is,* AC : CB :: P : W, *or* AC × W = CB × P.

Let S be the center of the earth. Then because the weights W and P gravitate towards the center S, the three forces in equilibrio act in the directions AS, BS, and SC. Draw CR, CD perpendicular to AS, BS, respectively, and CH, CG parallel to BS, AS; and the three forces will be as the sides of the triangle SCG, or CHS (320);

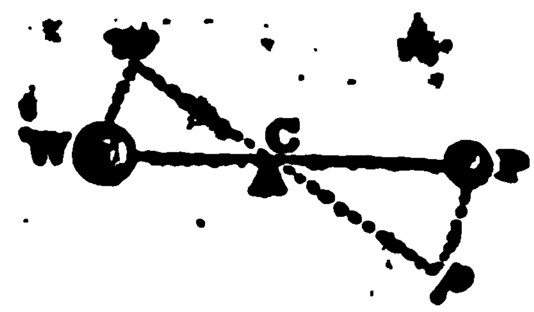


hence CH : HS (or CG) :: P : W.

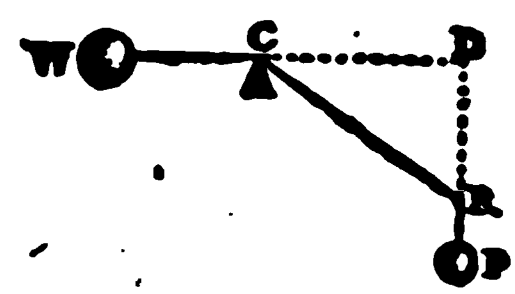
But in the parallelogram CHSG, the opposite angles at H and G are equal, consequently the angles CHR, CGD are also equal, and therefore the triangles CHR, CGD are similar,

whence CH : CG :: CR : CD.
But CH : CG :: P : W,
therefore by equality, CR : CD :: P : W;
or CA : CB :: P : W,

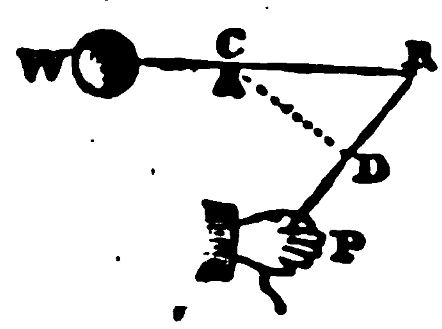
For CA, and CB, and the respective perpendiculars CR, and CD, are not sensibly different either in length or position.



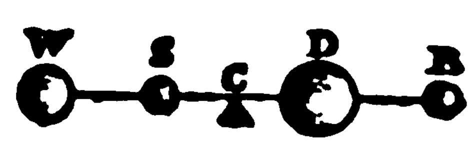
Corol. 1. Hence if the weights W, P, or the weight W and power P in equilibrio, move on the fulcrum or center of motion C, the arcs or spaces W*w*, P*p*, described in the same time, will be as the radii CW, CP, therefore the weight W × W*w* = power (or weight) P × P*p*; and since the velocities will be as the arcs W*w*, P*p*, the momenta of W and P are equal.



Corol. 2. And if WCR be a bended lever, and the power P act perpendicular to WCD, the weight W and power P will be in equilibrio when WC, DC, the perpendicular distances from the center of motion C, are reciprocally as the weight and power: that is, WC : DC :: P : W.



Corol. 3. Therefore when the power P acts obliquely against the end of the lever WR, the weight W and power P are reciprocally as WC and the perpendicular CD, the two distances of the directions of the forces from the center of motion C; that is WC : DC :: P : W. Hence, if WCD be a bended lever, and the weight W, and the power P, act perpendicularly to the arms CW, CD, then WC × W = CD × P, as in the straight lever.



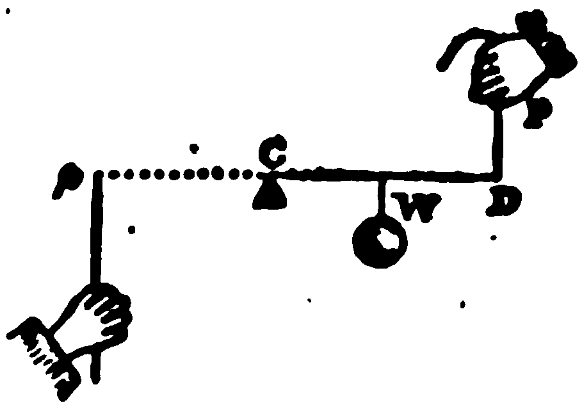
Corol. 4. When several weights W, S, D, B, acting on a straight lever WB, are in equilibrio, the sum of the products of the weights multiplied by their respective distances from the support C on one side, will be equal to the sum of the products on the other:

that is, $SC \times S + WC \times W = DC \times D + BC \times B$.

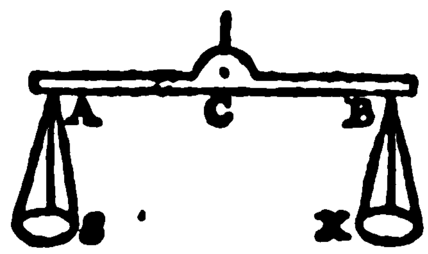
For the effort of each weight to turn the lever, is as the weight multiplied by its distance from the fulcrum C, and therefore the sum of the efforts on one side must be equal to those on the other, in the case of an equilibrium.

Corol. 3. Hence the place of the fulcrum is readily determined when the length of the lever WP, and the weights W, P, are given (see *fig.* to corol. 1). For W : P :: CP : CW; and by composition, W+P : WP (the length) :: P : CW; that is, the length must be divided into two parts having the proportion of the weights.

360. A lever of the second, or third kind, may be reduced to the first, thus,



Conceive the lever Cp to be equal to CD, then it is manifest, that if the power P were removed to p, but acted in a contrary direction, the equilibrium would still remain, and we should have $p\mathrm{C} \times p = \mathrm{CW} \times \mathrm{W}$, that is, $\mathrm{DC} \times \mathrm{P} = \mathrm{CW} \times \mathrm{W}$. Hence, in a lever of either kind, if the weight, and the power, are multiplied by their respective distances from the fulcrum, the products will be equal when there is an equilibrium.



Scholium. The beam of a pair of Scales is a lever of the first kind. Its arms CA, CB ought to be exactly of the same length; for should there be any difference, equal weights when placed in the scales S and K, will not rest in equilibrio. The obvious method of trial however, is to weigh any body, very accurately, in one scale, then if the weight and body change places, and either end preponderates, the scales are imperfect or false.

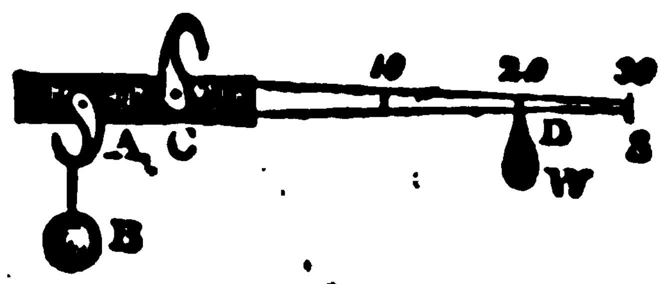
But when we know what the body weighs in each scale, its

true weight may be found thus——Let W and w denote the weights of the body b in the scales S and K respectively,

$$\text{then } CA \times b = CB \times W,$$
$$\text{and } CB \times b = CA \times w,$$

whence $b^2 = Ww$, or $b = \sqrt{Ww}$; that is, its true weight is a geometrical mean between the least and greatest weights found by the false scales.

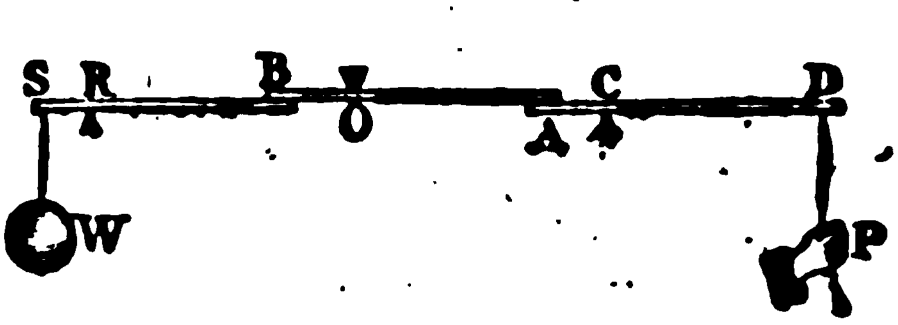
361. The steelyard or Roman balance is also a lever of the first kind. But the arms or *brachia* CA, CS are very unequal in length. The weight or counterpoise W is moveable backwards and forwards on the arm CS, which is divided and numbered. The distances of the divisions (which are notches in the beam) from the fulcrum C, are determined by repeated trials. Thus for example, suppose the weights B and W are in equilibrio, then if the weight B is 20*lb.* a notch is made in the beam at D, and that division is numbered 20. And in like manner, by suspending different weights at A, the other divisions are found.



If the arm CS be three quarters of a yard long, and CA one inch, then (neglecting the weight of the beam) a weight (W) of 2*lb.* at S, would weigh the body B of 54*lb.* For $27 \times 2 = 54 \times 1$.

362. *Let the compound lever* SD *be composed of three levers of the first kind*, DA, AB, BS, *acting upon one another; the fulcrums being at* C, O, R;

Then $P : W :: CA.OB.RS : CD.OA.RB$, *when the power P and weight W are in equilibrio.*



For $CA : CD :: P : \frac{CD.P}{CA}$ the force at A;

$$OB : OA :: \frac{CD.P}{CA} : \frac{CD.OA.P}{CA.OB} \text{ the force at B;}$$

$$RS : RB :: \frac{CD.OA.P}{CA.OB} : \frac{CD.OA.RB.P}{CA.OB.RS} = W, \text{ the}$$

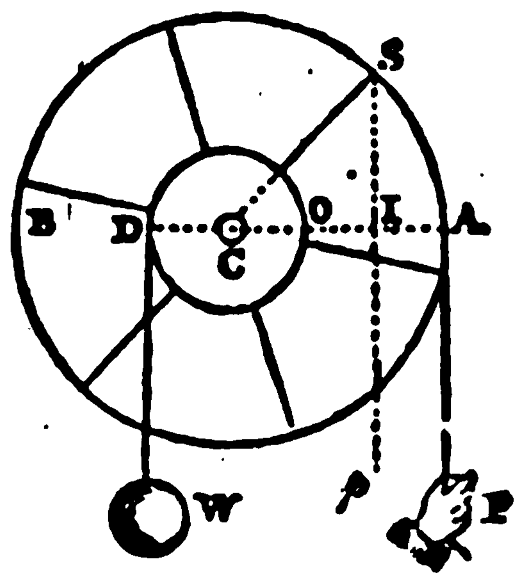
the force at S;

Therefore $CD.OA.RB.P = CA.OB.RS.W.$

And a similar conclusion is derived in the other kinds of levers, by making use of the respective distances from the props or fulcrums.

The heavy lever will be considered when we treat of the Center of Gravity.

WHEEL AND AXLE.



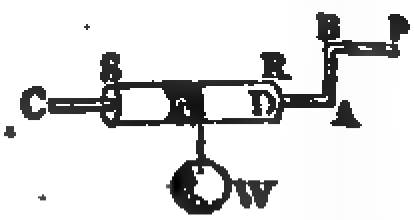
363. THIS instrument is a wheel AB fixed on a roller or axle OD, the axle being supported at its extremities so as to turn round freely with the wheel. It may be considered as a perpetual lever of the first kind. For when the weight W attached to a rope DW that goes round the axle, and the power P applied at the circumference of the wheel, are in equilibrio, then, as AC the radius of the wheel : CD the radius of the axle :: W : P, or $CD \times W = CA \times P$, as in the lever.

This will be obvious by considering the radii AC, CD as forming one line or lever AD, and C the fulcrum or center of motion upon which it turns.

Corol. 1. While the weight W, is drawn up by the power P, their velocities will be as the radii of the axle, and wheel, respectively.

Corol. 2. When the direction (S*p*) of the power applied to he wheel, is not perpendicular to its diameter, the radii SC, CD form a bended lever SCD; and if CI be drawn at right angles to S*p*, we shall have CI : CD :: W : power at *p*.

Corol. 3. If a roller or cylinder SD is turned on the axis CA by means of the handle ABP, and the power at P acting perpendicularly to AB, and the weight W, are in equilibrio; then



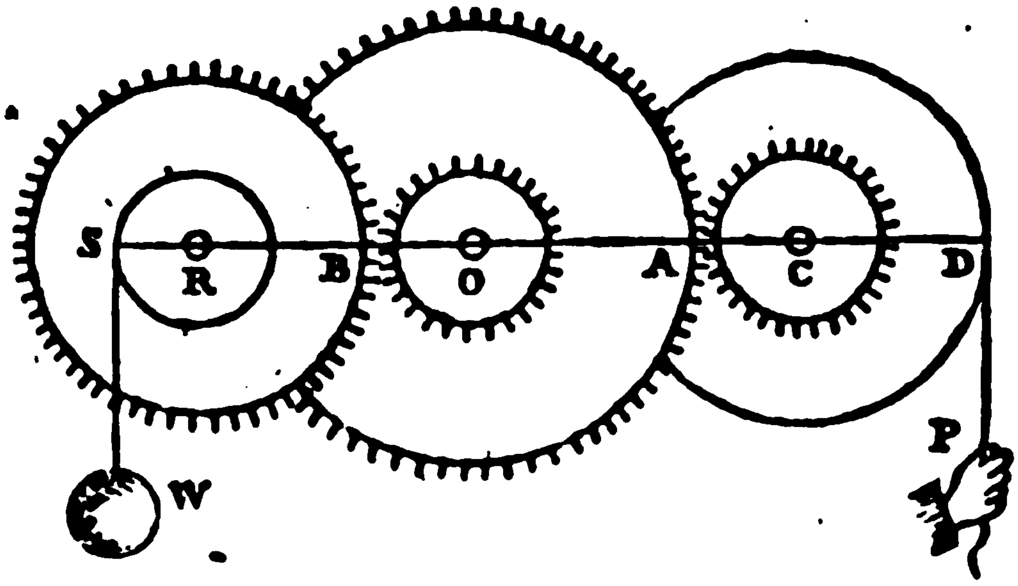
P : W :: DR the radius of the roller : AB the length of the handle. For when the roller is turned round, the point B describes a circle whose radius is AB.

SCHOLIUM. But the weight and thickness of the rope to which the body (W) is appended, ought to be taken into the account. Thus, every fold of rope on the axle or roller, may be said to increase its diameter; the radius therefore is always the distance from the axis of the roller to the middle of the outside rope or fold; for which reason it will sometimes be found necessary to increase the force. The weight however, will evidently diminish as the rope (which makes part of the weight) shortens, or is wound on the axle.

To the wheel and axle may be refered several kinds of machines or instruments, as the crane, windlass, capstan, gimblet, auger, &c.

364. *In a combination of wheels with teeth, if the power* P *be to the weight* W, *as the product of the radii of all the axles or pinions, to the product of the radii of all the wheels; the power and weight will be in equilibrio.*

That is, P : W :: CA.OB.RS : CD.OA.RB.



For the radii CD, OA, RB, of the wheels, with the radii CA, OB, RS of the pinions (or smaller wheels on the axes of the larger) act as three levers DA, AB, BS on the centers or fulcra C, O, R; therefore the three together may be considered as a compound lever SD (art. 362).

Corol. 1. And when the wheels are in motion, the velocity of the power P : velocity of the weight W :: CD.OA.RB : CA.OB.RS. Or the number of teeth in the wheels and pinions may be substituted for the respective radii.

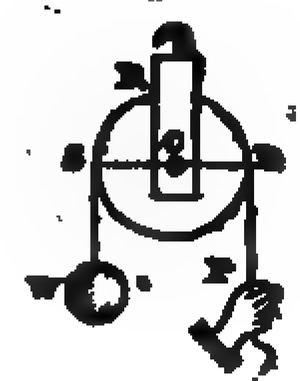
Corol. 2. Hence also, as the number of revolutions of the first wheel, is to the number of revolutions of the last wheel in the same time, so is the product of the number of teeth in the pinions, to the product of the number of teeth in the wheels. For as often as the number of teeth in any pinion, is contained in the number of teeth of the wheel that drives it, so many revolutions will the pinion make, for one revolution of the wheel.

Suppose the radii of the pinions are each = 4, CD = 25, OA = 32, RB = 25, and the power P = 10*lb*.

Then 4 × 4 × 4 : 25 × 32 × 25 :: 10 : 3125*lb*. the weight W when the weight and power are in equilibrio. But were the wheels in motion, the velocities of the power and weight would be as 3125 to 10.

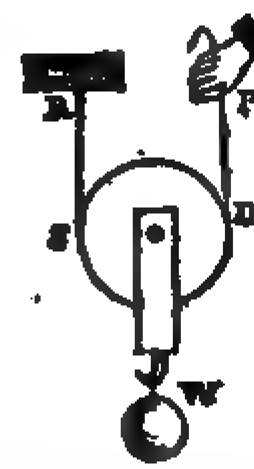
By the addition of another wheel and pinion, it is manifest a much greater weight might be raised by the same power, but the velocity of the weight would be proportionably diminished: Hence the maxim in Mechanics, *what is gained by power is lost in time.*

The Pulley.



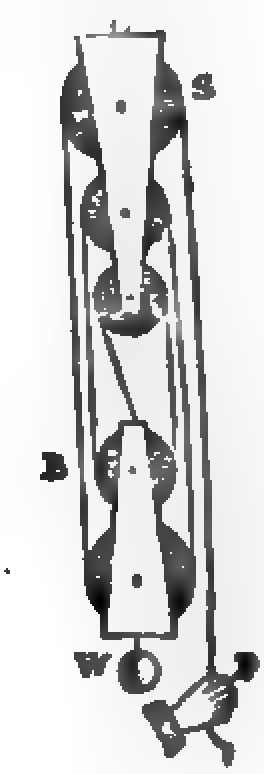
365. A Pulley is a small wheel SD of metal or wood, moveable round an axis C fixed in a block; a groove is cut round the edge of the wheel to receive the rope WSRDP.

If the pulley is fixed, and the weight W and power P are in equilibrio, the power and weight will be equal. For SCD is a lever of the first kind, where the arms CS, CD are equal.



When the pulley, and weight W, are supported by the rope RSDP, the power at P will be but half the weight in case of an equilibrium. For the rope being fastened at R, the two parts RS, PD will be equally stretched, and consequently each will bear half the weight.

366. *In a combination of pullies drawn by one rope going over all the pullies, if the power P, is to the weight W, as 1, is to the number of parts of the rope proceeding from the moveable or lower block and pullies (B.); then the power and weight will be in equilibrio.*



For the lower or moveable block (B) and the weight W together with the power P, are supported by 6 ropes, or rather 6 parts of the same rope, all equally stretched; but the part SP sustains the power P, and therefore each of the other parts is stretched by $\frac{1}{5}$ of the weight, consequently (in the annexed figure) 5 : 1 :: W : P.

Corol. Hence if the weight W be raised by the power P, the latter will descend 5 feet (for example) while the former is raised 1 foot. For each of the parts of the rope proceeding from the lower or moveable block to the upper

or fixed one, is shortened by 1 foot, and consequently the weight will be raised that distance.

There are various other combinations of blocks with pullies, but the equilibrium is determined in a similar manner; namely, by comparing the spaces described in the same time by the power, and the weight or body moved.

But pullies, and other machines seldom work without considerable friction; and ropes are never perfectly flexible; therefore we must not expect that computations will always agree with experiment.

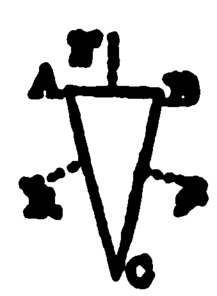
Of the WEDGE.

367. The form of this instrument is sufficiently known (art. 280, vol. 1). It is commonly made of wood or iron; and used in splitting blocks of stone, wood, &c. and sometimes in lifting or raising very heavy bodies. The force is communicated to the wedge by the blow of a sledge-hammer or heavy mallet.

When the force F *acting perpendicularly to the back of the wedge* AB, *is in equilibrio with the resistances* R, *r*, *which act perpendicular to the sides* AC, BC,

Then AB : F :: AC + BC : R + *r*.

For the three forces in equilibrio will be as the sides of the triangle ABC which are perpendicular to the directions of those forces, (320):



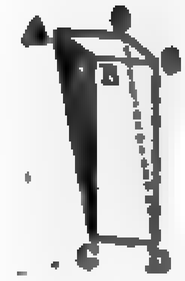
That is AC : BC :: R : *r*;

or AC + BC : BC :: R + *r* : *r*.

Also AB : F :: BC : *r*;

And by equality AB : F :: AC + BC : R + *r*.

368. If the wedge be rectangular, or the triangular sides ABC, OGD perpendicular to the back or end ABGO, those sides are parallel to the direction of the force (F) acting against the end AG; and therefore any resistances against the planes ABC, OGD would have no effect in producing an equilibrium except what arose from friction. The friction however, on the quadrangular sides only is so very great that it retains the wedge in its situation after the force (F) is removed, and hence we must conclude that it is, at least, equal to the force which drives the wedge.

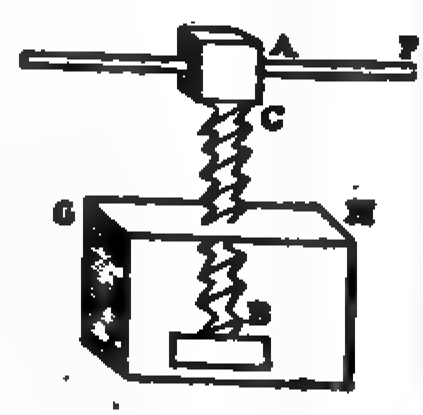


But we cannot compare percussive force with weight or pressure. A common iron wedge is sometimes wholly forced into a block of tough wood without splitting it, by a few blows with a heavy mallet; now it is impossible to discover by calculation, and it would be extremely difficult to determine from experiment, the enormous weight or pressure necessary for producing a like effect.

The wedge is a very simple mechanic power; and to this may be reduced most edge tools, and those that have a point; as the axe, chissel, spade, &c. and nails, bodkins, needles, &c.

Of the SCREW.

369. The screw is a cylinder CB round which is cut a spiral groove; the part that rises above the groove also forms a spiral, and is called the thread or threads of the screw; these make the same, or equal angles with the length BC.



When the screw is made use of as a press, &c. CB is called the male screw; the female, in which the other turns, is concave or hollow, and fixed in a block or frame GH.

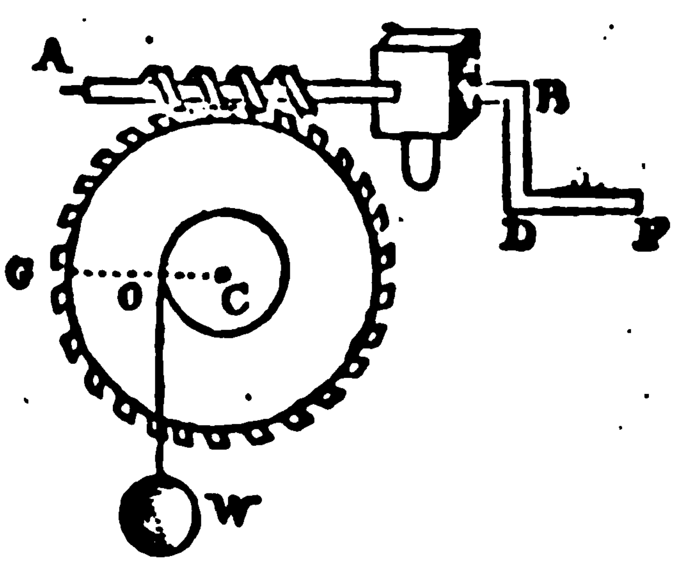
If the screw be turned by a power P acting at the end of a lever AP; then (abstracting from friction), *As the distance between any two contiguous threads measured in the direction of the length* CB, *is to the circumference of the circle which the power* P *describes, so is the power* P, *to the force at* B, *when there is an equilibrium.*

For the screw moves the distance between two threads in the direction CB at every revolution; and in case of an equilibrium, the forces (at P and B) are reciprocally as the spaces described in the same time: Which is a general property in all mechanical powers.

Suppose the force at P = 50*lb.* the distance of the threads = $\frac{1}{3}$ of an inch, and AP = 9 *feet.* Then $18 \times 12 \times 3.1416 = 678.58$ *inches* the circumference described by P;

And $\frac{1}{3}$: 678.58 :: 50 : 101787*lb.* the force exerted by the lower end of the screw.

But the friction is generally so great that it prevents the screw from receding when the power (P) ceases to act; and therefore it would probably require a force = 2P on the lever to produce the computed effect.



370. The endless or perpetual screw AB has square threads adapted to the spaces between the teeth of a wheel, which are cut oblique to fit the spiral groove whose sides are perpendicular to the axis of the screw. The screw is turned by means of a handle BDP.

Suppose the weight W to be supported on the pinion or roller whose radius is CO; and let *n* denote the number of teeth in the wheel, *C* = its circumference, and *c* = the circumference of the circle described by the handle or power P.

Then since the wheel is moved forward by one tooth of the wheel at every revolution of the screw or handle, P will make n revolutions while the wheel makes 1, or the power P moves through the spaces nc while the teeth describes the circumference C:

Hence, *velocity* of P : *vel.* of teeth :: nc : C :: $n \times$ BD : CG, (because the circumferences c, C, are as the radii BD, CG).

But CO : CG :: *vel.* of W : $\frac{\text{CG} \times \textit{vel.} \text{ of W}}{\text{CO}}$ the velocity of the point G or of the teeth;

hence by equality, *vel.* of P : $\frac{\text{CG} \times \textit{vel.} \text{ of W}}{\text{CO}}$ :: $n \times$ BD : CG,

or *vel.* of P : $\frac{\textit{vel.} \text{ of W}}{\text{CO}}$ :: $n \times$ BD : 1;

That is, *vel.* of P : *vel.* of W :: $n \times$ BD : CO:

But in the case of an equilibrium, the weight and power are reciprocally as their velocities,

Therefore W : P :: $n \times$ BD : CO, or $\text{W} = \frac{\text{P} \times n \times \text{BD}}{\text{CO}}$.

Let BD = 12 *inches*, CO = 3 *inches*, the number of teeth in the wheel = 100, and P = 40*lb*.

Then $\frac{40 \times 100 \times 12}{3} = 16000$*lb*. the weight (W) that a power of 40*lb*. at the handle would sustain, supposing no friction. This however, is always very considerable; but less on square than on sharp threads.

From the two preceding examples it is easy to perceive that screws may be made to act with prodigious force. The Instrument called a Vice or *Vis* probably was distinguished by that name on account of its great power.

Respecting the *Inclined Plane*, it may be sufficient to refer to articles 343, 346, and 351.

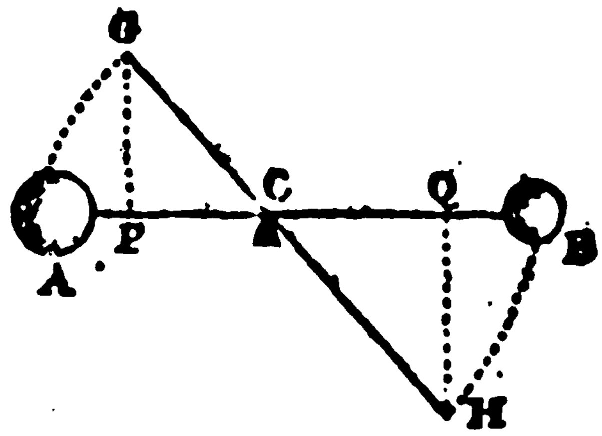
OF THE CENTER OF GRAVITY.

371. THE center of gravity is that point by which if a body be suspended, it shall hang or rest in any position.

Thus the center of a globe, and the middle of the axis of a cylinder, are their centers of gravity, if the bodies are uniformly dense.

Also the center of a circle, and the intersection of the diagonals of a parallelogram, are the centers of gravity. In speaking of the center of gravity of a surface however, we suppose that surface to be an indefinitely thin uniform lamina of matter.

372. *Suppose the centers of the globes* A *and* B *are connected by an inflexible horizontal line or lever* AB *(without weight), and let* C *be the fulcrum or support; then if* $CA \times A = CB \times B$, *the bodies will be in equilibrio* (369); *and the fulcrum* C *is their center of gravity.*



For suppose the lever with the bodies to be turned on the fulcrum C into any other position GH; and let fall the perpendiculars GP, HQ. Then since the bodies are supported at the centers A and B, they gravitate or act on those points or the ends of the lever only, but in the directions GP, and QH when the lever is in the position GH. Now the triangles, CPG, CQH being similar, it follows that CP and CQ have the same ratio as CA and CB; hence $CP \times A = CQ \times B$, therefore the equilibrium still remains; and the bodies would rest in any position of the lever, provided it was always supported at the point C.

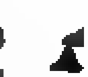
Corol. 1. Hence if the fulcrum or point
shifted towards B (for example), it is mani
A would preponderate, and the lever turn
vertical position, and consequently the cent
the bodies would, in that case, be below the
but in the same vertical line.

Corol. 2. Hence also, if a body be sus
about which it can move freely by its own
of gravity will rest in the vertical line passi
of suspension. This suggests a method of f
the center of gravity of a thin flat body, th

Suspend the body by a string SP at an
point P on the edge, and mark the vertic
PO on the surface by means of a plumb-lin
SO. Hang it up again by some other poi
Q, and draw the vertical QR; then becaus
the center of gravity is in the line PO, an
also in QR, it must be at their intersectio
G, or rather within the surface opposite tha
point.

And it follows that the body would remai
at rest on a prop at R were the string S
removed; but a very small weight or forc
when applied on either side of the vertical li
it fall.

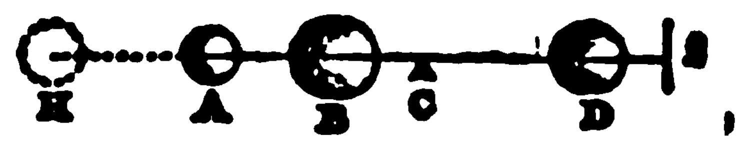
Corol. 3. Therefore when the vertical line (GP) passing through the center of gravity (G) of a body (AB) falls without the base (AD), the body will



not stand on an horizontal plane (AH), b
part of the support or base in that vertic
quently the body must turn on the base, a
This is also the reason why a globe will not
cept it be horizontal.

373. *If three bodies,* A, B, D *of known weights, are connected by means of an inflexible line or lever* AD *passing through their centers of gravity, and the distances* AB, BD, *from each other are given; to find the common center of gravity* (C) *of the whole.*

This is the same thing as determining the fulcrum C when they rest in equilibrio.



Therefore (359, corol. 4) it will be $CB \times B + CA \times A = CD \times D$.

Let the distance $AB = a$, $BD = b$, and $CD = x$; then $CB = b - x$; and we have $(b-x)B + (b-x+a)A = Dx$, which reduced

gives $x = \dfrac{Aa + Ab + Bb}{A + B + D} = CD$, the distance of the center of gravity C from the center of the body D.

Let S be a point in the line or lever AD produced; and put $DS = d$;

Then

$$CS = \frac{Aa+Ab+Bb}{A+B+D} + d = \frac{Aa+Ab+Bb+Ad+Bd+Dd}{A+B+D};$$

$$\text{or } CS = \frac{(a+b+d)A+(b+d)B+dD}{A+B+D} = \frac{AS.A+BS.B+DS.D}{A+B+D}.$$

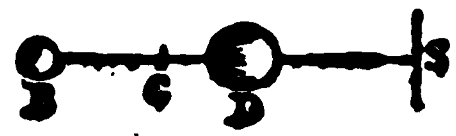
That is, if we consider S as a fulcrum or point of suspension, the sum of the forces or products $DS.D + BS.B +$ &c. divided by the sum of the bodies $D + B +$ &c. gives the distance of the common center of gravity of all the bodies from that point.

Corol. 1. Hence the center of gravity of any number of bodies in the same right line is readily determined:

For $\dfrac{HS.H+AS.A+BS.B+DS.D}{H+A+B+D}$ is the distance of the centre of gravity of the bodies H, A, B, D, from S; therefor

making $DS = o$, we have $DS.D = o$, $HS = HD$ &c. and the expression becomes $\frac{HD.H + AD.A + BD.B}{H + A + B + D}$, for the distance from the body D.

Corol. 2. And if C be the center of gravity of the bodies B and D; then $(B + D)\,CS = BS.B + DS.D$:

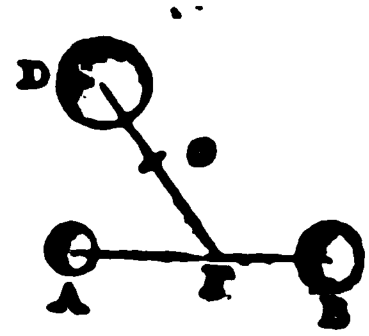


$$\text{For } \frac{BD.B}{B+D} = CD, \text{ and } CS = \frac{BD.B}{B+D} + DS;$$

$$\text{and } (B + D)\left(\frac{BD.B}{B+D} + DS\right) = (BD + DS)\,B + DS.D;$$

That is $(B+D)\,CS = BS.B + DS.D$.

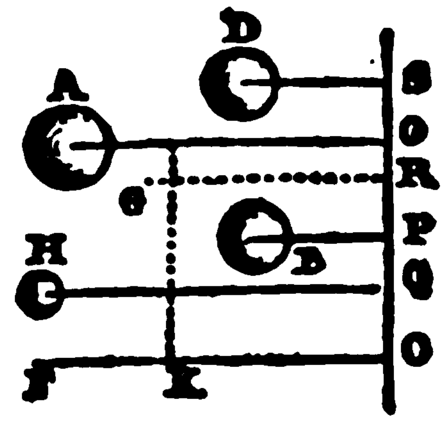
374. *The centers of gravity of three bodies* A, B, D, *any how situated, being given; to find their common center of gravity* C.



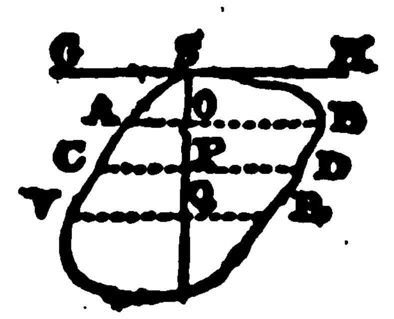
Join the centers of gravity of any two of them, suppose A and B: divide the distance AB reciprocally as the weights, that is take FB (for example) so that $A + B : AB :: A : BF$; then F is the center of gravity of the two bodies, or the place of the fulcrum upon which they would rest in equilibrio.

We now may consider F as the place of both bodies A and B, because that point sustains, or is pressed by their weight. Hence if DF be divided into two parts CD and CF having the ratio of $A + B$ to D, the point C will be the center of gravity of the three bodies. And in this manner, by taking two at a time, &c. the center of gravity of any number, or system of bodies, may be found.

375. *But the center of gravity of several bodies* A, B, D, H, *not situated in the same plane, may be determined thus,*



Let DS, AO, BP, HQ, drawn from the centers of gravity of the bodies, be perpendicular to a plane passing through SC. Then since all the points of suspension are in that plane, if we proceed with the distances as in Art. 373, we shall have $\frac{HQ.H+AO.A+BP.B+DS.D}{H+A+B+D}$ for the distance of the center of gravity from the plane. Suppose a plane GK parallel to SC to be at that distance from SC; and let the distance of the center be found from another plane CF, then the point which is the required center will be somewhere in the intersection of these two planes; and therefore a third plane, found in like manner, if it cuts that intersection, will determine the point.



Corol. Hence if a body be suspended at a point S in the plane GH, and, after the manner of indivisibles, we suppose it to be composed of innumerable sections, AB, CD, VR, &c. parallel to the plane GH; then if SQ be perpendicular to GH,

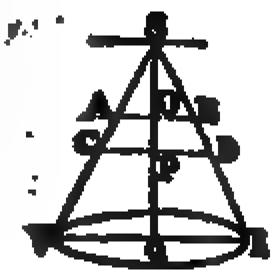
$$\text{the sum of} \left\{ \begin{array}{l} SO \times \textit{section } AB \\ SP \times \textit{section } CD \\ SQ \times \textit{section } VR \\ \&c. \qquad \&c. \end{array} \right\} \text{divided by the body}$$

will be the distance of the center of gravity from the plane GH. For if the body be homogeneous, the magnitude of the whole, or any part, is proportional to its weight.

Therefore if $d =$ the distance SO, or SP, &c. $s =$ the section AB, or CD, &c. and $B =$ the body,

Then $\frac{\text{all the } ds}{B} =$ the distance of the center of gravity from the plane GH. And by finding two other planes in which the center lies, its exact situation will be determined.

Example. Let SVR be a right cone suspended at the vertex. Then since the center of gravity of all the sections parallel to the base are in the axis SQ, the center of gravity of the cone must also be in that line.



Put the diameter of the base VR $= b$, the diameter CD of the section indefinitely near the base $= a$, the diameter AB of the next section $= c$, &c. $h =$ the height or axis QS, and $m = .7854$.

$$\text{Then } b : h :: a : \frac{ah}{b} = \text{SP},$$

$$b : h :: c : \frac{ch}{b} = \text{SO},$$

&c. &c.

$$\left.\begin{array}{l} \text{And } mb^2 \times h \text{ or } b^3 \times \dfrac{mh}{b} \\ ma^2 \times \dfrac{ah}{b} \text{ or } a^3 \times \dfrac{mh}{b} \\ mc^2 \times \dfrac{ch}{b} \text{ or } c^3 \times \dfrac{mh}{b} \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{array}\right\} \text{are the sections multiplied by the distances from the point of suspension:}$$

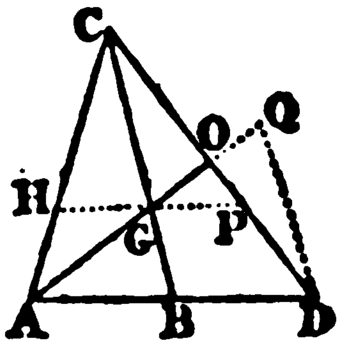
$$\text{That is } (b^3 + a^3 + c^3 + \text{\&c.}) \times \frac{mh}{b} = \text{all the } ds.$$

But since the indefinitely small distances QP, PO, &c. are supposed to be equal, the differences of VR, CD, AB, &c. will also be equal; and therefore $b^3 + a^3 + c^3 +$ &c. constitute an infinite series of cubes whose roots are in arithmetical progression; the greatest cube being b^3, and the least, or that at the vertex $= o$.

Now the sum of such a series is $\frac{n^4}{4}$, where n denotes the number of terms (179); and substituting b for n, we have $\frac{b^4}{4}$ for the sum of $b^3 + a^3 + c^3 +$ &c. *in infin.* Therefore $\frac{b^4}{4} \times \frac{mh}{b}$ or $\frac{mhb^3}{4} =$ all the ds; this divided by $\frac{mb^2h}{3}$ (or B) the content of the cone, gives $\frac{3}{4}h$, the distance of the center of gravity from the vertex S.

Corol. Hence the center of gravity of an upright pyramid is also $\frac{3}{4}$ of its axis distant from the vertex.

To find the center of gravity of a Triangle (ACD).



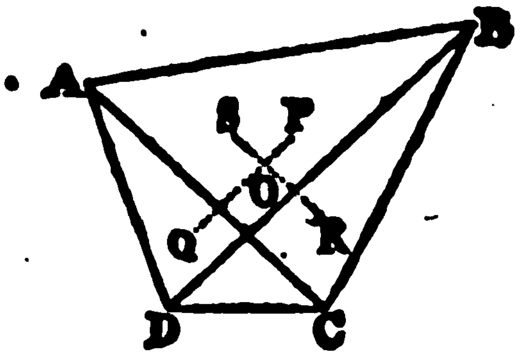
376. Bisect any two of its sides AD, CD, by lines CB, AO, drawn from the opposite angles; and the point of intersection G is the center of gravity.

Conceive the triangle to be composed of an indefinite number of right lines AD, HP, &c. parallel to the side AD; then since CB bisects those lines, or passes through their centers of gravity, the common center of gravity of the whole, or the centre of gravity of the triangle, must also lie in that line. And in the same manner it is proved that it falls in the bisecting line AO; consequently it must be at the intersection G.

If DQ be parallel to BC, the triangles OQD, OGC are equiangular, and because CO=OD, they are also equal, therefore DQ=CG; and since AB=BD, it follows, from similar triangles, that GB=½QD=½GC, therefore CB is trisected in G, and consequently CG=⅔CB. In like manner AG=⅔AO.

Corol. If CAD be the base of an upright prism, the center of gravity of the prism will be in the middle of the line drawn perpendicular to the base at the point G.

To find the center of gravity of a Trapezium (ABCD).



377. Draw the diagonals AC, DB; and find Q, P, the centers of gravity of the triangles ADC, ABC; and R, S, those of the triangles DCB, DAB; join QP, and RS: then, as the center of gravity of the trapezium lies in QP, and also in RS, its situation must therefore be at O the intersection of those lines.

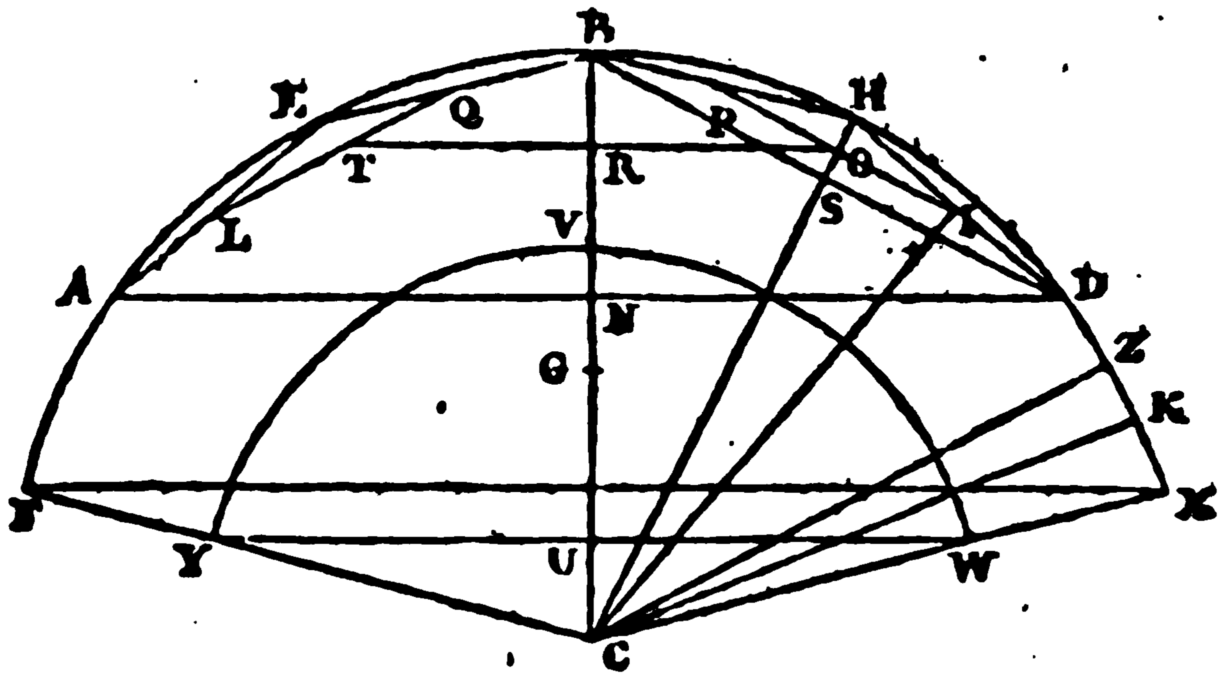
Or the center O may be found by dividing QP reciprocally as the areas of the triangles ACB, ACD; that is,

triang. ACB : *triang.* ACD :: OQ : OP.

To find the center of gravity of a Polygon.

378. The center of gravity of any regular polygon is evidently that of its inscribed or circumscribing circles. But if the polygon be irregular, divide it into triangles, and find their centers of gravity; then if we consider the magnitude of each triangle to be a weight placed at its center of gravity, the common center of gravity of the whole may be found by proceeding as in art. 374.

379. *If sides of a regular polygon* (AE, EB, BH, HD, &c.) *be inscribed in the segment of a circle; then, as half the sum of the sides* (DH + HB), *is to* (CI) *their distance from the center of the circle, so is* (ND) *half the chord of the segment, to* (CR) *the distance of the center of gravity of those polygonal sides from the center of the circle.*



Let the sides be bisected in L, Q, P, I; then if LQ and IP are also bisected in T and O, the intersection of TO and CB, or the point R, will evidently be the center of gravity of the chords or polygonal sides AE, EB, BH, HB.

The triangles DSH, COI are similar,

whence CO : CI :: DS : DH,

:: 2DS (= BD) : 2DH (= DH + HB);

or CO : BD :: CI : DH + HB.

And from the similar triangles ORC, BND,
we get CO : BD :: CR : ND;
Therefore by equality DH + HB : CI :: ND : CR.

Corol. 1. If we suppose the sides of the polygon to be diminished indefinitely, so as to coincide with the arc, then half the sum of the sides is equal to half the arc, and CI the perpendicular becomes equal to the radius; hence

As half any arc of a circle, to half its chord, so is the radius of the circle, to the distance of the center of gravity of the arc from the center of the circle.

Corol. 2. Conceive the sector FBXC to be divided into an infinite number of triangles CXK, CKZ, &c. the bases XK, KZ, &c. being considered as right lines; then (376) their centers of gravity will be $\frac{2}{3}$ CX distant from the center C. Let CW $= \frac{2}{3}$CX, and describe the arc WVY; then the centers of gravity of all the triangular spaces will be in that arc; consequently the center of gravity of the arc will also be that of all the triangles, or of the sector FBCX; therefore if G be the center of gravity of the arc WVY, or of the sector,

arc VW : UW (half its chord) :: CW : CG.

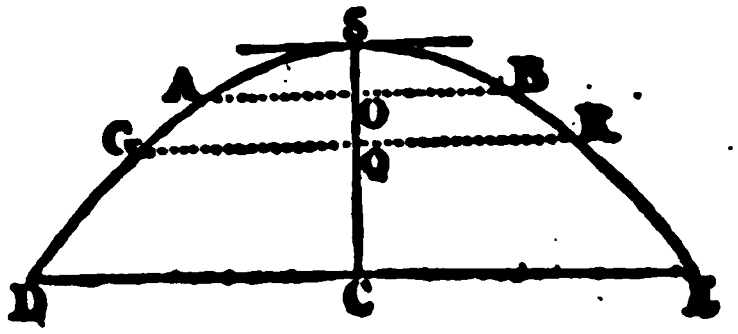
But the sectors BCX, VCW are similar, and CW $= \frac{2}{3}$CX,
whence, *arc* BX : $\frac{1}{2}$ *chord* FX :: $\frac{2}{3}$CX : CG,
or, as the *arc* FBX : *chord* FX :: $\frac{2}{3}$ *radius* CX : CG.

Corol. 3. If t and x respectively denote the distances of the centers of gravity of the triangle CFX, and the circular segment FBX, from G,

Then *triang.* $\times\, t =$ *segm.* $\times\, x$, whence $\dfrac{triang. \times t}{segm.} = x$ the distance of the centre of gravity of the segment from that of the sector.

To find the center of gravity of a Parabola.

380. It is manifest the center of gravity lies in the axis.



Let the parabola be suspended at its vertex S: and put $a=$SC the axis, and $p=$ the parameter. Also suppose the surface to be composed of an infinite number of lines AB, GR, &c. parallel to DE, and at equal distances SO, OQ, &c. from one another:

Then, from the nature of the parabola,

$$2\sqrt{pSO} = AB,$$
$$2\sqrt{pSQ} = GR,$$
$$\&c. \qquad \&c.$$

And (375. corol.) $2\sqrt{pSO}\times SO + 2\sqrt{pSQ}\times SQ + \&c.\ldots.2\sqrt{pSC}\times SC,$

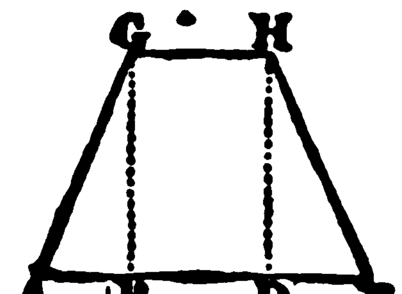
$$\text{or } 2\sqrt{pSO^3} + 2\sqrt{pSQ^3} + \&c.\ldots\ldots\ldots\ldots 2\sqrt{pa^3}$$

$$\text{That is } (SO^{\frac{3}{2}} + SQ^{\frac{3}{2}} + \&c.\ldots\ldots\ldots a^{\frac{3}{2}})\,2\sqrt{p} = \text{all the } ds.$$

Now, by Art. 179, the sum of the series $SO^{\frac{3}{2}} + SQ^{\frac{3}{2}} + \&c.\ldots\ldots\ldots a^{\frac{3}{2}}$ (that is, from $o^{\frac{3}{2}}$ at S, to $a^{\frac{3}{2}}$ at C) will be $\frac{a^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{a^{2\frac{1}{2}}}{2\frac{1}{2}}$; therefore $\frac{a^{\frac{5}{2}}}{2\frac{1}{2}}$ $\times 2\sqrt{p} =$ all the ds; which divided by $\frac{4}{3}\sqrt{pa^3}$ the area of the parabola, (or B), gives $\frac{\frac{4}{5}\sqrt{a^5}}{\frac{4}{3}\sqrt{a^3}} = \frac{3}{5}a$, the distance of the center of gravity from the vertex S.

To find the center of gravity of the frustum of a square Pyramid.

381. Let $l=$GH the side of the less end, $g=$AB the side of the base, or greater end, and $h=$ the height of the frustum.



If the frustum be cut through the 4 sides of the less end by planes perpendicular to the base, it will be divided into 4 equal square pyramids, 4 equal triangular prisms, and a parallelopiped GHDR, all of the same altitude. Each pyramid having AR or DB or $\frac{g-l}{2}$ for the side of its

base; and AE or DB, and GH or ED or their equals $\frac{g-l}{2}$ and l, are the sides of the bases of the prisms *.

Now $\left(\frac{g-l}{2}\right)^2 \times \frac{1}{3}h \times 4 = (g^2-2gl+l^2)\frac{1}{3}h =$ the cubic contents of the 4 pyramids.

$\frac{g-l}{2} \times \frac{1}{2}h \times l \times 4 = (gl-l^2)h$..............of the 4 prisms.

$l^2 \times h$..........of the parallelopiped.

And the aggregate is $(g^2+gl+l^2)\frac{1}{3}h$, the content of the frustum.

Suppose the frustum to be suspended at the least end GH;

Then $\frac{3}{4}h$ is the distance of the center of gravity of the pyramids } from that end.
$\frac{1}{2}h$ .... of the prisms
$\frac{1}{2}h$ .... of the parallelopiped

And $\frac{(g^2-2gl+l^2)\frac{1}{3}h \times \frac{3}{4}h + (gl-l^2)h \times \frac{1}{2}h + l^2 \times h \times \frac{1}{2}h}{(g^2+gl+l^2)\frac{1}{3}h}$ will be, the distance in the axis, of the center of gravity of the frustum from the least end, or plane of suspension GH, (375). This expression reduced becomes $\frac{3g^2+2gl+l^2}{g^2+gl+l^2} \times \frac{1}{4}h$.

Corol. And the same expression will answer for the frustum of a cone or any upright pyramid, if g and l denote the diameters, or other similar lines of the greater and less ends: because the surfaces will be as the squares of those diameters, or lines.

Example. Let the frustum be a squared piece of timber 30 *feet* long, and the sides of the greater, and less ends $= 2$, and $1\frac{1}{2}$ *feet*, respectively;

Then $\frac{12+6+2\frac{1}{4}}{4+3+2\frac{1}{4}} \times \frac{30}{4} = 16\frac{31}{74}$ *feet*, the distance of its center of gravity from the least end.

382. *Suppose* AB *is a squared beam, or lever of oak,* 30 *feet long, each end being a foot square; now what weight* W *at the end* A, *would keep it in an horizontal position, on a fulcrum* C, 3 *feet from that end, if each cubic foot of the beam weighs* 57lb.?

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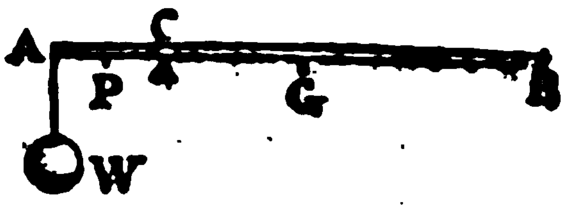
* These sections will readily be comprehended if the exterior lines are drawn on a model, which may be cut from a common cork, or soft wood.



Since 1 foot in length is also a cubic foot, we have $27 \times 57 = 1539lb.$ the weight of the arm CB, and $3 \times 57 = 171lb.$ that of CA.

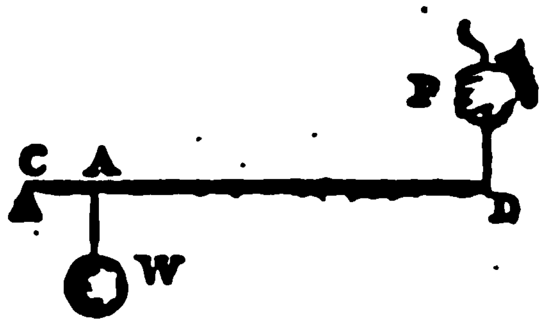
$CG = 13\frac{1}{2}$, and $CP = 1\frac{1}{2}$ feet, the distances of the centers of gravity of the arms from the fulcrum or prop C.

We may now consider a weight of 1539*lb.* at G, another of 171*lb.* at P, and a third at A, all suspended on a lever void of gravity, and resting in equilibrio on the support C: and we have

$$W \times CA + P \times CP = G \times CG, \text{ (359 corol. 4)}$$

whence $W = \dfrac{G \times CG - P \times CP}{CA} = \dfrac{1539 \times 13\frac{1}{2} - 171 \times 1\frac{1}{2}}{3} = 6840lb.$ the weight or force required.

383. *If* CD *be an uniform iron bar, or lever of the second kind,* 6 *feet in length, weighing* 36lb. *it is required to find what power* P *would be sufficient to sustain the lever, and weight* W = 64lb. *in equilibrio, if the distance* CA = 16 *inches?*



The lever being supposed a right line, we have $DC \times P = CA \times W$ (360), whence $P = \dfrac{CA \times W}{CD}$, which, in that case, would be the power; but half the weight of the lever, or $\frac{1}{4}CD$, is sustained by the power P, (CD being inches); therefore P is $= \frac{1}{4}CD + \dfrac{CA \times W}{CD} = 18 + \dfrac{16 \times 64}{72}$ $= 32\frac{2}{9}lb.$ the power required.

Scholium. Were the lever without gravity, it is manifest, by increasing its length, the necessary power would be diminished. But when the lever is an iron bar, beam of wood, &c. there is a certain determinable length which admits of a *minimum* power that will sustain, or raise a given weight (W) at a given distance (CA) from the prop C. Thus in the present example, let the length $CD = x$ inches; then $\frac{1}{4}x =$ half the weight of the lever in pounds; and $\frac{1}{4}x + \dfrac{CA \times W}{x} =$ the power P.

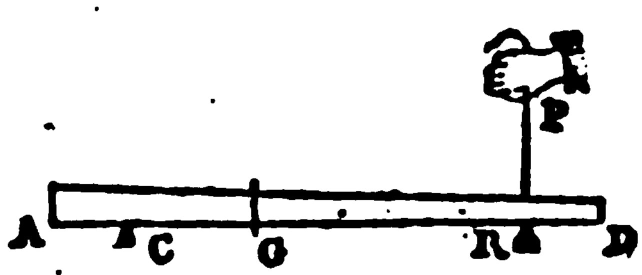
Suppose $\frac{1}{4}x + \dfrac{CA \times W}{x} = m$, then by reduction we have the quadratic equation $x^2 - 4mx = -4CA \times W$;

whence $x = 2m \pm \sqrt{(4m^2 - 4CA \times W)}$:

Now the least possible value of m is when $4m^2 - 4CA \times W = 0$, or $4m^2 = 4CA \times W$; for if $4m^2$ be *less* than $4CA \times W$, the expression $\sqrt{(4m^2 - 4CA \times W)}$ becomes impossible. Hence, when $4m^2 = 4CA \times W$, we get $m = \sqrt{(CA \times W)}$; and $x = 2m \pm 0 = 2\sqrt{(CA \times W)} = 2\sqrt{(16 \times 64)} = 64$ *inches*, the length of the lever.

And $P = \frac{64}{4} + \frac{16 \times 64}{64} = 32$ *lb*. The least power by which the weight can be sustained in equilibrio, when the lever weighs half a pound per inch.

384. *A piece of timber nearly in the shape of a conic frustum, 40 feet long, is supported in an horizontal position on two props* C *and* R, *6 feet from the ends; now if the diameter of the greater end be 2 feet, and that of the less 1 foot, what is the pressure on each prop?*



Let g and l denote the diameters of the ends A and D, and h the length,

Then (381) $\frac{12+4+1}{4+2+1} \times 10 = 24\frac{2}{7}$ feet, $=$ DG the distance of the center of gravity G from the less end; therefore $GC = 9\frac{5}{7}$, and $GR = 18\frac{2}{7}$.

If $s =$ the weight, or the cubic contents of the frustum, we may conceive that weight, or contents, to be suspended at the center of gravity G on a lever AB void of gravity, and supported in equilibrio on the fulcrum C by a power P instead of the prop R;

Then (360) $CG \times s = CR \times P$; whence $\frac{CG \times s}{CR} = P$;

or 28 (CR) : $9\frac{5}{7}$ (CG) :: s : P, the pressure on R;
and 28 : $18\frac{2}{7}$ (GR) :: s : the pressure on C:

That is, the whole weight must be divided reciprocally as the distances of the props from the center of gravity of the body.

385. *Let* DB *be a heavy body in the form of a parallelopiped, standing on the base* ACB *perpendicular to the horizon; to find what power* P *acting parallel to the horizon, at a given height* CO *or* BN *above the base, would be sufficient to turn it over.*

Let the perpendicular VS bisec CB; then as the center of gravity of the rectangle CQ is in that perpendicular, we may consider the surface CQ as being collected into a weight suspended at the point S directly under the center of gravity. to move about the point C, the forc SC × the weight at S, or SC × surf dering SCO as a bended lever, and C SC × surface CQ = CO × power at libriums; hence $\frac{SC \times surface\ CQ}{CO}$ =

Now the body is composed of in and equal to CQ, therefore substitut body or of all those planes, for C

or CO : SC :: w : P, the force planes or the body in equilibrio on t vided it rested on that edge only.

Let CB = CA = 3 *feet*, the height BQ = 6 suppose the body to be heavy stone or mar foot. Then, 3 × 3 × 6 × 160 = 8640*lb.* =

$$\text{And } \frac{SC \times w}{CO} = \frac{1\frac{1}{2} \times 8640}{4} = 3240lb.$$

Therefore it would require a force somethi applied at the height of 4 feet, either to pull

386. *Let* BQAH *be the perpendic earth; to find the thickness* CB *of an necessary to support it.*

If the bank consisted of loose earth out any support on the side QB, QBA would slide down, leaving the AB inclined to the horizon CH in a g or less angle, according as the eart more, or less tenacious. Sand and fi

the angle ABQ is less than 30°; but a slope greater than 60° may be formed with some stiff soils. On these accounts, the inclination of AB is usually taken at 45° in computations, as a sort of medium.

Let $h = BQ$ the height of the wall, and $x = BC$ its thickness; then if $QA = QB$, $\frac{1}{2}h^2$ is the area of the triangle QBA, and $hx =$ that of the rectangle DB or section of the wall.

Now if we consider the triangle QBA as a body at liberty to descend down the plane AB without friction, its force against QB in an horizontal direction RN will be equal to its weight $\frac{1}{2}h^2$ (denoting its weight by the area or surface): for it is sustained in equilibrio by the resistance of BQ, which resistance is perpendicular to BQ: therefore (344, corol. 2) BH (or QA) : BQ :: *weight* $\frac{1}{2}h^2$: $\frac{1}{2}h^2$, the force in the horizontal line NR; R being the center of gravity of the triangle.

Let hx be considered as a weight at S the middle of BC (as in the preceding article); then C being the fulcrum of the benthed lever NBC, and $BN = \frac{1}{3}BQ$ (376),

we have $CS \times hx = BN \times \frac{1}{2}h^2$, in the case of an equilibrium;

$$\text{or } \tfrac{1}{2}x \times hx = \tfrac{1}{3}h \times \tfrac{1}{2}h^2; \text{ whence } x = h\sqrt{\tfrac{2}{3}},$$
$$\text{or } BC = .816\,BQ:$$

That is, when the wall is built with materials of the same weight as the earth, its thickness must exceed $\frac{8}{10}$ of the height. This is according to the example in Dr. Hutton's Course of Math. vol. II.

Muller however, (*Practical Fortification*) by allowing $\frac{1}{3}$ of the pressure for friction on the plane AB, reduces the force of the triangle to $\frac{2}{3} \times \frac{1}{2}h^2$ or $\frac{1}{3}h^2$ against the point N;

$$\text{Hence } \tfrac{1}{2}x \times hx = \tfrac{1}{3}h \times \tfrac{1}{3}h^2; \text{ and } x = h\sqrt{\tfrac{2}{9}} = .47h,$$
$$\text{or } BC = .47\,BQ:$$

That is, the thickness is nearly half the height.

But M. Belidor (*Science des Ingenieurs*) endeavours to prove that the triangle QAB should first be diminished to half its pressure or weight on account of the tenacity of the earth. He then considers the parts of the triangle as acting separately against QB in directions parallel to the slope AB, and reduces all their forces to the point Q. The same conclusion nearly however, is obtained by taking $\frac{1}{2}$ of the triangle, or $\frac{1}{2} \times \frac{1}{2}h^2$ for

the force acting in an horizontal direction against the point N; and therefore we shall have

$$\tfrac{1}{2}x \times hx = \tfrac{2}{7}h \times \tfrac{2}{3} \times \tfrac{1}{2}h^2, \text{ whence } x = h\sqrt{\tfrac{4}{21}} = .436h;$$
$$\text{or } BC = .436\,BQ:$$

which is not greatly different from the conclusion according to Muller.

To compute the thickness when the wall or revetment is of brick, or of stone: Let $e = 124 lb.$ the weight of a cubic foot of common earth; $b = 125 lb.$ that of a cubic foot of brick; and $s = 158 lb.$ the weight of stone per cubic foot. Then 124, 125, 158, or any three numbers in the same proportion, will denote their specific gravities. And since the weights of bodies are as their specific gravities, if the wall be of brick we shall have

$$\left.\begin{array}{l} \tfrac{1}{2}hx^2 \times b = \tfrac{1}{3}h^3 \times e, \text{ and } x = h\sqrt{\tfrac{2e}{3b}} = .813h = \tfrac{13}{16}h \\ \tfrac{1}{2}hx^2 \times b = \tfrac{1}{9}h^3 \times e \ldots\ldots\ldots x = h\sqrt{\tfrac{2e}{9b}} = .47h = \tfrac{8}{17}h \\ \tfrac{1}{2}hx^2 \times b = \tfrac{2}{21}h^3 \times e \ldots\ldots\ldots x = h\sqrt{\tfrac{4e}{21b}} = .43h = \tfrac{3}{7}h \end{array}\right\} \text{nearly.}$$

That is, if the wall be 16 *feet* high, its thickness, according to the first hypothesis, should be rather more than 13; but Belidor's makes it about 7 *feet*.

If the wall be of stone, then

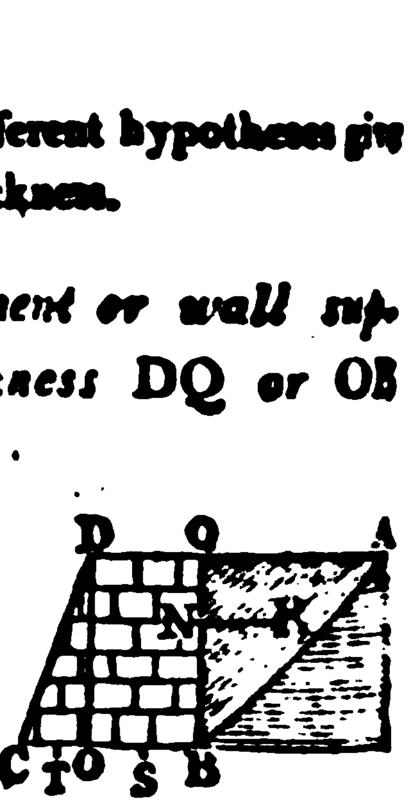
$$x = h\sqrt{\frac{2e}{3s}} = .72h$$

$$x = h\sqrt{\frac{2e}{9s}} = .42h$$

$$x = h\sqrt{\frac{4e}{21s}} = .39h$$

Hence, supposing the height $BQ = 16$ *feet*, the different hypotheses give about $11\frac{1}{2}$, $6\frac{7}{10}$, and $6\frac{1}{4}$ *feet*, respectively, for its thickness.

387. *If* CDQB *be the profil of a revetment or wall supporting the earth* QBA; *to find the thickness* DQ *or* OB *when the slope* DC *is given.*



Let $h =$ OD or BQ, $nh =$ CO the base of the triangle CDO, $x =$ BO, also suppose e, b, s to denote the same specific gravities as in the last article.

Then $hx =$ the rectangle DB, and $\frac{1}{2}h^2 =$ the triangle QBA (as above); also $\frac{1}{2}nh^2 =$ the triangle CDO. Now instead of finding the center of gravity of the trapezoid CDQD, we shall conceive

the surfaces of the rectangle DB, and the triangle CDO to be weights at S and T directly under their centers of gravity, S being the middle of OB (as before); but T will be $\frac{2}{3}$ CO distant from C; that is CT $= \frac{2}{3}nh$, and CS $= nh + \frac{1}{2}x$.

Then C being the fulcrum of the bended lever CBN, we have

$\frac{2}{3}nh \times \frac{1}{2}nh^2b + (nh + \frac{1}{2}x)\,bhx = \frac{1}{3}h^3e$,

or $\frac{1}{3}n^2h^3b + nbh^2x + \frac{1}{2}bhx^2 = \frac{1}{3}h^3e$ in the case of an equilibrium when the wall is of brick, according to the first hypothesis in the preceding article. This expression reduced

gives $x + nh = \sqrt{\dfrac{2h^2e + n^2h^2b}{3b}}$, and $x = h\sqrt{\left(\dfrac{2e}{3b} + \dfrac{n^2}{3}\right)} - nh$.

Suppose CO $= \frac{1}{5}$ of the height OD, that is, let $n = \frac{1}{5}$,

then $x = h\sqrt{\left(\dfrac{2e}{3b} + \dfrac{n^2}{3}\right)} - nh = .62h$, nearly.

But adopting the second hypothesis (Muller's), it will be

$\frac{1}{3}n^2h^3b + nbh^2x + \frac{1}{2}bhx^2 = \frac{1}{9}h^3e$; whence $x = .28h$.

And taking $\frac{2}{27}h^3e$ instead of $\frac{1}{9}h^3e$, we get $x = h\sqrt{\left(\dfrac{4e}{27b} + \dfrac{n^2}{3}\right)} - nh = .23h$; or DQ $= \frac{1}{4}$QB, according to Belidor.

If the revetment be of stone, then substituting s for b, we get

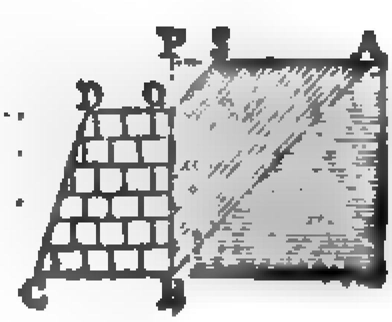
$$\left.\begin{aligned} x &= h\sqrt{\left(\frac{2e}{3s} + \frac{n^2}{3}\right)} - nh = .33h \\ x &= h\sqrt{\left(\frac{2e}{9s} + \frac{n^2}{3}\right)} - nh = .23h \\ x &= h\sqrt{\left(\frac{4e}{27s} + \frac{n^2}{3}\right)} - nh = .2h \end{aligned}\right\}\ \text{nearly.}$$

Suppose the height BQ $= 15$ feet,

$$\left.\begin{aligned} \text{then }\ .33 \times 15 &= 5 \text{ feet} \\ .23 \times 15 &= 3\tfrac{1}{2} \\ .2 \times 15 &= 3 \end{aligned}\right\}\ \text{nearly, for DQ the thickness at top.}$$

And by adding CO $= 3$ feet ($\frac{1}{5}$ of BQ) we shall have CB the thickness at bottom.

388. *When the revetment supports a bank of earth* QBAS *raised above the top* DQ: Let $T =$ the area of the triangle PBA, and $t =$ that of the triangle PQS; also suppose r to denote 1, or $\frac{1}{2}$, or $\frac{2}{7}$.



Then $\frac{1}{3}$ BP $\times$ Tr = the force of the triangle F lever PBC about the fulcrum C; and (BQ + $\frac{1}{3}$ QI angle PQS;

and their difference $\frac{1}{3}$ BP $\times$ Tr — (BQ + $\frac{1}{3}$ QP) quadrilateral QBAS.

Hence if the wall be of stone, we shall hav

$\frac{1}{2}nb^2s + nsh^2z + \frac{1}{2}hsz^2 = \frac{1}{3}\text{BP} \times Trs — (\text{BQ} +$
revetment and bank are in equilibrio.

Suppose QP = 9 feet, and the talus QS paral PQS = 45°; the other dimensions remaining as in

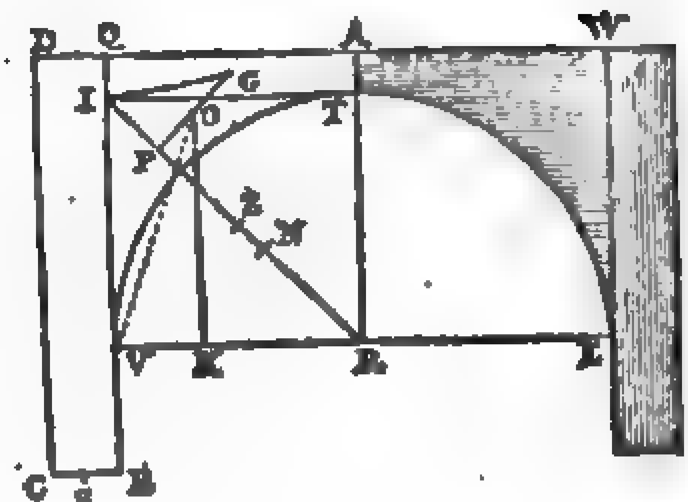
Then the equation reduced gives $x^2 + 6x + 6 =$

If r = 1, then x = 17.1 feet, } nearly, t
$r = \frac{1}{2}$ ...... x = 8.6 } ing to the
$r = \frac{1}{3}$ ...... x = 7.0

Remark. If the triangle QBA, (*fig*. p rather the triangular prism of earth of wh dicular section, were a solid body on a p AB, its force in the horizontal direction by $\frac{1}{2}$BQ^2 at all inclinations of the slope A = the area of the triangle QBA; consequ QA : QB :: $\frac{1}{2}$QB $\times$ QA : $\frac{1}{2}$BQ^2 its force as in the first of the preceding methods. gives the thickness of walls nearly dou found by experience to be necessary. S various, that we cannot expect any rule, ev which will give a satisfactory result in all c

In these computations, the wall is consi block, or the joinings as strong as the soli its resistance arises from the weight only. firmly attached to a foundation sunk in the some other *data* derived from the *streng* enter the computations.

309. *To find the thickness of the piers a semicircular Arch* VQWETV.



Let VTAQ be a perpendicular section of half the arch, O its center of gravity; and CBQD the corresponding section of the pier supporting that half arch.

Now the arch is supposed to be of such materials, that were it not opposed by the pier DB, its own weight would break it at TA: the arch therefore exerts its force or weight in three directions, namely in the perpendicular direction OK, in that of OV, and in an horizontal direction KV; and the forces will be as those three lines: but OK, which is in the direction of gravity, must therefore be proportional to the weight. Hence if w = the weight of the arch, or the surface VTAQ (to which it is proportional),

Then $OK : w :: KV : \frac{wKV}{OK}$ = the lateral pressure at V, or that in the horizontal direction KV.

We now consider C as the fulcrum of the bended lever CBV, and suppose a weight at S the middle of CB, equal to the surface DCBQ, or $= CB \times BQ$;

Then $VB \times \frac{wKV}{OK}$ is the effort of the arch at V in the direction KV to turn it on the point C, and $CS \times CB \times BQ$ that of DB on the same point C, in a perpendicular direction: consequently, in case of an equilibrium, those forces must be equal; that is,

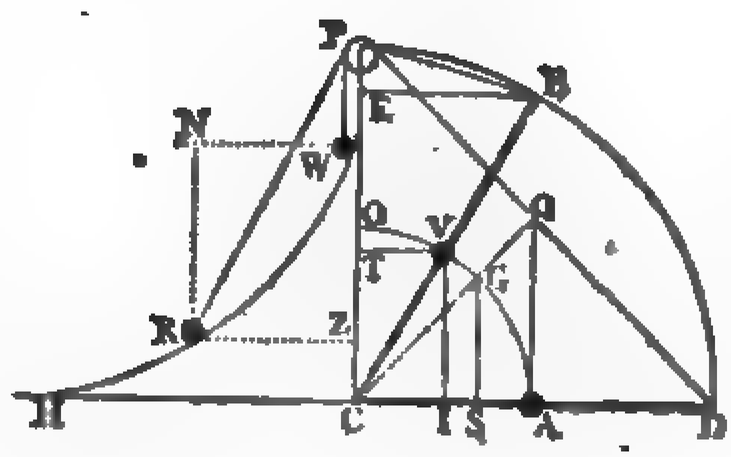
$$\tfrac{1}{2}CB \times CB \times BQ = VB \times \frac{wKV}{OK}; \text{ whence } \tfrac{1}{2}CB^2 = \frac{VB \times wKV}{BQ \times OK}.$$

Let the radius RV or RT = 30, TA = 6, and VB = 12 *feet*; then QB = 48. Now to find O the center of gravity of the surface VTAQ:

Let TI be parallel to AQ; draw IR and IG, G being the middle or center of gravity of the parallelogram IQAT. And if N be the center of gravity of the quadrant RTV, RN = 18 *feet* nearly (379 corol. 2). Also the area of the quadrant RTV = 706.86 *feet*, that of the square ITRV = 900, and the difference 193.14 = the area VTI; and since Z the middle of IR, is the center of gravity of the square ITRV, NZ is = 3.213; hence, if P be the center of gravity of the space ITV, we shall have (372) PZ $= \frac{706.86 \times 3.213}{193.14} = 11.76$, therefore IP = 9.452 nearly. Moreover, since GI is = ½ the diagonal of the parallelogram QT, we get (by trigonom.) PG = 12.094, which being divided reciprocally as the two surfaces IQAT, and VTI, gives GO = 7.414, and O is the common center of gravity of both surfaces or of the quadrilinear space VTAQ or section of the half arch. And hence OK and KV are readily found to be 26.64, and 9.87 *feet*, respectively.

These values being substituted in the above expression, we shall have $\frac{1}{2}BC^2 = \frac{12 \times (193.14 + 120) \times 9.87}{46 \times 26.64}$, whence BC = 7.8 *feet*, nearly, the thickness, when the pier just prevents the arch from falling, consequently the dimensions should be somewhat greater. Also, when the pier stands in water, its pressure will be lessened, and an allowance ought to be made on that account, except it be supported on the side DC.

390. *Suppose* CD *is a beam of wood moveable about the end* C, *and supported by the weight* W *attached to a flexible line* DPW *passing over a pulley at* P; *to determine the curve* WRH, *along which the weight* W *must move, so that the beam and weight shall always be in equilibrio.*



Let CD be horizontal, and the perpendicular CP = CD; also suppose the beam is of uniform thickness; then we may consider it as a line or lever without gravity, having a weight at the middle point A (its center of gravity) equal to that of the beam. Make CQ perpendicular to PD, and QA will be perpendicular to CD.

Since CQ and CA are respectively perpendicular to DP and QA the directions in which the weights W and A act on the lever to turn it about the end C, the weights or forces in equilibrio, will be reciprocally as those perpendiculars CQ and CA, (by the properties of the lever);

that is, CQ : CA :: *weight* A : *weight* W;
or CG : CS :: *weight* A : *weight* W, (by sim. triang.).

Now suppose CB to be another position of the beam or lever, and R the corresponding place of the weight W: Draw RZ, VT, BE parallel to CD, and VI parallel to PC; then WZ is the perpendicular descent of the weight W, and IV the corresponding vertical ascent of the weight A, and those spaces are reciprocally as the weights or forces, in the case of an equilibrium, (345);

That is, WZ : IV :: *weight* A : *weight* W :: CG : CS (by equality);

Hence it appears that WC = CQ: for WZ : IV : CG : CS :: CQ : CA or CO; that is, WZ : IV :: CQ : CO; but when the beam is vertical or in the position CP, V and O coincide, and IV becomes = CO, therefore the antecedents WZ and CQ are also equal.

Let WZ $= x$, ZR $= y$, CD or CP $= h$, PW $= p$, OT $= v$, $s =$ the length of the line DPW, $m =$ CG, $n =$ CS or GS:

Then WZ : IV :: CG : CS (or CQ : CO);

that is, $x : \frac{1}{2}h - v :: m : n$; whence $v = \frac{\frac{1}{2}mh - nx}{m}$; but the sectors OCV, PCB are similar, and PC $= 2$OC, whence $2\text{OT} = 2v = \frac{mh - 2nx}{m}$.

And because the triangle PCB is isoceles, $\text{PB}^2 = 2\text{PC} \times \text{PE}$ $= 2h \times \frac{mh - 2nx}{m} = \frac{2mh^2 - 4nhx}{m}$, therefore $\text{PB} = \sqrt{\frac{2mh^2 - 4nhx}{m}}$;

consequently $s - \sqrt{\frac{2mh^2 - 4nhx}{m}} = \text{PR}$; and $\text{PZ} = p + x$.

whence $\left(s-\sqrt{\frac{2mh^2-4nhx}{m}}\right)^2-(p+x)^2=y^2$ $(=RZ^2)$, the equation of the curve, exhibiting the relation of an ordinate WZ or NR (x), to its corresponding abscisse ZR or WN (y).

Suppose CD = CP = 20 *feet*, its weight or the weight of A = 600*lb.* then CS being the side of a square, and CG its diagonal, m and n may be denoted by 1.414 &c. and 1; and we have

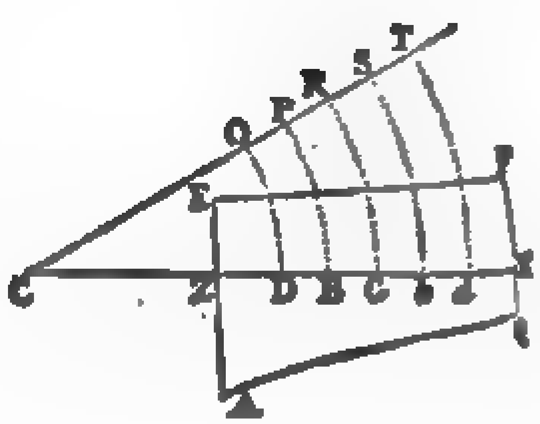
1.414 : 1 :: 600 : 424*lb.* the weight W.

Also $\sqrt{200} = 14.14$ &c. *feet* = CQ = CW; and PW = 5.86; where s = DPW = 34.14 *feet*; and CH = $\sqrt{(34.14^2-20^2)} = 27.67$ *feet*, nearly.

If we assume x, and find the corresponding values of y, the curve may be traced by means of points; Thus, suppose WZ or $x = 10$, which being substituted for x in the equation of the curve, gives ZR or $y = 10.16$ *feet*, nearly.

If CV be the radius, the perpendicular ascent of the weight A will always be denoted by (VI) the sine of the inclination of CB to the horizon; hence M. Belidor calls this curve the *Sinusoid*: see his *Science des Ingenieurs*, where a weight (W) moveable along the curve, is made the counterpoise to a Draw-bridge (CD) that turns on the end C.

301. *Let the plane* AEFQ *be perpendicular to the horizon,* G *its center of gravity, and* CGH *parallel to the horizon; then if the plane revolves about* C *as a center, always retaining its vertical position, the solid it generates, is equal to the said plane drawn into the arc* (GR) *described by its center of gravity.*

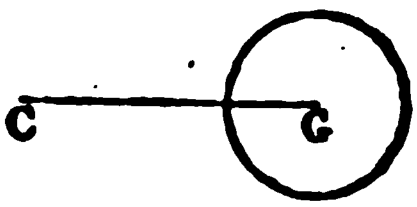


Conceive the whole surface AEFQ to be collected in, or reduced to the axis or line ZH by an indefinite number of perpendiculars to that line drawn through the surface; and suppose the distances GB, Gb; GD, Gd, &c. are reciprocally as the number of particles in the points B, b, D, d,

&c. respectively; then if B, b, D, d, &c. denote the number of particles in those points, we have (by prop. of the lever) $GB.B = Gb.b$, $GD.D = Gd.d$, &c. and therefore (373, corol. 2) $CB.B + Cb.b = CG(B+b)$, $CD.D + Cd.d = CG(D+d)$, &c. But the arcs DO, BP, &c. are respectively as the radii CD, CB, &c. hence, by taking those arcs instead of their radii, we get $BP.B + bS.b = GR(B+b)$, $DO.D + dT.d = GR(D+d)$, &c.; whence $BP.B + bS.b + DO.D + dT.d$, &c. $= GR(B+b+D+d$, &c.$)$: now the whole solid is made up of all the $BP.B$, $bS.b$, &c. taken together, therefore GR (B + b + &c.), or GR × *surface* AEFQ is the solid. And the like is also true of surfaces described by lines.

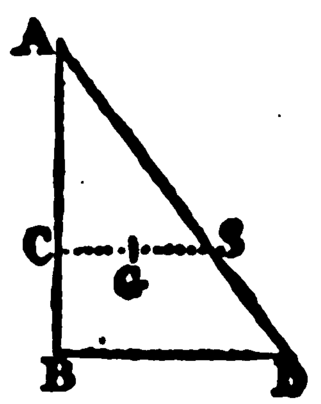
Examples.

1. If the circle whose center is G, revolve about C, it will generate a ring (like the ring of an anchor); its solid content will therefore be = the surface of the circle multiplied by the space described by the center of gravity G, or the circumference of the circle whose radius is CG. This is also known from other principles.



2. To find the content of a cone, or the solid generated by the revolution of a right angled triangle ABD about the perpendicular BA.

If AC = ⅔AB, and CS parallel to B, then G, the middle of CS, is the center of gravity of the triangle; that is, CG = ⅓BD. And the circle described by the point G in one revolution = ⅔BDc (c being = 3.1416), this multiplied by ½AB × BD (the area of the triangle), is ⅔BDc.½AB.BD = BD²c.⅓AB; but BD²c is the area of the base of the cone, or of the circle described by BD; therefore the base multiplied by ⅓ of the height gives the solid content.



3. To find the content of a parabolic spindle, or the solid generated by the revolution of a parabola DVS about an ordinate DS.

If G be the center of gravity of the parabola, $GC = \frac{2}{5}CV$ (380); hence, putting $c = 3.1416$, we shall have $\frac{4}{5}CVc$ = the circumference of the circle described by the point G. And since $\frac{2}{3}DS \cdot CV$ is the area of the parabola (304) or the generating surface, $= \frac{8}{15}CV^2c \cdot DS$: but $CV^2c \cdot DS$ is the content o der. Therefore a parabolic spindle is $\frac{8}{15}$ of it when its axis (DS) is at right angles to (CV) parabola.

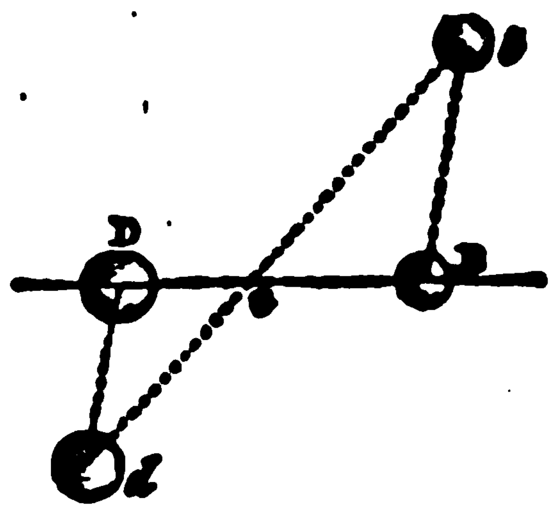
4. Let it be required to find the surfac by the revolution of a semicircular arc D DS.

Suppose C to be the center of the circle, G gravity of the arc DBS; then (379, corol. 1) $\frac{1}{2}$DS : $\frac{CS^2}{\frac{1}{2}DBS} = CG$; and (putting $c = 3.1416$), $\frac{4CS}{DB}$ or circumference described by the center of grav revolution; this multiplied by DBS the generati $4CS^2c$; that is, 2CS the diameter of the spher circumference, gives the superficies.

5. By a reverse operation, the center surface may sometimes be determined. posed to find the center of gravity G of th DS being the transverse axe, and CB the

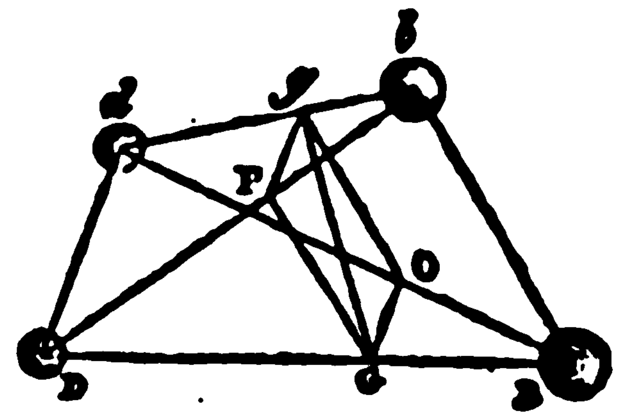
Put $t = CD$ the semitransverse, $g = CB$ the jugate, $c = 3.1416$, and $x = CG$. Then (274. $\frac{1}{2}tgc$ is the area of the semiellipse. And $2cx =$ cumference described by the center of gravity posing the ellipse to revolve on the axis DS; $2cx \times \frac{1}{2}tgc$ or tgc^2x is the generated solid or but this is also equal to $\frac{2}{3}$ of the circumscribing (281); that is $tgc^2x = \frac{4}{3}tg^2c$; whence $x = \frac{4g}{3c} =$ C (379, corol. 2), $CG = \frac{\frac{1}{2}CB \cdot AP}{arc\ ABP} = \frac{4g}{3c}$. Theref the semicircle ABP, and semiellipse DBS are the ner it is found that the center of gravity of th that of the semicircle described with the radius C

392. *If two or more bodies move uniformly in straight lines, their common center of gravity will either be at rest, or move uniformly in a right line.*



1. Let G be the center of gravity of the bodies D and B. Then (372) D : B :: GB : GD. Now if the bodies move from the positions D and B in any right lines whatever, but in opposite directions, and their velocities are as GD and GB, the center of gravity G will remain at rest. For suppose D moves to *d* while B describes B*b*, then the directions being opposite, D*d*, and B*b* are parallel and have the same ratio as GD and GB, and consequently the triangles GD*d*, GB*b* are similar; hence GB : GD :: G*b* : G*d* :: *body* at *d* : *body* at *b*; therefore G is the center of gravity of the bodies when at *d* and *b*.

When D*d* and B*b* coincide with the line passing through D and B, the bodies move directly towards, or from each other.



2. Suppose G the center of gravity of the bodies D and B, as before, and let D be stationary while B moves uniformly from B to *b*; then if GP is parallel to B*b*, the center of gravity G will describe that line GP with an uniform motion in the same time. Again, if B be stationary while D moves uniformly along D*d* in the same time that B described B*b*, G will then describe GO which is parallel to D*d*.

Join *db*, and draw P*g* parallel to D*d* or GO; then the triangles DGP, DB*b*; and also P*bg*, D*bd*, are respectively similar.

Hence $Dd : GO :: DB : GB :: Db : Pb :: Dd : Pg$: now the antecedents Dd, Dd, being the same, the consequents GO, Pg must be equal, and consequently Og is parallel and equal to GP. Moreover, since DB and db are divided proportionally in G and g, the latter point g is the center of gravity of the bodies at d and b.

It therefore follows, that if D and B move uniformly together, and describe Dd, Bb in the same time, their center of gravity G, which is urged in the directions GO, GP, will describe the diagonal Gg of the parallelogram $GPgO$ with the same kind of motion, in that time, whether they move in the same, or different planes; for the points O and P will, in both cases, fall in Bd, and Db, respectively.

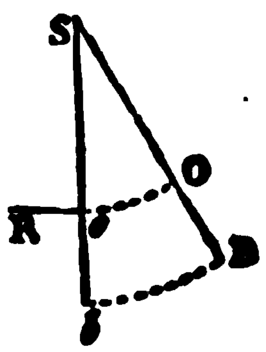
We may now consider D and B as one body at G, moving in the given direction Gg, while a third body describes some other line; and the track of their common center of gravity being determined, as above, a fourth may be added; and so on.

Corol. Hence we conclude that the center of gravity of two or more bodies is not affected by any action of the bodies upon one another. For suppose D and B attract each other, then their center of gravity G is the center of that attraction; the bodies therefore in approaching G must move through spaces proportional to GD and GB, whether they continue in the same line DB, or are urged in the directions Dd, and Db, and consequently G will remain at rest, or describe the line Gg.

It may also be observed, that when a body is projected with a whirling motion, the rotation is made round an axis passing through the center of gravity. So a body if quiescent in free space, may be said to rest on its center of gravity, but an oblique impulse would destroy the equilibrium by turning it on that center.

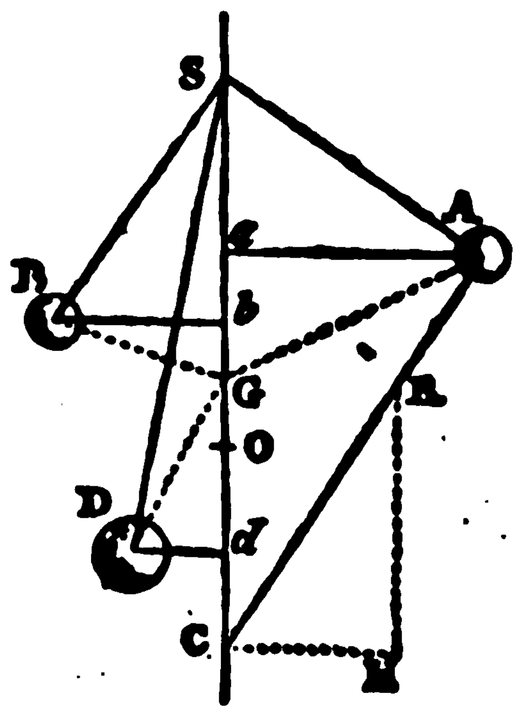
OF THE CENTERS OF PURCUSSION, OSCILLATION, AND GYRATION.

393. THE center of Percussion of a body, or a system of bodies, moving about an axis or point of suspension, is a point which being stopped by an immoveable obstacle, the body or system is quiescent without acting on the axis of motion.



Thus if a rod of wood or metal, SB, vibrating about the end S, strike a fixt obstacle Ro, and the momenta of oS and ob are equal (O being in the axis of the rod), then all motion will be destroyed; for neither of the parts oS, ob would have a tendency to move round the point o, which in that case, is the center of percussion, or the point in which all the moving force of the body or rod is collected.

394. *To find the center of percussion* (O) *of a system of bodies* A, B, D, &c. *connected by inflexible lines without gravity, and revolving about the point of suspension* S, *in a plane passing through their centers of gravity.*



Through G the common center of gravity of the bodies, draw SC, upon which let fall the perpendiculars Aa, Bb, Dd, &c. and let AC be perpendicular to SA; also make CR = SA, and draw CH perpendicular and RH parallel to SC. Then AC is the direction of A's motion as it revolves about S; and the system being stopt at O, the body A will urge the point C forward with a force proportional to its velocity into the quantity of matter, that is, as A.SA or A.CR. Now if the force A.CR be resolved into the two forces

$A.RH$, and $A.CH$, the latter $A.CH$ will represent the effort of A in a direction perpendicular to SC at the point C; but the triangles CHR, SaA are similar and equal, and therefore $CH = Sa$, consequently $A.Sa$ is the force of A in the direction HC; and (by prop. of the lever) this force drawn into CO, or $A.Sa.CO = Sa.A\,(SC - SO) = Sa.A.SC - Sa.A.SO = A.SA^2 - Sa.A.SO$ is the effort of A to turn the system or bodies about the point O.

In the same manner we get $B.SB^2 - Sb.B.SO$, and $D.SD^2 - Sd.D.SO$, the forces of B and D to turn the mass about the same point O. But when O is quiescent, the forces on contrary sides of that point destroy one another, or their sum is $= o$, that is,

$$A.SA^2 - Sa.A.SO + B.SB^2 - Sb.B.SO + D.SD^2 - Sd.D.SO, \&c. = o;$$

whence $SO = \dfrac{A.SA^2 + B.SB^2 + D.SD^2, \&c.}{Sa.A + Sb.B + Sd.D, \&c.}$, the distance of the center of percussion O from the point of suspension S.

It must be remarked, that when perpendiculars, Aa, Bb, &c. fall on both sides of S, the expressions for those forces which have a tendency to turn the system in a contrary direction, must have contrary signs.

Corol. 1. The common center of gravity of the bodies being G, we have $(A + B + D, \&c.)\,SG = Sa.A + Sb.B + Sd.D$, &c. (373. corol. 2.) hence by substitution,

$$SO = \frac{A.SA^2 + B.SB^2 + D.SD^2, \&c.}{(A + B + D, \&c.)\,SG}.$$

Corol. 2. But (Geom. art. 83) $SA^2 - Sa^2 = (Aa^2) = GA^2 - Ga^2$; whence

$$SA^2 = GA^2 + Sa^2 - Ga^2 = GA^2 + (Sa + Ga)(Sa - Ga) = GA^2 + SG\,(SG - 2Ga)$$
$$= GA^2 + SG^2 - 2SG.Ga \text{ (because } Sa = SG - Ga);$$

that is, $SA^2 = SG^2 + GA^2 - 2SG.Ga$.

In like manner $SB^2 = SG^2 + GB^2 - 2SG.Gb$;

and $SD^2 = SG^2 + GD^2 + 2SG.Gd$, &c.

those values of SA^2, SB^2, &c. being substituted in the preceding fraction which is equal to SO, and its numerator will be

$$= \begin{cases} A\,(SG^2+GA^2)-(2SG.Ga)A \\ +B\,(SG^2+GB^2)-(2SG.Gb)B \\ +D\,(SG^2+GD^2)+(2SG.Gd)\,D, \text{ \&c.} \end{cases}$$

Again, G being the center of gravity of A, B, D, &c. the sum of the products of the bodies by their perpendicular distances from that center on one side, is equal to the sum of the like products on the other (373); that is, $Ga.A+Gb.B$, &c. $=Gd.D$, &c.

Therefore $-(2SG.Ga)\,A-(2SG.Gb)\,B+(2SG.Gd)\,D$, &c. $=0$;

hence $SO=\dfrac{A\,(SG^2+GA^2)+B\,(SG^2+GB^2)+D\,(SG^2+GD^2)\text{ \&c.}}{(A+B+D\text{ \&c.})\,SG}$

$=\dfrac{(A+B+D,\text{ \&c.})\,SG^2+A.GA^2+B.GB^2+D.GD^2,\text{ \&c.}}{(A+B+D,\text{ \&c.})\,SG}$;

But if we conceive the plane passing through A, B, D, to be the section of any single mass or body, and all the particles of the body reduced to this plane by perpendiculars falling from them upon the plane, then, considering A, B, D, &c. as particles, the sum $A+B+D$, &c. will be the whole mass or body, which put $=b$, and the last expression becomes

$$SO=SG+\frac{A.GA^2+B.GB^2+D.GD^2,\text{ \&c}}{b.SG}$$

And $SO-SG\,(=GO)=\dfrac{A.GA^2+B.GB^2+D.GD^2,\text{ \&c.}}{b.SG}$, the distance of the center of percussion below the center of gravity.

Corol. 3. Hence also, $SG.GO=\dfrac{A.GA^2+B.GB^2+D.GD^2,\text{\&c}}{b.}$; therefore GO is reciprocally as SG, since the bodies A, B, D, &c. and their distances from G are given; consequently if the distance SG is known, GO will also be given.

Corol. 4. If a circle be described a
GS, the point of suspension (S) may b
cumference, and the distance between th
percussion will continue invariable, th
maining as before.

395. *Suppose the body* A (preceding
about S *by the constant force f, acting*
dicular to SC, *at a given point* C; *to*
placed in C, *would receive the same*
same time by the force f acting at C, *a*

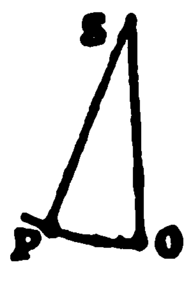
By considering CSA as a bended leve
have SA : SC :: f : $\frac{f.\text{SC}}{\text{SA}}$ the force at
force f at C, or it is the effect of the
the forces f and $\frac{f.\text{SC}}{\text{SA}}$ acting separately
would therefore have equal effects on
acting at C, and the latter at A.

Let x denote the mass required at C
of the revolving masses or bodies x a
$\frac{f.\text{SC}}{\text{SA}} = F$. Then by art. 317, (the ti
$\times \frac{W}{w}$, where W and w denote the bo
V and v, that is, $W = \text{A}$, and $w =$
Moreover, when the angular motions o
equal, their velocities will be as SC and

whence by substitution, the equatio
$\frac{\text{SC}}{\text{SA}} = \frac{\text{SA}}{\text{SC}} \times \frac{\text{A}}{x}$, which gives $x = \frac{\text{SA}}{\text{S}}$
acts at any other point O instead of C
be $\frac{\text{SA}^2.\text{A}}{\text{SO}^2}$.

Corol. 1. In like manner, the masses or bodies $\frac{SB^2.B}{SO^2}$ and $\frac{SD^2.D}{SO^2}$ if collected in O, would acquire the same angular motion from any constant force acting at O, as the bodies B and D receive from the same force acting at the same point. Consequently instead of the motion of a system of bodies A, B, D, &c. arising from a force f acting at a given point O, we may consider the motion of the mass $\frac{SA^2.A}{SO^2}+\frac{SB^2.B}{SO^2}+\frac{SD^2.D}{SO^2}$, &c. or $\frac{SA^2.A+SB^2.B+SD^2.D,\ \&c.}{SO^2}$ when concentrated in O, as an equivalent.

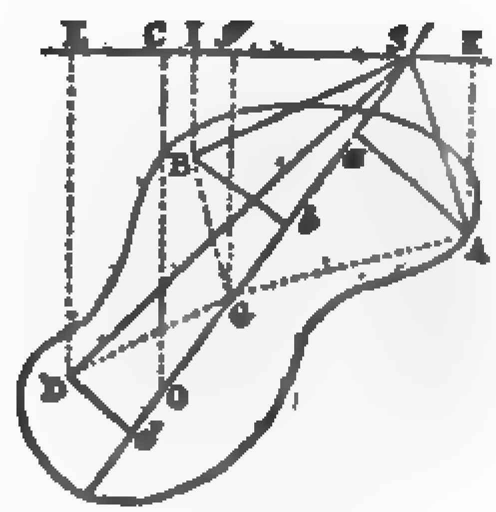
Corol. 2. Let $\frac{SA^2.A+SB^2.B+SD^2.D,\ \&c.}{SO^2}=m$; then the motive or moving force being f, and m the mass or body moved, the absolute velocity of the point O (or of m, or the whole system A + B + D &c.) will be directly as f, and inversely as m, that is as $\frac{f}{m}$; but the angular velocity is directly as the real or absolute velocity, and reciprocally as the distance SO from the center of motion S *; that is, as $\frac{f}{m}$ divided by SO, or as $\frac{f}{m.SO}$, or $\frac{f.SO}{SA^2.A+SB^2.B+SD^2.D,\ \&c}$.



* Let m be a body at O moveable about the point of suspension S; then if it be urged through the arc OP by any force f in a certain time t, a greater force would move it through a greater arc in the same time; its velocity therefore will be directly as the moving force f. But if the body be increased, the force f will not be sufficient to urge it through the arc OP in the time t, consequently the velocity, or the space described, diminishes as the body is increased, the velocity therefore will vary as the fraction $\frac{f}{m}$ varies, for a fraction is enlarged by increasing the numerator, but diminished by augmenting the denominator. Again, the angular velocity is measured by the angle OSP or by the arc OP or space described by the body, which space is as the real velocity; but if the radius OS be augmented, the angular velocity is diminished, that is, a less angle is described with the same absolute velocity, in the same time; and therefore the angular velocity is directly as the real velocity, and inversely as the distance OS.

CENTER OF OSCILLATION.

396. The center of oscillation of a body vibrating by the force of gravity, is that point in which if any quantity of matter be placed, it will perform its vibrations in the same time, and with the same angular velocity as the body itself.



Let G be the center of gravity of the body, DBSAG the plane in which it vibrates, S the point of suspension, O the center of oscillation, ESK an horizontal line, and suppose the matter of the body to be reduced to the plane of vibration by perpendiculars let fall from all its particles A, B, D, &c. upon that plane. Draw Aa, Bb, Dd, &c. perpendicular to Sd the line passing through the centers of gravity and oscillation, and AK, Gg, BI, OC, DE, perpendicular to EK.

Since A, B, D, act by the force of gravity in the directions KA, IB, ED, their efforts to move about S, will (by prop. of the lever) be A.SK, B SI, D.SE; but the effort of A is opposed to that of B and D, and therefore subtractive; whence (corol. 2 preceding art.) the sum B.SI + D.SE − A.SK will be equal to f.OS; therefore substituting B.SI + D.SE − A.SK for f.OS in the expression denoting the angular velocity (395. corol. 2) we have $\frac{B.SI + D.SE - A.SK}{A.SA^2 + B.SB^2 + D.SD^2}$ the angular motion generated by A, B, D. But if A + B + D, &c. were concentrated in O, the numerator and denominator would become (A + B + D, &c.) SC, and (A + B + D, &c.) SO², respectively; consequently $\frac{(A + B + D)\, SC}{(A + B + D)\, SO^2}$ or $\frac{SC}{SO^2}$ is the angular motion generated by a body at O: now the angular motions are supposed to be equal,

therefore $\frac{B.SI+D.SE-A.SK}{A.SA^2+B.SB^2+D.SD^2} = \frac{SC}{SO^2} = \frac{Sg}{SG.SO}$ (from the sim. triang. SCO, SgG), whence $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{B.SI+D.SE-A.SK} \times \frac{Sg}{SG}$. But it follows from art. 373, that $B.SI+D.SE-A.SK = Sg\,(A+B+D)$ whence, by substitution $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{(A+B+D)\,SG}$; and by the same article, $(A+B+D)\,SG = A.Sa+B.Sb+D.Sd$, therefore $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{A.Sa+B.Sb+D.Sd}$, being the same expression as that for the distance of the center of percussion from the point of suspension. Hence the centers of percussion and oscillation are in the same point. And therefore whatever has been demonstrated in art. 394 respecting the center of percussion, holds equally true for the center of oscillation.

And here it must be observed, that when any of the perpendiculars (Aa, Bb, &c.) fall above the point S, the expressions for the corresponding forces are to be negative.

Corol. 1. If the center of oscillation O be made the point of suspension, S becomes the center of percussion or oscillation, the plane of vibration remaining the same. For let $n = A.GA^2 + B.GB^2 + D.DG^2$; then (394, corol. 2), $\frac{n}{b.SG} = GO$, and $\frac{n}{b.SG} + SG = SO$ the distance of the point of suspension and center of oscillation; therefore if O be the point of suspension, $\frac{n}{b.OG} + OG$ is also the distance of that point from the center of oscillation; but $OG = \frac{n}{b.SG}$, which substituted for OG, and $\frac{n}{b.OG} + OG$ becomes $\frac{n}{b.SG} + SG = SO$, the distance from the point of suspension O to the center of oscillation, as before.

Corol. 2. If p be any particle, as A, B, or D, &c. of the vibrating body, d its distance from the axis of motion S, and $b =$ the body or sum of all the particles $A+B+D$, &c.

then $SO = \frac{A.SA^2 + B.SB^2 + D.SD^2, \&}{(A+B+D \&c.) SG}$ the distance of the center of oscillatio pension.

Or if d = the distance of any particle vity G, then (394, corol. 2) we have $\frac{su}{}$ the distance of the center of oscillati gravity.

CENTER OF GYRA

397. THE center of gyration of a bo is that point in which if the whole m same angular velocity would be generat a given force acting at any place, as in t

Thus suppose the body mn to be mo a certain angular velocity about the axi a force f acting at P, then if all the A, B, D, &c. of the body were collect the center of gyration, the same force at generate, in the same time, an equal motion in the mass at R.

To find the point R, we have $\overline{A.SA}$ angular motion generated in the parti system, by the force f acting at P (39 the system is concentrated in the poin comes $\frac{f.SP}{(A+B+D)SR^2}$ for the angular the definition) those expressions are equ

$= \frac{f.SP}{A.SA^2+B.SB^2+D.SD^2}$, whence SR = the distance of the center of gyration R f sion at S.

Corol. 1. Since (by the last corol.) $A.SA^2 + B.SB^2 + D.SD^2 = SO.b.SG$, we get $SR^2 = \frac{SO.b.SG}{b}$, or $SR^2 = SO.SG$, that is, SR is a mean proportional between SO and SG the distances of the centers of oscillation and gravity from the axis of motion.

Corol. 2. If $d =$ the distance from the axis of motion of any particle p of a body b (or $A + B + D$, &c.) then $SR = \sqrt{\frac{\text{sum of all the } p.d^2}{b}}$.

Hence if a body *nn* moves about an axis by the force of gravity, its whole momentum is, or may be considered as in one point O, the center of percussion or oscillation; but when the body is urged by any other extraneous force, that point changes, and is called the center of gyration.

398. *Examples.*

1. To find the center of gyration of a right line or very small cylinder SP, moving about the end S.

Suppose the line or cylinder to be composed of innumerable contiguous particles p, p, p, &c. and o, m, n, &c. their respective distances from S;

Then $po^2 + pm^2 + pn^2 +$ &c.......pSP^2,
or $p(o^2 + m^2 + n^2$, &c...........$SP^2) =$ all the pd^2;

Now (179) the sum of the infinite series of squares $o^2 + m^2$ &c. from o^2 to SP^2, is $\frac{SP^3}{3}$; and since the body $b = SP$, we have $\frac{\text{all the } pd^2}{b} = \frac{pSP^3}{3SP} = \frac{1}{3}SP^2$, (rejecting p as inconsiderable); therefore $SR = \sqrt{\frac{1}{3}SP^2} = SP\sqrt{\frac{1}{3}}$ the distance of the center of gyration R from the axis of suspension S.

2. Let it be required to find the center of percussion or oscillation of the line or small cylinder SP, the axis of motion being at S, as before.

Let O be the middle of SP, or its center of gr
oscillation: then if p, d, and b denote the same
ple, we have (394. corol. 9) $SO = \frac{pSP^3}{b \times 3SG}$ =
tance of O from the axis of motion.

Corol. Since (356. corol. 3) the length of a
seconds in the latitude of London is 39,13 in
$= 39.13 + \frac{39.13}{2} = 58.69$ *inches*; which is the len
that would vibrate by its own weight, once in a s
vibration being supposed small.

8. To find the center of oscillation of
suspended at the circumference, and vibr

Let S be the point of suspension, SP a diamet
the center of the circle or its center of gravity,
the center of oscillation.

If we suppose the surface of the circle to be com
of the circumferences of innumerable concentric
DI, BK, &c. and p a particle in the circumfere
D, or B, &c. then pGD^2 is the product of the p
p by the square of its distance from the center of
and (putting $\pi = 3.1416$), $2\pi GD$ is the circumf
radius is GD; therefore $pGD^2 \times 2\pi GD$ or $2p\pi$
the circumference drawn into the squares of th
like manner $2p\pi GB^3$ will denote the products i
and so on:

Therefore $2p\pi o^3 + 2p\pi GD^3 + 2p\pi GB^3$ &c.......
or $2p\pi\ (o^3 + GD^3 + GB^3$ &c............C
in the surface of the circle SP;

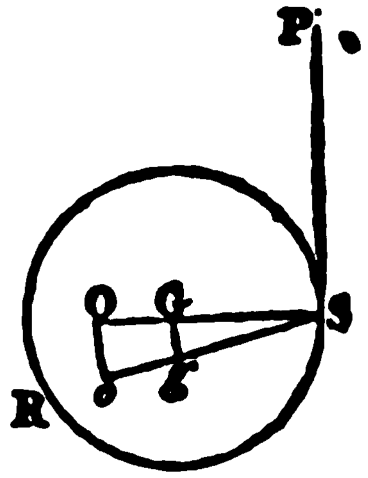
that is, (179) $2p\pi \times \frac{GS^4}{4}$ or $\frac{p\pi GS^4}{2}$ = sum

body) equal area of the circle $= \pi GS^2$ theref
distance of the center of oscillation from the cente
that is, $SO = \frac{3}{4}$ of the diameter SP.

Corol. Hence if a cylinder be suspended at th
cular section passing through its center of gravi

plane of that section, the center of oscillation will be at the distance of ¾ of the cylinder's diameter from the point of suspension. For we may conceive the cylinder to be composed of an infinite number of circular sections or planes.

399. *If one end of a string* PSR, &c. *wrapped round a cylinder, be fastened at* P, *and the cylinder left to descend by its own weight, it will move with a whirling motion; and the space descended will be to the space described in the same time by a body falling freely, as* 2 *to* 3.



Let RS be the circular section of the cylinder through G its center of gravity, O the center of oscillation, S a momentary point of suspension, and SO parallel to the horizon.

Now if all the matter of the cylinder were concentrated in the point of oscillation O, its angular velocity about the point of suspension S at the beginning of the motion, would be the same as that of the cylinder (396); but the initial velocity of a body at O would be the same as that of a body left to descend freely; hence, if O*o*, G*g* be indefinitely small arcs described by the centers of oscillation and gravity in the same time, their perpendicular velocities (and distances described) will be as the arcs O*o* and G*g*, or as OS and GS; and since the center of oscillation (O) is always in the horizontal line drawn from the point of contact S through the center of gravity G, the velocities of O and G will have the same constant ratio in all stages of the body's descent; but the absolute space descended by the cylinder, is the line described by its center of gravity; therefore, as SG is to SO, so is the perpendicular descent when it turns round its center of gravity, to the space it would describe freely in the same time.

A body descends from rest 16.13 *feet* in the first second of time; therefore SO : SG, or as 3 : 2 :: 16.13 : 10.75 *feet*, the distance which the cylinder would fall in that time by the constant unwinding of the string.

Corol. 1. The tension of the string
the cylinder. For conceive a support a
gravity G is prevented from descending
sequently, by the nature of the lever,
:: *weight* sustained at O : *weight* su
tension is constant; for the point O ge
out acting on the point S.

Corol. 2. Hence it appears that wh
an inclined plane, the space it descend
the space it would describe freely in t
plane perfectly smooth, as GS to OS
generate their motions are both dimini
absolute to the relative gravity upon the
described will therefore retain the same
OS. And the friction on the plane ha
cylinder's motion, as a string wound ro

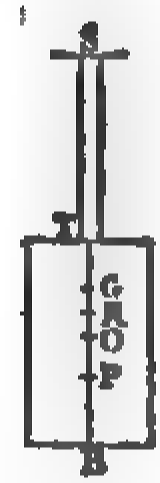
Corol. 3. Since the progressive m
uniformly accelerated, the rotation abo
an uniformly accelerated motion.

SCHOLIUM. If a simple pendulu
vibrate together in small arcs by their
of gravity, and the oscillations are perfo
the length of the pendulum is the distan
lation of the body below the point of
pendulum however, is imaginary. But
or its distance from the axis of suspen
by counting the number of vibrations
thus:

Suppose by a good clock or watch a
a minute; then (355), $41^2 : 60^2$ :: 39.1
which is the distance of the center of
below the point of suspension.

In an experiment of this kind, the body should always describe small arcs; and be suspended freely, so that the least force is sufficient to move it.

400. *Suppose S to be the axis of suspension of a pendulum SB, composed of a block of wood TB and a strong bar ST; to find the velocity of a bullet, which being discharged against the block at a point P, shall cause the pendulum to describe a given arc.*

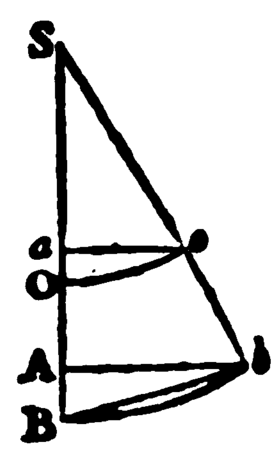


Let G, R, O, be the centers of gravity, gyration, and oscillation of the pendulum; and put SR $= a$, SO $= b$, SP $= c$, the weight of the pendulum $= m$, that of the bullet $= n$, and $x =$ the velocity of the bullet when it strikes the pendulum.

Conceive the whole mass of the pendulum to be collected in the center of gyration R; then (397) the same motion would be generated in the point R by the stroke at P as the pendulum receives; but (395, corol. 1) $\frac{SR^2 . m}{SP^2}$ or $\frac{a^2 m}{c^2}$ is the mass, which if collected in P, the pendulum would receive the same motion as before, or when all its matter was supposed to be in R. Hence if n and $\frac{a^2 m}{c^2}$ are considered as two nonelastic bodies, the former moving with the velocity x, and striking the latter at rest, we shall have (383, cor. 1) $nx \div \left(n + \frac{a^2 m}{c^2}\right)$ or $\frac{c^2 nx}{c^2 n + a^2 m}$ the velocity at P with which the pendulum and bullet (as one mass) begin their motion together. But a simple pendulum vibrating in a given arc has the same velocity in the lowest point of that arc as the velocity acquired by a heavy body in its perpendicular descent through the versed sine of the arc (354, corol. 1); and since the velocities acquired by bodies falling freely are as the square roots of the spaces descended (317), if $v =$ the versed sine of the arc described by the center of oscillation O, and $s = 16.12$

feet, it will be $\sqrt{s} : \sqrt{v} :: 2s : 2s\sqrt{\frac{v}{s}}$ = the velocity of O at the lowest point of the arc of vibration; and $b : c :: 2s\sqrt{\frac{v}{s}} : \frac{2c}{b}\sqrt{sv}$ the velocity of the point of impact P when the pendulum begins to move, which therefore must be equal to the former velocity, that is, $\frac{2c}{b}\sqrt{sv} = \frac{c^2nx}{c^2n+a^2m}$; whence $x = \left(1+\frac{a^2m}{c^2n}\right)\frac{2c}{b}\sqrt{sv}$, the velocity of the bullet when it strikes the pendulum.

This is called the Ballistic Pendulum, contrived by that eminent mathematician Mr. Benj. Robins, for the purpose of determining nearly the initial velocities of shot. We shall give an example in numbers from the pendulum described in his New Principles of Gunnery, Prop. 8.



Let SB be the pendulum in a vertical position, O the center of oscillation, and B*b* the arc which B described by the force of the stroke: the chord B*b* of this arc was measured by means of a ribon, one end of which was fastened at B.

$SB = 71\frac{1}{8}$ inches, length of the pendulum.
$Bb = 17\frac{1}{4}$ inches, chord of the arc B*b*.
$SO = 62\frac{2}{3}$ inches, center of oscillation from the point of suspension.

The chord B*b* is a mean proportional between 2SB and the versed sine AB (Geom. art. 216. vol. 1), therefore $\frac{Bb^2}{2SB} = AB$; and the sectors SO*o*, SB*b* being similar, we have $SB : SO :: \frac{Bb^2}{2SB} : \frac{SO.Bb^2}{2SB^2} = aO = \frac{62\frac{2}{3}\times(17\frac{1}{4})^2}{2\times(71\frac{1}{8})^2}$ = 1.83038 *inches*, nearly, = *a*O the versed sine of the arc O*o* described by the center of oscillation.

Weight of the pendulum .... $56\frac{3}{16}$ *lb.* = *m*
Weight of the bullet .... $\frac{1}{12}$ *lb.* = *n*
Center of oscillation from the point of suspension... $62\frac{2}{3}$ *inches* = *b*
Center of gravity of the pendulum 52 *inches* from the same point,
Whence the distance of the center of gyration = $\sqrt{(52\times 62\frac{2}{3})}$ = *a*
Point of impact P from the axis of suspension .... 66 *inches* = *c*
16.13 *feet* = 193.56 *in*...... = *s*
1.83038 *in*... = *v*

Then, by substitution, $x = \left(1+\frac{a^2m}{c^2n}\right)\frac{2c}{b}\sqrt{sv} = \left(1+\frac{52\times 62\frac{1}{2}\times 56\frac{1}{4}}{66^2\times\frac{1}{12}}\right)$ $\times \frac{132}{66\frac{1}{2}} \times \sqrt{(193.36 \times 1.83038)} = 20108$ *inches* or 1676 *feet*, the velocity per second, with which the bullet moved when it struck the pendulum. Mr. Robins, by computing with the velocity of the pendulum at the point of impact, instead of the velocity at the center of oscillation, brings out 1641 *feet*: this mistake is noticed by Euler in his comment on Robins's Gunnery.

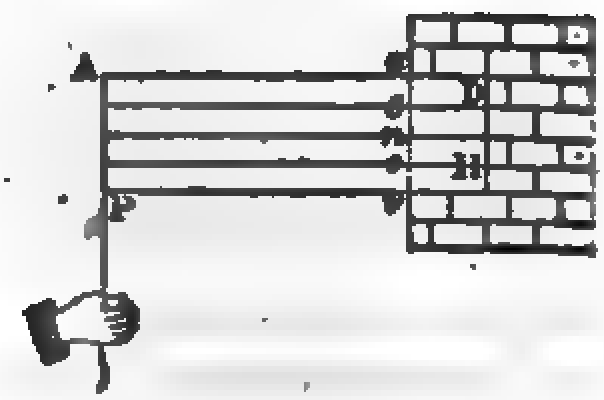
Corol. Since in the same pendulum, and with the same weight of ball, all the quantities in the expression for x are constant, except the versed sine v whose value depends on the length of the chord (Bb), therefore the velocity of the ball is directly as the chord of the arc described by the pendulum.

SCHOLIUM. In the foregoing conclusions it is supposed that the pendulum begins its motion at SB by the stroke of the ball with the same velocity as it acquires in falling back from the position Sb to the perpendicular SB by its own weight: this would be exactly the case did the bullet communicate all its motion to the pendulum at the moment of impact. The ball however, continues to act during the time it is penetrating the wood; and since the motion of the pendulum is circular, and the bullet endeavours to proceed nearly in a right line, its action on the pendulum must produce a shock, or stress on the axis: now both these circumstances may affect the velocity deduced from the rule. A small variation will also result from the augmented weight of the pendulum by the ball, but this is too minute to be of consequence.

OF THE STRENGTH AND STRESS OF TIMBER.

401. THE *lateral strength of squared Timber, is proportional to its breadth drawn into the square of the depth.*

Let PADH represent a vertical section of a beam of timber AH, the end DH being fixed in a wall; and conceive this section to be composed of innumerable parallel fibres a, c, n, o, &c.



Now a force P acting perpendicularly at the end AP sufficient to break the beam at ao will bend it downwards, and the uppermost fibre a will be first broken; this done, a less force will bend all the remaining fibres, but the fibre c is the next that will break; and then a less force would break the following fibre n; and so on: consequently the forces just sufficient to break the fibres in succession will diminish as their number or the depth of the beam is diminished; that is, the strength of the section is as the number of fibres lying upon one another.

Therefore taking DH as the first or greatest force, and calling a fibre f, the successive forces will be represented by the infinite arithmetical series DH, $DH-f$, $DH-2f$, $DH-3f$, &c. . . . . to $DH-DH$ or o: but the number of terms is DH, and therefore the sum of the series will be $(DH+o)\frac{1}{2}DH$ or $\frac{1}{2}DH^2$. Hence if B be the breadth or number of perpendicular sections in the beam, its strength or the force necessary to break it, will be as $\frac{1}{2}DH^2.B$, or as $DH^2.B$, because the wholes are proportional to their halves.

Corol. 1. Hence a rectangular beam is stronger with the broadest side vertical than when that side is horizontal, in the

proportion of the depth to the breadth. For let D the broadest side be the depth, and B the breadth; then the strength is as D^2B when D is vertical, and as B^2D when it is horizontal; but D and B have the same ratio as D^2B and B^2D.

For example, suppose the depth DH = 4 inches, and the breadth = 1; and that it can just support a weight at P = 600*lb*, then $4^2 \times 1 : 600lb. :: 1^2 \times 4 : 150lb.$ the weight it would bear were DH placed horizontal.

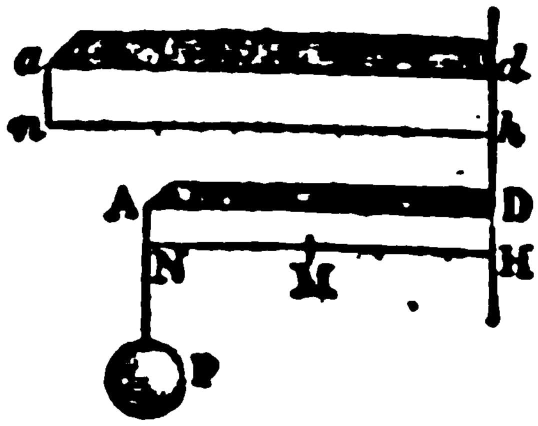
Corol. 2. And the strength of beams of the same depth are as their breadths. For let B and b denote the breadths, and D the common depth, then D^2B and D^2b will represent the strengths, which expressions are as B and b.

Corol. 3. Hence also, the lateral strengths of beams whose sections are similar, will be as the cubes of their breadths or depths: thus if D^2B, and d^2b denote the strengths, then the sides being proportional, we have $D : B :: d : \frac{Bd}{D} = b$, which substituted for b, and d^2b becomes $\frac{Bd^3}{D}$; now D^2B and $\frac{Bd^3}{D}$ multiplied by D, give D^3B and Bd^3, which are as D^3 and d^3.

Corol. 4. Also, since D^2B and d^2b are the areas of the sections multiplied by the depths; therefore the strengths of beams having similar sections, will be as their areas multiplied by the depths: Or as the cubes of the depths, when those depths are homologous.

Thus, if a cylinder AH whose diameter DH is 4 inches, can just sustain a force at P = 800*l*, then a cylinder of the same material, 1 inch in diameter, and of the same length, will bear only 12½*lb*. for $4^3 : 1^3 :: 800 : 12\frac{1}{2}$.

402. *If the beam* AH *of a given length, and depth, when fixed horizontally at the end* DH, *can just support a given weight P at the other end; to find the dimensions of a similar beam (ah) of the same material, that will break by its own weight, or only just sustain itself.*



Suppose the beam to be of uniform thickness, and let $W=$ its weight. Then we may consider the end DH as the fulcrum of a lever NH void of gravity, supporting a weight at N equal to P, and another weight $= W$ at M the middle of NH directly under the center of gravity of the beam. And by the nature of the lever, the effort of the weight W to bend or break the lever at DH, will be as $\text{MH}.W$, or $\frac{1}{2}\text{NH}.W$, and that of the weight P as $\text{NH}.P$, therefore $\frac{1}{2}\text{NH}.W+\text{NH}.P$, or $(\frac{1}{2}W+P)\,\text{NH}$ is the whole effort, or stress on the fulcrum DH.

But since the beams are similar, their weights will be as the cubes of the lengths, or depths; hence $\text{DH}^3 : dh^3 :: W : \frac{dh^3.W}{\text{DH}^3} =$ the weight of the beam ah; and $\text{DH} : \text{NH} :: dh : \frac{\text{NH}.dh}{\text{DH}}$ its length; therefore $\frac{dh^3.W}{2\text{DH}^3} \times \frac{\text{NH}.dh}{\text{DH}}$, or $\frac{dh^4.W.\text{NH}}{2\text{DH}^4}$, is the stress of the beam ah on the end or fulcrum $dh : dh$.

Now the strengths of the beams must be as the stresses, but the strengths are as DH^3 and dh^3; therefore $\text{DH}^3 : dh^3 :: (\frac{1}{2}W+P)\text{NH} : \frac{dh^4.W.\text{NH}}{2\text{DH}^4}$, whence $W : W+2P : \text{DH} : dh$ the depth of the beam; and $\text{DH} : \text{NH} :: dh : nh$ its length.

Let the ends of AH be squares, the side $\text{DH}=1$ *inch*, length $\text{NH} = 1$ *foot*, its weight $= .3lb$, and the weight $P = 100lb$.

then $W : W+2P :: \text{DH} : dh$,

or $.3 : .3+200 :: \frac{1}{12} : 55.14$ *feet*, nearly, $= dh$; and $55.14 \times 12 = 661.67$ *feet*, nearly, the length nh.

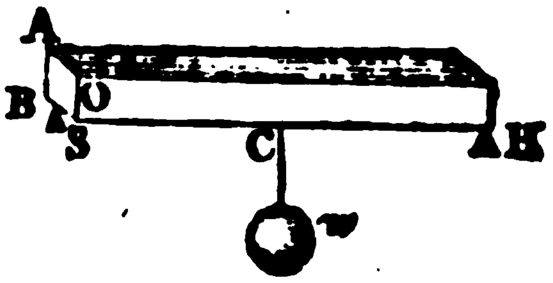
Corol. Hence, if it be required to find the length of a spar having the same depth and breadth as AH, that would break by its own weight, let $l =$ the length in feet, then $.3l =$ the weight in lb. and $\frac{1}{2}l \times .3l$ is the effort of its own weight; therefore $\frac{1}{2}l \times .3l = (\frac{1}{2}W+P)\,\text{NH}$ the effort of AH (together with the weight P) on the end or fulcrum DH,

That is, $.15l^2 = (\frac{1}{2}\times.3+100)\times 1 = 100.15$, and $l = \sqrt{\frac{100.15}{.15}} = 25.8$ &c. *feet*, the length required.

SCHOLIUM. From these computations it appears that in the construction of works, &c. it is possible to take a beam of such dimensions that the stress by its own weight may exceed its strength. Machines may also be made too large to be useful, for the less are stronger in proportion to their bulk than the greater when the dimensions of both are similar. Thus we find that small animals are stronger and more active in proportion to their weight or size than large ones.

403. *It is found by experiment that a spar of oak* (AH) *an inch square, and* 1 *foot in length* (SH), *when supported horizontally at the ends, will bear about* 670lb. (w) *suspended at the middle* (C) *before it breaks; hence it is required to find what weight a piece of the same oak will bear, which is* 10 *feet long,* ½ *a foot deep, and* ⅓ *of a foot broad, the weight being also suspended at the middle?*

Let the depth OS $= d$, breadth OA $= b$, length SH $= l$.
depth ½ foot $= D$, breadth ⅓ foot $= B$, length 10 feet $= L$, required weight $= W$.



Then d^2b will denote the lateral strength of the spar AH, and D^2B that of the other piece. And since each prop sustains a weight $= \frac{1}{2}w$ acting at C, if C were the fulcrum, and a force $= \frac{1}{2}w$ acted vertically at each end A and H, their efforts to break the spar would be the same as that of the weight w when the spar is supported on the props; but (by prop. of the lever) the effort of $\frac{1}{2}w$ is $\frac{1}{2}w \times \frac{1}{2}l$, and $\frac{1}{2}W \times \frac{1}{2}L$ that of $\frac{1}{2}W$; and since the two pieces just support the weights, their strengths must be as the efforts or the greatest forces they resist,

that is, $d^2b : D^2B :: \frac{1}{2}w \times \frac{1}{2}l : \frac{1}{2}W \times \frac{1}{2}L$,

whence $d^2bWL = D^2Bwl$, and $W = \dfrac{D^2Bwl}{d^2bL} = \dfrac{\frac{1}{4} \times \frac{1}{3} \times 670 \times 1}{\frac{1}{144} \times \frac{1}{12} \times 10} = 9648$lb the answer.

*403. *If the spar* AH (preceding art.) *break with* 660lb, *suspended at* C; *then what will be the length of another piece of the same wood,* ½ *a foot square, that will support* 17620lb, *at its middle?*

Here $d = \frac{1}{12}$, $b = \frac{1}{12}$, $l = 1$, $w = 660$, $D =$
from the equation $d^2bWL = D^2Bwl$, we get

$$L = \frac{D^2Bwl}{d^2bW} = \frac{\frac{1}{2} \times 660 \times 1}{\frac{1}{12^3} \times 17820} = 8\ \textit{feet}, \text{ the l}$$

404. ***A deal spar 1 foot long, and*** ***supported horizontally at the ends, wi*** ***pended at the middle; then what wei*** ***deep, and 10 inches wide, sustain in*** ***rest on two props 9 feet asunder.***

In this example $d = \frac{1}{12}$, $b = \frac{1}{12}$, $l = 1$, $w =$
which substituted in the equation $W = \frac{D^2Bw}{d^2bL}$
quired weight. And the same equation or the
the strengths of prisms, or bars of metal one

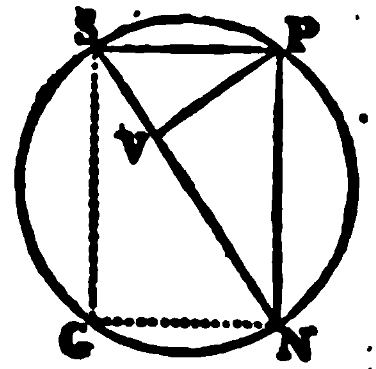
405. ***If the beam* HD *in a position*** **HO *be loaded with a weight* P *at the*** ***perpendicular to* HO; *then the str*** **AH × *P*.**

For suppose the horizontal line HB be connected with HD at H, then B may be considered as a bended lever w the force P acts perpendicular to the h son BO, and therefore (359, corol. AH × P is the effort of P to turn the l about the fulcrum H.

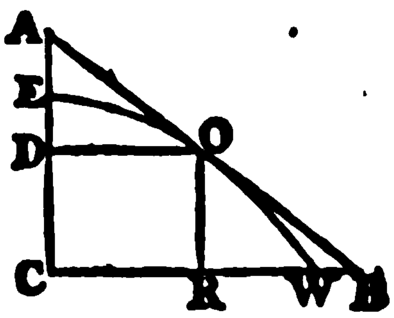
Corol. Hence if HO were another ends H and O, and of the same materi the two beams would require just the sa C and A to break them.

When the beams are large, it may b allowance by including their weight in t

406. *To cut the strongest scantling timber* SPNG *from a cylindric one.*



Let the circle whose diameter is SN be a section of the cylinder. Put $d = SN$, and $y = SP$ the breadth of the beam; then $d^2 - y^2 = PN^2$ the square of the depth: and since the strength is as the breadth into the square of the depth, it will be denoted by $(d^2 - y^2)y$ or $d^2y - y^3$, which is to be the greatest possible, or a *maximum*.



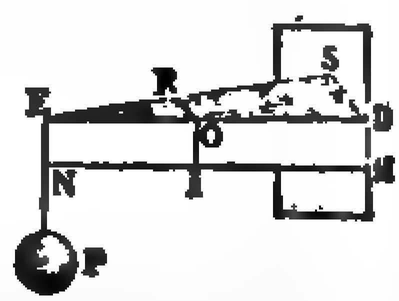
Suppose ED an absciss, DO the corresponding ordinate of the parabola EOW, OA a tangent at O, OB = OA, and CB parallel to DO. Then because AB is bisected in O, the rectangle DR is a maximum or the greatest that can be inscribed in the parabola, or in the triangle CAB, (243, corol. 2); and since ED = EA (297, corol. 2), and DC = DA, therefore ED = $\frac{1}{3}$EC.

If the absciss $ED = x$, ordinate $DO = y$, $p =$ the parameter, and $a = EC$; then $a - x = DC$, and $(a-x)y =$ the rectangle DR; but $px = y^2$ (by prop. of the parabola) and $x = \frac{y^2}{p}$ which substituted for x, and $(a-x)y$ will be $\left(a - \frac{y^2}{p}\right)y$: let the parameter $p = 1$, then $\left(a - \frac{y^2}{p}\right)y$ becomes $ay - y^3 =$ the rectangle DR; but when $p = 1$, then $x = y^2$, therefore $y^2 = \frac{1}{3}a$, that is, when $ay - y^3$ is a maximum, $y^2 = \frac{1}{3}a$, and therefore the expression $d^2y - y^3$ is a maximum when $y^2 = \frac{1}{3}d^2$, or $SP^2 = \frac{1}{3}SN^2$, and consequently $PN^2 = \frac{2}{3}SN^2$; that is, SP and PN are in the same ratio as the side of a square and its diagonal. Hence this construction; Make $SV = \frac{1}{3}SN$, and erect the perpendicular VP; then PS is the breadth of the rectangular end. For $VP^2 = SV \times VN = \frac{1}{3}SN \times \frac{2}{3}SN = \frac{2}{9}SN^2$, and $PS^2 = SV^2 + VP^2 = \frac{1}{9}SN^2 + \frac{2}{9}SN^2 = \frac{1}{3}SN^2$.

Corol. Because $SV.VN = VP^2$, therefore $SV.VN + VN^2 = PN^2$. Also by sim. triangles, SN : SP :: SP : SV, whence

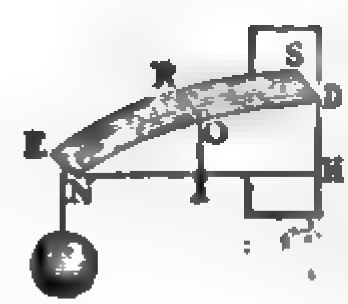
$\surd(SN.SV) = SP$; and the maximum $SP.PN^2 = \ldots\ldots\ldots$ $\surd(SN.SV)\ (SV.VN+VN^2) = \surd(SN.SV.VN^2)\ (SV+VN)$; but $SV+VN$ is given, therefore $\surd(SN.SV.VN^2)$, or its square $SN.SV.VN^2$ is also a maximum; consequently (because SN is given) $SV.VN^2$ is a maximum: hence when a given line (SN) is divided so, that the solid under one part and the square of the other, is the greatest possible, the least part (SV) will be half the other (VN).

407. *Suppose a weight P is supported at one end of a beam* EH *of a given depth* EN *or* DH, *the other end* SDH *being fixed in a wall; to find the figure of the beam when its strength at any vertical section* (IOR) *is equal to the stress at that section: Or that it shall be as liable to break at any one place as at another.*



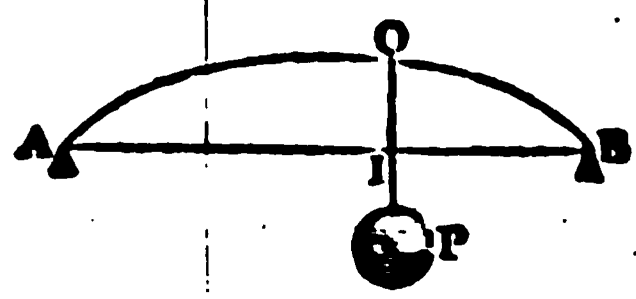
Let the section IOR be parallel to the end SDH; then (401, corol. 2) the strengths at IOR and HDS, are as OR and DS; and the stress at IOR and HDS, as $EO \times P$ and $ED \times P$, or as EO and ED; but the strength is supposed to be equal to the stress; therefore OR and DS have the same ratio as EO and ED, and consequently ESD is a triangle; therefore EH is a prism; the upper side ESD (and also the lower) being parallel to the horizon.

408. *To determine the figure of the beam when the breadth* EN *or* SD *is given.*



If the sections IOR, SDH are parallel, as in the last article, the strengths or stresses at these sections will be as $IO^2 \times OR$ and $HD^2 \times DS$, or as IO^2 and HD^2, because $OR = DS$; but the stresses are as NI and NH, that is, the squares of IO and HD are as NI and NH, and therefore NOD is the curve of a common parabola whose axis is NH, and vertex N.

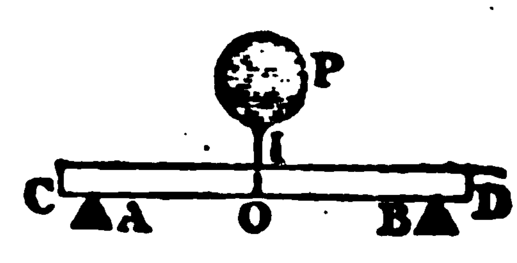
409. *Let* AOBI *be a vertical section or one side of a beam of timber of a given breadth, supported horizontally on the props at* A *and* B; *to find the figure of the beam so that its strength shall be as the stress when a given weight* (P) *is suspended any where between the ends* A, B.



Since the weight P is supported by both props, we have (384) $AB : P :: IB : \frac{IB.P}{AB}$ = that part sustained by A, and $\frac{IA.P}{AB}$ the other part sustained by B; and therefore $IA \times \frac{IB.P}{AB}$ is the stress at IO arising from the weight $\frac{IB.P}{AB}$, and $IB \times \frac{IA.P}{AB}$ the stress from the other weight, and the sum of both, or $IA \times \frac{IB.P}{AB} + IB \times \frac{IA.P}{AB} = IA \times IB \times \frac{2P}{AB}$ is the whole stress at IO; but $\frac{2P}{AB}$ is given, therefore the stress every where is as the rectangle $IA \times IB$.

Now the beam being of an equal breadth, its strength will be as the square of the depth, or as OI^2; but the strength must be as the stress, that is, OI^2 is as $IA \times IB$, or the rectangle of the two abscisses, is as the square of the corresponding ordinate. Therefore the section OIB is a semiellipse, (267).

410. *If the beam* CD *when resting loose on the props* A *and* B *near the ends, can just support a weight* P *at the middle, it will bear double that weight when the ends* C, D *are fixed down so as to be immoveable.*



When the ends are loose on the props, each bears half the weight, and the beam will break in the middle I where the stress is greatest. Hence, if the supports A and B were removed, and the beam rested on a prop at O, it follows that half the weight P suspended at each

of the points A and B would also break the beam at the middle O, that is, the pressure on the fulcrum O when the beam broke, would be equal to the weight P. Now if the ends C and D are fastened down, and the weight P doubled, each of the props A and B will support half the weight, or a weight equal to P, therefore double the weight P would just break the beam at A and B when the ends are fixed.

Or thus. When the ends are fixed, we may conceive forces acting at C and D sufficient to produce an equilibrium on the fulcra A and B, so that $CA \times \textit{force at } C = AO \times \frac{1}{2}P$, and $DB \times \textit{force at } D = BO \times \frac{1}{2}P$, and hence it appears than an additional weight equal to P will be necessary to produce a like effect.

Remark. Pieces of wood of equal dimensions, cut from the same beam or plank, are found to differ in strength, and therefore computations from theory seldom agree with experiment.

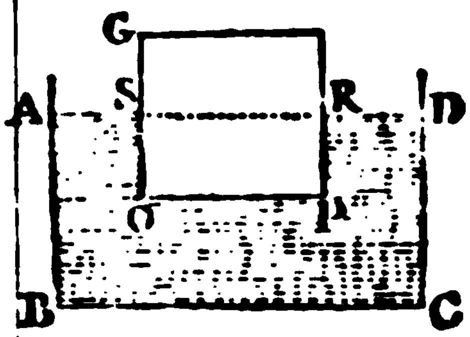
By experiments made on deal spars an inch square, resting loose on two props 1 foot asunder, it was found that they bore from 460 to 690*lb*. at the middle before they broke. Oak spars from 660 to 970*lb*. And cast iron bars from 730 to 990*lb*. the bars being an inch square, and the supports 3 feet apart.

HYDROSTATICS.

411. IF *a body of uniform density of the same specific gravity as water, or whose weight is equal to the weight of the same bulk of water, be immersed in that fluid, it will rest in any position.*

For when the specific gravities of the body and fluid are equal, the latter is pressed by the former just as much as it is by the like bulk of water when it fills the space occupied by the body. The body therefore in any position can have no more tendency to rise or sink than the water itself.

412. *A body heavier than the fluid will sink to the bottom. But if it be lighter it will float with only a part immersed.*

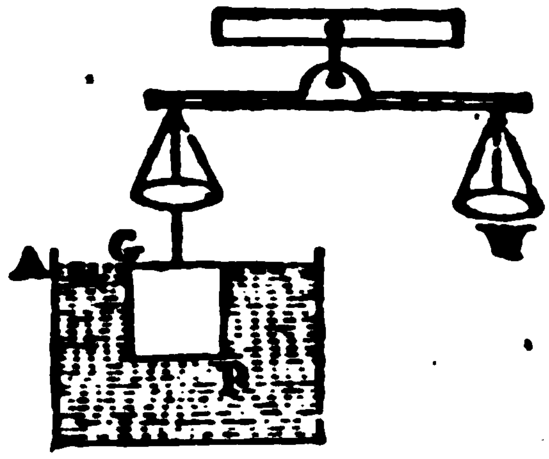


Let AD be the surface of the water in the vessel AC. Then if the body GP be heavier than the fluid, its pressure on the water underneath is greater than that of an equal bulk of water, or the resistance in the fluid is less than the force by which the body endeavours to descend; and since the parts of fluids are easily moved among themselves, the body will sink by its superior gravity.

But when the specific gravity of the body is less than that of the water, the body can only sink till the weight of the fluid displaced is equal to the weight of the body, or till the pressure and resistance are equal. Thus if GP be a cube of wood weighing 500 *ounces* avoirdupois, and the side OP or OG a foot, it will sink just 6 inches in spring water, which weighs 1000 *ounces* per cubic foot. Hence it appears that fluids press upwards as well as downwards, for the action of the water in a vertical direction under the body GP is equal to the compressing force or pressure of the body downwards.

If the weight of the cube were 1000 *ounces*, it would sink 'till its upper surface became level with the surface of the water, and the water would exert a force against the bottom OP equal to the weight of the cube. The pressure of fluids vertically is therefore the same as in the direction of gravity at the same depth; hence a body lighter than the fluid, when immersed, will rise till it floats at the surface, for the force of the fluid against the body underneath is greater than its weight together with that of the column of water directly over it.

413. *Bodies suspended in a fluid lose as much weight as the weight of the fluid displaced.*



Thus the cube GP whose side is a foot, and its weight = 1000 *ounces*, when put in water, loses all its weight, that is, it will require no force to keep it suspended in the fluid, because it is just as heavy as the same bulk of water. But if the cube were of brick (the specific gravity being double that of water) its weight would be 2000 *ounces*, and a counterpoise = 1000 *ounces* in the scale W would be necessary to keep it from sinking, because in that case the water sustains half the weight of the cube. A body of lead of the same dimensions is 11325 *ounces*, therefore its weight would be 11325 — 1000 or 10325 *ounces* when weighed in water. But a cubic foot of cork which weighs only 240 *ounces* would require a weight on the upper side, or a force equivalent to 1000 — 240 or 760 *ounces* to sink it even with the surface of the water.

Corol. Hence to determine the specific gravity of a body heavier than water; first let it be weighed in air, after the usual method, then weigh it in water (which may be done by suspending it from the balance or scales), and the difference of the results will be the weight of water equal in bulk to the body; and that difference, and the weight of the body in air, will be the specific gravities of water and the body: for the specific gravities of bodies are denoted by their weights when the magnitudes are equal.

Thus suppose a mass of lead to weigh $226\frac{1}{4}$ and $206\frac{1}{4}$ *ounces*, respectively, in air, and in water, then $226\frac{1}{4}-206\frac{1}{4}=20$ *ounces* is the weight of a volume of water equal in magnitude to the lead; hence the specific gravity of water to that of lead is as 20 to $226\frac{1}{2}$, or as 1000 to 11325.

Hence, if A = the absolute weight of a body,
a = its weight in water,
s = its specific gravity;
w = the specific gravity of water.

Then $A-a$ is the weight of a volume of water equal in magnitude to the body:

And $A-a : A :: w : \dfrac{Aw}{A-a} = s$, the specific gravity of the body A.

The value of w, which denotes the specific gravity of water, is arbitrary. In some tables it is supposed to be 1. But it is much more convenient to make it $= 1000$, because a cubic foot of pure water weighs 1000 *ounces* avoirdupois.

414. *To find the specific gravity of a body (B) lighter than water.*

Let a heavy body whose weight is A be attached to it, so that both will sink together;

and put $B =$ the weight of the body B,
$r =$ its specific gravity,
$c =$ the weight of the compound in water,
$a =$ the weight of A in water,
$w =$ the specific gravity of water:

Then, proceeding as in the last corollary, we have

$c-a$, the weight of the lighter body B in water, which, in this case, is negative because a is greater than c.

$B-(c-a)$ or $B+a-c$, the weight of a volume of water equal in magnitude to the body B:

And $B+a-c : B :: w : \dfrac{Bw}{B+a-c} = r$, the specific gravity of B.

Example. Suppose 755 ounces of lead is attached to a block of deal weighing 275 ounces, and that the weight of the whole together in water is $463\frac{1}{3}$ ounces. What is the specific gravity of the deal?

$11325 : 10325 :: 755 : 688\frac{1}{3}$ ounces, the weight of the lead in water:

$B = 275 \qquad a = 688\frac{1}{3}$

$c = 463\frac{1}{3} \qquad w = 1000.$

And $\dfrac{Bw}{B+a-c} = \dfrac{275 \times 1000}{275 \times 688\frac{1}{3} - 463\frac{1}{3}} = 550$, the specific gravity required.

This result is ounces: and it is equal in bulk to 1000 ounces of water with which it is compared. The wood therefore weighs 550 ounces per cubic foot.

415. *The weights of bodies are proportional to the products of their masses and specific gravities.*

Thus if a mass of lead be l cubic inches; then since a cubic foot or 1728 cubic inches weigh 11325 ounces,

we have, $1728 : 11325 :: l : \frac{11325l}{1728}$, weight of the mass.

Or if $d =$ the cubic inches in a piece of deal,

then $1728 : 550$ *ounces* $:: d : \frac{550d}{1728}$ its weight in ounces:

But 11325 and 550 denote the specific gravities of lead and deal: and $11325l$ and $550d$ are in the same ratio as $\frac{11325l}{1728}$, and $\frac{550d}{1728}$.

416. *To determine the quantities in a mass compounded of two ingredients when its weight, and the specific gravities are given.*

Let A and B denote the magnitudes of the two ingredients,
a and b their specific gravities, respectively;
C the weight of the compound, or the weight of $A+B$,
c its specific gravity:

Then by the last article, aA, bB, and $cA+cB$, will denote, or be proportional to the weights of A, B, and $A+B$, respectively:

$$\text{consequently } aA+bB=cA+cB$$
$$\text{or } aA-cA=cB-bB,$$

whence $A : B :: c-b : a-c$, that is, the magnitudes A and B are as $c-b$ and $a-c$:

And therefore the weights of A and B will be as $a(c-b)$ and $b(a-c)$, or as $ac-ab$ and $ab-ac$:

Consequently we have to divide the weight C into two parts having the proportion of $ac-ab$ to $ab-ac$:

Therefore

$ac-ab+ab-bc : C :: ac-ab$:

$$\text{or}\quad ac-bc : C :: ac-ab : \frac{(c-b)\,aC}{(a-b)\,c},\ \text{the weight of } A.$$

$$ac-bc : C :: ab-bc : \frac{(a-c)\,bC}{(a-b)\,c},\ \text{the weight of } B.$$

Example. Suppose a composition of Copper and Tin to be 42*lb.* and its specific gravity 8000; what is the quantity of each metal, the specific gravity of Copper being 9000, and that of Tin 7320?

$a = 9000$ $\qquad\qquad c = 8000$

$b = 7320$ $\qquad\qquad C = 42.$

And $\frac{(c-b)\,aC}{(a-b)\,c} = 19\frac{1}{8}$*lb.* the weight of Copper. Consequently $42 - 19\frac{1}{8} = 22\frac{7}{8}$*lb.* the weight of Tin.

Corol. Hence we can find the specific gravity of a mass compounded of two ingredients when their weights and specific gravities are given. For suppose A and B to denote the weights of A and B, respectively: then $\frac{(c-b)\,aC}{(a-b)\,c} = A$, whence $\frac{(A+B)\,ab}{bA+aB} = c$, the specific gravity required.

And in the same manner a, or b, may be found when the values of the other letters are given.

417. *To find the specific gravity of a fluid.* Let a body whose specific gravity is known, be weighed both in, and out of the fluid, then the specific gravity may be found from the expression $\frac{Aa}{A-a} = s$ (Art. 413, corol.) or $w = \frac{(A-a)\,s}{A}$, where w denotes the specific gravity of the fluid:

Thus suppose 28*lb.* 5 ounces of lead when weighed in water, is 25*lb.* 13 ounces; then $A = 28\frac{5}{16}$, $a = 25\frac{13}{16}$, and $s = 11325$ the specific gravity of lead;

and $w = \frac{(A-a)\,s}{A} = 1000$, the specific gravity of water.

Corol. Hence to compare the specific gravities of two fluids, weigh the same body in both, and we shall have,

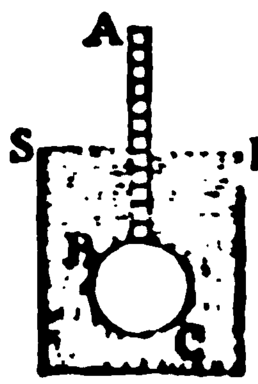
As the loss of weight in one fluid, is to the loss in the other,

So is the specific gravity of the former fluid, to that of the other:

Thus if $31\frac{1}{4}$ *ounces* be the loss in water, and $27\frac{1}{8}$ *ounces* the loss in rectified spirit of wine, the specific gravities of the fluids are as $31\frac{1}{4}$ and $27\frac{1}{8}$, or as 1000 to 868 which therefore is the specific gravity of the spirit, that of water being 1000.

Platina and Gold are the only known substances that will sink in Quicksilver or Mercury: but its specific gravity may be determined by putting it in a small open glass vessel suspended from the scale, and weighing it in water: the vessel however, must be first balanced in water.

418. The instrument used in these experiments for weighing, is an *Hydrostatical Balance*, which however, differs but little in the construction from a pair of scales. But for comparing the specific gravities of fluids that are nearly of the same density, another instrument has been contrived, called the *Hydrometer*. The common sort consist of a graduated cylindric stem AB fixed to a hollow globe of copper BC; its weight being adjusted so, that it may swim in the fluid with part of the stem above the surface SR. When this is put in proof spirits (for example) it will sink to a particular division on the stem; but if the spirit be under proof or weaker than proof spirit, (as common brandies, &c.) it will not sink to that division; on the contrary, should the fluid be rectified spirit of wine, the hydrometer will descend to a division on the stem much above the proof point.



SCHOLIUM. The specific gravity of bodies however, must necessarily vary with their temperature, for most bodies expand by heat, but are contracted again in cooling. And in computing the specific gravity of a compound, an error may arise in

consequence of some uncertainty in its bulk. Thus a pint of spirits of wine and another of water when mixed together will be less than a quart. To account for this diminution, or *penetration of dimensions*, it has been supposed that the constituent particles of the fluids are globular, but of a size much less in one fluid than in the other, because in that case, a considerable number of the less particles might fall into the spaces between the greater particles in the operation of mixing. Mr. Ramsden, in 1792, published an account of a new instrument called the *Balance Hydrometer*, by which he could determine the exact quantity of spirit and of water in any compound of the two, its specific gravity, the diminution in the volume, &c. &c. at any temperature.

419. *A Table of Specific Gravities.*

Platina	23000	Nitre	1900
Fine Gold	19400	Ivory	1825
Standard Gold	17724	Sulphur	1810
Pure Mercury	14000	Chalk	1790
Common Mercury	13600	Dry Lignum vitæ	1327
Lead	11325	Pit Coal	1250
Fine Silver	11091	Dry Mahogany	1063
Standard Silver	10535	Dry Boxwood	1030
Copper	9000	Sea Water	1030
Copper halfpence	8915	Spring Water	1003
Gun metal	8784	Distilled Water	993
Cast Brass	8000	Gunpowder close shaken	937
Steel	7850	Proof Spirits, in the temperature of 55°	927
Wrought Iron	7645	Dry Oak	925
Cast Iron	7425	Ice	900
Tin	7320	Rectified Spirits of Wine	866
Diamond	3517	Dry Ash	800
Marble	2700	Dry Beech	700
Common Green Glass	2600	Dry Elm	600
Flint	2570	Dry Fir or Deal	550
Common Stone	2520	Cork	240
Clay	2160	Air in a mean state	1$\frac{1}{5}$
Brick, from 2000 to	2400		or 1$\frac{1}{4}$
Common Earth	1984		

The numbers in the foregoing table denote the specific gravities of the several bodies, and also the weight of a cubic foot of each in *ounces* avoirdupoise. Thus a cubic foot of common water weighs 1000 *ounces* or $62\frac{1}{2}$*lb.* A cubic foot of cast iron 7425 *ounces*, &c. &c. Hence we have a ready method of finding the magnitude of a body from its weight, or the weight from its magnitude, as in the following examples.

1. Suppose an irregular piece of *Oak timber* to weigh $14\frac{1}{2}$ hundred weight; how many cubic feet does it contain?

$14\frac{1}{2} \times 112 = 1624$*lb.* $= 25984$ *ounces*, the weight, which divided by 925 (the ounces per cubic foot) gives $28\frac{84}{925}$ cubic feet, the *answer*.

2. What is the weight of 600 square feet of *sheet Lead* $\frac{1}{10}$ of an inch thick?

$\frac{1}{10} \times \frac{1}{12} = \frac{1}{120}$ of a foot, the thickness:
and $\frac{1}{120} \times 600 = 5$ cubic feet, the contents:

Then, as 1 *foot* : 11325 *ounces* :: 5 : 56625 *ounces* the weight, which divided by 16 gives $3539\frac{1}{16}$*lb.* the *answer*.

3. What is the diameter of a cast *Iron Shot* whose weight is 9*lb.*?

9*lb.* $\times$ 16 $=$ 144 *ounces*:

Then, 7425 : 1728 *in.* :: 144 : 33.5127 *inches* nearly, the cubic contents:

And .5236 being the contents of a sphere whose diameter is 1, we have .5236 : 1^3 :: 33.5127 : 64 *inches*, very nearly, the cube of the diameter; Therefore the diameter of a 9*lb.* iron shot may be taken at 4 inches without sensible error.

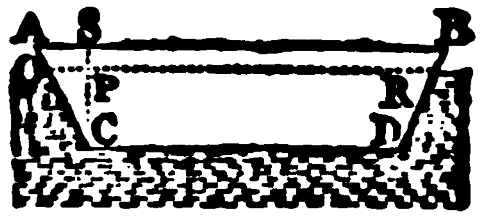
4. To find the weight of a *Lead Shot* of a given diameter.

Put $d =$ the diameter in inches; then $.5236d^3$ is the cubic contents in inches:

And 1728 *in.* : 11325 *ou.* :: $.5236d^3$: $\frac{11325 \times .5236d^3}{1728}$, or $\frac{5930d^3}{1728}$ nearly, is the weight in ounces, of the ball whose diameter is d; hence if $d = 12$ inches the weight will be 5930 *ounces*.

If the diameter $= 3.6$ or $3\frac{3}{5}$ *inches*, the weight is 10*lb.* or 160 *ounces*, nearly; for $12^3 : 5930 : (3\frac{3}{5})^3 : 160$; which proportion is more exact than that given in the Arith. *art.* 188, examp. 7.

Or the cube of the diameter in *inches*, multiplied by 3.432 gives the weight in *ounces*. And if the weight in *ounces* be multiplied by the decimal .2914, the cube root of the product will be the diameter in *inches*.



5. Let ABDC be the profil of a *Pontoon* floating in water, the water mark being OR. Put n = CD the external length at bottom, r = the difference of CD and the length at top AB, d = SC the depth, s = PC the depth of the part immersed, and b = the breadth.

By similar triangles, $d : \frac{1}{2}r$ (or AS) :: $s : \frac{sr}{2d}$ = OP, whence $n + \frac{sr}{d}$ = OR, and $\left(n + \frac{sr}{2d}\right)s$ = the area of the trapezoid OCDR, therefore $\left(n + \frac{sr}{2d}\right)sb$ is the cubic contents of the immersed part. Suppose the dimensions are in inches, and let $f = 1728$, $l = 62\frac{1}{2}$ *lb.* avoirdupoise the weight of 1728 cubic inches of water, and w = the weight (in pounds) of the water displaced, or the weight of the pontoon together with the weight it bears; then $\left(n + \frac{sr}{2d}\right)\frac{sbl}{f}$, or $\frac{nbl}{f}s + \frac{rbl}{2df}s^2 = w$.

If the weight w be given, and the depth PC required, the equation gives $s = \sqrt{\left(\frac{2dfw}{rbl} + \frac{d^2n^2}{r^2}\right)} - \frac{dn}{r}$ = PC.

Let the outward dimensions be

AB	= $21\frac{1}{2}$ feet,		
CD	= $17\frac{1}{6}$	....	= 206 *in.* = n,
SC	= $2\frac{1}{4}$	....	= 27 *in.* = d, depth,
r	= $4\frac{1}{3}$	....	= 52 *in.*
breadth	= $4\frac{3}{4}$	....	= 57 *in.* = b:

And suppose the water mark OR to be 9 inches from the top, or PC = 18 *in.* = s;

Then $\frac{nbl}{f}s + \frac{rbl}{2df}s^2 = 8288$*lb.* = w, the weight (including the pontoon) that sinks it 18 inches.

Again, if the weight, including the pontoon = 6000*lb.* = w; what is the depth in the water.

$$s = \sqrt{\left(\frac{2dfw}{rbl} + \frac{d^2n^2}{r^2}\right)} - \frac{dn}{r} = 13.3 \textit{ inches} = \text{PC, the depth required.}$$

Hence, for pontoons of the above dimensions, the expressions will become

$$424.7s + 1.985s^2 = w.$$

$$\text{And } \sqrt{(.50371w + 11441)} - 107 = s = \text{PC}.$$

That is, multiply the depth sunk (in *inches*) by 424.7, and its square by 1.985, and the sum of the products is the weight of the loaded pontoon in pounds.

And to find the depth when the weight (including the pontoon) is given, Multiply the weight in *lbs.* by the decimal .30371, and add the product to 11441, then 107 subtracted from the square root of the sum, gives the depth PC in *inches.*

A pontoon of the above dimensions weighs about 900*lb*.

420. *If a body is at rest when floating in a fluid, its center of gravity and the center of gravity of the fluid displaced, are in the same vertical line.*

Suppose the body to be a globe whose center is C, and let its density be unequal so that G (instead of the center C) is its center of gravity; then since the centre of gravity of the fluid displaced or segment SRB, is in the vertical diameter VB, the center of pressure of the fluid upwards is in that vertical line, the center C may therefore be considered as the point upon which the body is suspended, consequently it must move round that point till the diameter AD becomes vertical. (372, corol. 1 and 2). And the same method of reasoning will apply to bodies of any form.

Corol. Hence if a body be left to float in a fluid, it will turn by its own gravity till the heaviest side is downwards.

421. *Suppose* HBOPGS *to be a bended funnel or glass containing a fluid, and open at both ends, and* GB *a vertical section through the lowest point; then the two parts of the fluid,* BOPG *and* BHSG, *press equally against that section.*

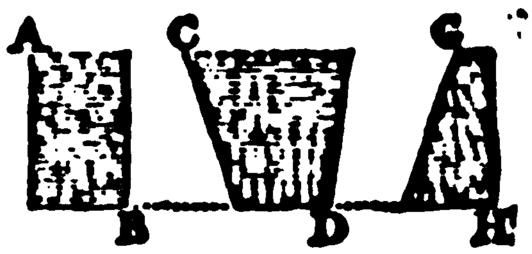
For if the pressure of BHSG the largest body of the fluid were greatest, the other must ascend in consequence of that superior force, but the two surfaces HS and PO will always be in the same horizontal line HO when the fluid is quiescent, whatever be the difference in the diameters HS and PO; that is, the sur-

faces are constantly at the same vertical height above the lowest part of the fluid (B).

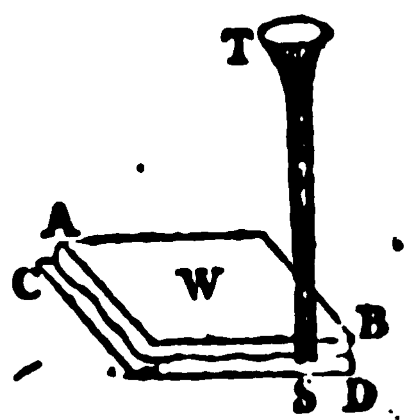
And the pressures are equal in opposite directions at any other section AC; for the pressure of the volume AOPC upwards at AC, is equal to that of AHSC downwards, otherwise the fluid would not rest with the two surfaces HS, PO in the same horizontal line.

Corol. 1. Hence water conveyed to a reservoir by means of a pipe, will rise to the same horizontal level as the spring or head from which it is supplied.

Corol. 2. The pressure of fluids therefore, on equal surfaces alike situated, is the same at equal perpendicular heights, whatever difference there may be in the quantities of the incumbent fluid. Thus suppose AB, CD, GH are three vessels of the same height filled with water, then if the bases are equal, each base will sustain a weight equal to that of the prism of water in the vessel AB whose sides are perpendicular.



Corol. 3. Hence a small quantity of fluid may be made to float or raise a great weight. And on this principle the *hydrostatical paradox* is constructed. Thus AB and CD are two boards connected at the edges by leather so that they can be moved to, and from one another like the top and bottom of a pair of bellows: TS is a small tube open at both ends, that communicates with the inside between the boards. Now when water is poured into the tube at the top T, the board or top AB will be forced upwards by the pressure of the fluid against it on the inside, if all the joinings are water-tight. For the force against the board AB is equal to the weight of a prism of water whose base is AB and height equal to the height of the tube TS, and consequently it would raise



a weight placed on the top at W equal
prism of water, Thus suppose AB and
side being 2 feet, and the length of the tu
= 24 cubic feet is the contents of the
= 1500*lb.* its weight. Hence if the in
tube be $\frac{1}{4}$ of an inch, about 5 pints of w
top T will be sufficient to raise AB the up
if it be loaded with 18 hundred weight.

422. *If* TBA *be a bent funnel, or
fluids, and* HSRO *the horizontal plane
contact, the pressure of the denser flui
equal to that of the other in* TS.

For if TS and AR were empty, the
HBO would remain quiescent, because t
tions (or surfaces) HS and RO are in th
horizontal plane; the fluids in TS and A
therefore press equally on those sections
faces when they are in equilibrio.

Corol. If the sections HS and RO a
dicular heights HT and OA will be reci
gravities of the fluids. For in that case th
HS and RO are pressed by equal weig
fluids whose heights are HT and OA;
HS and RO are equal, the specific gravi
be inversely as their heights.

423. *Let a cubical vessel* (PD) *whose
filled with water; then the pressure of th
side is equal to half its pressure on the b*

Suppose the square AD to be one
of the sides. Produce DB till BC
= BA; then the pressure on the
bottom, and against one side, will be
as the square AD to the triangle

ABC. For since fluids press equally in all directions, and that in proportion to their depths, an indefinitely small column AB will press at B in the horizontal direction BC with the same force as upon the base at that point B, therefore if the pressure of the column on the base be denoted by the depth AB, its equal pressure in the horizontal direction BC will also be represented by BC; in like manner, the horizontal pressure of the vertical column AR will be denoted by RS which is equal to AR, and so on: consequently all the BC, RS, &c. together, or the area of the triangle ABC will represent the whole horizontal pressure against the side at the line AB; and AB taken BD times (or the area of the square AD) is the pressure on the line BD; but the pressure on the whole base is the weight, or content of the contained fluid, or AD × side of the base (= AD × DQ); and the pressure on either side = triangle ABC × side of the base (= ABC × AP); that is, one pressure is double the other, because the square AD = twice the triangle ABC.

Corol. 1. But if BG (instead of BD) be any other breadth of the vessel, the pressure on the side BP will remain the same as before; the pressure of a fluid therefore, against any upright surface, is equal to half the weight of a prism of the fluid whose base is the surface pressed, and height equal to the perpendicular height of the fluid.

Thus, let the depth of the cubical vessel be 3 *feet*; then $\frac{9 \times 3}{2} \times 62\frac{1}{2} = 843\frac{3}{4}$*lb.* the pressure against one side.

Or suppose the gate or lock supporting water in a canal to be 12 *feet* broad and 10 *feet* deep, and we have $\frac{10 \times 12 \times 10}{2} \times 62\frac{1}{2} = 37500$*lb.* the pressure it sustains.

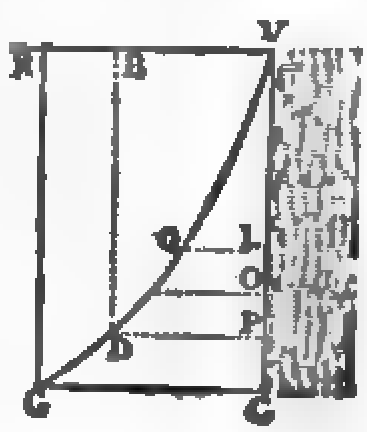
Corol. 2. Let O be the center of gravity of the triangle ABC; then since the pressures at B, R, &c. in an horizontal direction are denoted by BC, RS, &c. and the whole pressure against AB by the triangle ABC, the center of pressure of the fluid against AB, and center of gravity of the triangle are in the same horizontal line OI, because O is the center of pressure of

the triangle; and since $AI = \frac{1}{3}AB$, the centers of pressure and [illegible]sion of the upright surface of the fluid BPA coincide, the percu[illegible] axis of motion, [illegible] being supposed at the surface AP. Thus, if $BG = GD$, and $GT =$ [illegible] 1 foot or $\frac{1}{3}$ of the height of the vessel, then T is the center of pressur[illegible] of the fluid against the side AD, and a force $= 843\frac{3}{4}$ *lb.* acting perpe[illegible]dicularly against that side at the point T, would just sustain it in e[illegible]uilibrio with the fluid if the side were a loose board.

Scholium. If ABC be the vertical section of a bank or wall supporting a body of water, the thickness at bottom must exceed the height when its specific gravity and that of the fluid are the same. If it be constructed of brick, which is about twice as heavy as water, the breadth at bottom should therefore be rather more than half the height, supposing it to resist by its own weight only; for if the foundation be sunk below the water, the breadth may be less, but in that case, the thickness must depend on its strength.

Hence in computing the thickness of a wall (Art. 386, &c.) regard should be had to the consistence of the body it is to support, because the center of pressure will vary with its tenacity. Thus the center of pressure of loose earth in an horizontal direction, is usually taken at one third of its depth, but in a fluid it is at two thirds.

424. *Let* CVQG *be the perpendicular section of a wall supporting a fluid; to find the nature of the curve* VQG *when the strength of the wall is every where as the force it sustains, or that it shall be equally liable to break at all depths.*



Conceive any depth VP to be divided into innumerable equal parts PO, OL, &c. Then since the pressure of the fluid against any point L is as the depth VL, the number of particles acting by their weight against the wall at that point, will be denoted by VL, and their force at L on the lever PL (considering P as the

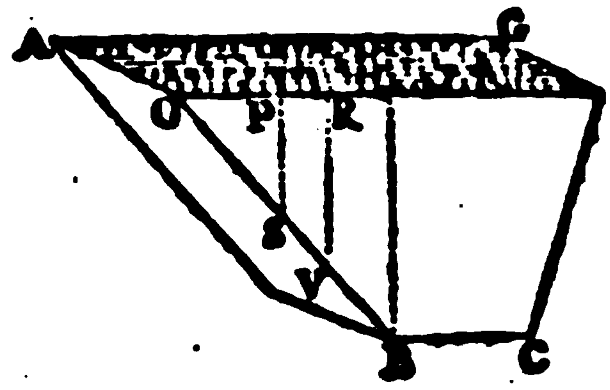
fulcrum) is PL.VL. In like manner PO.VO will denote the force at O of the particles represented by VO, and so on. Therefore all the PL.VL+PO.VO+ &c. will be the whole force tending to break the wall at P. Hence, if VP $= d$, and any other depth VL $= y$, then all the PL.VL+PO.VO+ &c. will be denoted by all the $(d-y)y$ or $dy-y^2$; now (179) all the $dy-y^2$ is $= \frac{dy^2}{2} - \frac{y^3}{3}$, which (when $y = d$ or VP) becomes $\frac{y^3}{2} - \frac{y^3}{3}$ or $\frac{1}{6}y^3$; therefore the force, or tendency of the fluid to break the wall at any depth, is as $\frac{1}{6}y^3$, or as y^3 the cube of the depth.

And if $b =$ the breadth of the wall, and $x =$ LQ its thickness at the depth y, it follows from Art. 401, &c. that the strength will be as bx^2 or as x^2 (the breadth b being given), that is, the square of the wall's thickness is always as the cube of the corresponding depth of the fluid; or the cubes of the ordinates BD, RG, &c. are as the squares of the abscissas VB, VR, &c. This curve is called a *semicubical parabola*: V is the vertex, and VR the axis.

The thickness of the wall must evidently depend on its strength or tenacity. Let the depth VC $= 10$ *feet*, thickness at bottom $x = 3$ *feet* $=$ CG,

Then $10^3 : 3^2 :: 6^3 : 3.4$, and the square root of 3.4 is 2.324 *feet*, nearly, the thickness at the depth of 6 *feet*, &c.

425. *If the side* AB *of a vessel containing a fluid, be oblique to the horizon, the pressure it sustains is equal to the weight of a body of the fluid whose magnitude is the surface* AB *multiplied by the perpendicular distance of its center of gravity below the surface of the fluid.*



Suppose the surface AB to consist of an indefinite number of minute surfaces represented by S, V, B, &c. that support the vertical columns PS, RV, &c. then the force or pressure on S, (or content of the column PS) will be denoted by PS $\times$ S, and that on V by RV $\times$ V, and so on; but all these forces or columns together make up the content of the incumbent fluid.

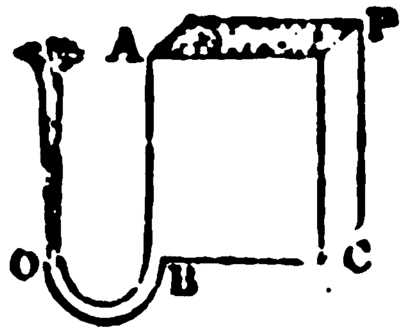
Now if we consider S, V, &c. as a number or system of bodies, it follows from Art. 373, corol. 2, that the aggregate S + V + &c. multiplied by (d) the distance of their common center of gravity from the plane OG, is equal to the sum of the products of those bodies drawn into their respective distances from the same plane; that is, AB $\times d =$ PS $\times$ S + RV $\times$ V + &c. = the pressure on AB, because S + V + &c. = the surface AB. And the same method of reasoning is applicable to any curve surface whatever that supports a fluid.

Let AB be a square whose side is 4 *feet*, and the depth of the vessel = 3 *feet*, then the center of gravity of the surface AB is $1\frac{1}{2}$ *feet* below the surface OG; and if the fluid be water, $16 \times 1\frac{1}{2} \times 62\frac{1}{2} = 1500$*lb*. the pressure on the side AB.

And if a hollow sphere be filled with a fluid; then as the center of the sphere is also the center of gravity of the surface, its distance from the highest part of the fluid is equal to the radius, therefore the internal surface multiplied by the radius of the sphere = the pressure; that is, the pressure on the internal surface = 3 times the weight of the contained fluid; as in the cubical vessel, Art. 423.

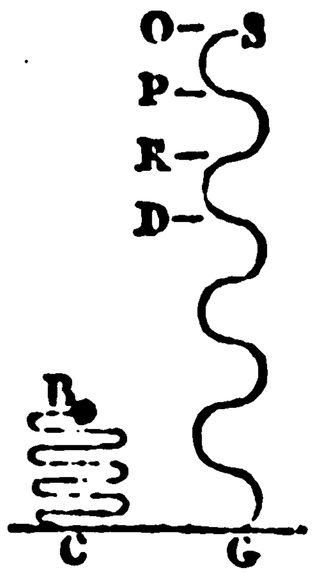
HYDRAULICS.

426. *If a vessel* AC *be full of water, the first or greatest velocity with which it issues through a hole at the bottom, is equal to that which a body would acquire in descending freely from rest through the perpendicular height* AB.



This appears from experiment. For a heavy body projected vertically with the velocity it acquired in descending from a given height, would rise to that same height, provided the motions were made in a non-resisting medium, (317); and when the orifice O of a pipe OB inserted at the bottom is horizontal, and the vessel kept full, the fluid is observed to spout up nearly as high as the surface AP: the stream however, or jet is impeded by the resistance of the air, and therefore its ascent is less than it would be in vacuo.

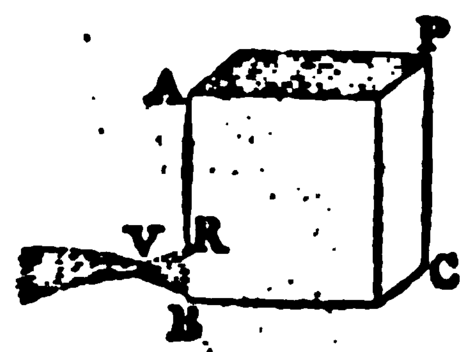
To determine the velocity from theory, we suppose the weight or pressure of the incumbent column of water to generate that velocity; this force therefore will vary as the perpendicular height of the fluid's surface above the orifice varies, and consequently the force and depth increase uniformly together, like the bending of a spring. Thus let SG be a spring of uniform strength, and suppose that 1*lb.* weight when laid on the end at S will bend it through the space of 1 inch, or from O to P, then it is found that 2*lb.* will bend it twice that space (= OR), and 3*lb.* three times OP or three inches (= OD), and so on; therefore the space through which the spring is bent increases uniformly with the compressing force: hence the velocity of the water may be compared with that of a bent spring when suddenly loosened.



Let the spring SG be compressed into the space BC, and a small weight B attached to the end or top B when it is disengaged, then the weight is an uniformly retarding force, and the height to which it will ascend by the force of the spring will be as the square of the initial velocity, as in bodies when projected upwards (317). The same thing however, is evident without supposing a weight at the end B; for since the spring is compressed by an uniformly accumulated weight or force, it must

regain its unbent position by a motion uniformly retarded. The initial velocities of the spring and effluent water are therefore analogous; and hence the velocity of the water through an orifice at any depth, will be as the square root of that depth.

It is not infered from this determination that water is an elastic fluid. But if the experiments of Mr. Canton are conclusive (Philos. Trans. vol. 53, 54) the whole mass of water belonging to our globe is in a constant state of compression by the weight of the surrounding atmosphere.



But the greatest velocity of the water is at a little distance from the orifice BR. For since the pressure of the fluid in the vessel is in all directions, it is observed that the effluent water converges after it quits the hole, so that when the orifice is circular, it forms the frustum of a cone BRV. Sir I. Newton found that the area of the section at V (which he called the *vena contracta*) is to that of the orifice BR as 1 to $\sqrt{2}$ nearly, and therefore the velocities of the water at BR and V, are as 1 to $\sqrt{2}$. Now it is the velocity of the stream at V which is equal to that acquired by the descent of a heavy body through the whole height AB, consequently the velocity *at* the orifice BR will be nearly the same as a body would acquire in descending through half that height.

Suppose the height BA $= 12$ *feet* $= S$, and let $s = 16\frac{1}{12}$ *feet*, $v = 32\frac{1}{6}$, and $V =$ the velocity at V; then (317) $V = v\sqrt{\frac{S}{s}} = \sqrt{4sS} = \sqrt{772} = 27.78$, &c. *feet*, the velocity per second at V. And $\sqrt{2} : 1 :: 27.78$, &c. $: 19.64$, &c. *feet* per second, the velocity at the orifice BR.

427. *Let the depth of the vessel be* 12 *feet, and the top and bottom* AP *and* BC *squares, each side being* 3 *feet, to find the time in which it would be exhausted through a circular orifice in the bottom whose diameter* $= \frac{1}{4}$ *an inch, supposing it filled with water.*

Put a = area of the orifice in feet, $b = 25$ feet the area of the top or bottom, $h = 12$ feet the depth, and $s = 16\frac{1}{12}$ feet. Then $\sqrt{2sh}$ will be the first velocity or that with which the water quits the orifice when it begins to run out. And since all horizontal sections of the vessel are equal, it is manifest the surface of the water in the vessel will descend with an uniformly retarded velocity, that is, the velocities will form an infinite arithmetical progression whose first term is $\sqrt{2sh}$, and last term $= 0$ the least or last velocity, hence $\frac{\sqrt{2sh}+0}{2} = \sqrt{\frac{sh}{2}}$ feet per second, will be the mean velocity.

Now the whole length of the cylinder of water that flows through the orifice is $= \frac{hb}{a}$ feet; hence $\sqrt{\frac{sh}{2}} : 1$ *sec.* $:: \frac{hb}{a} : \frac{b}{a}\sqrt{\frac{2h}{s}}$ $= 22397$ *seconds*, nearly, is the time it would require to pass through the orifice with the mean velocity, which therefore is the time of exhaustion,

Corol. 1. And if d = any other height from the bottom, the time of exhaustion will be $\frac{b}{a}\sqrt{\frac{2d}{s}}$; therefore the time of emptying any depth $h - d$ from the top is $\frac{b}{a}\left(\sqrt{\frac{2h}{s}} - \sqrt{\frac{2d}{s}}\right)$.

Corol. 2. The velocities through apertures in the side and bottom are the same at equal depths. And the velocities at different depths are as the square roots of those depths.

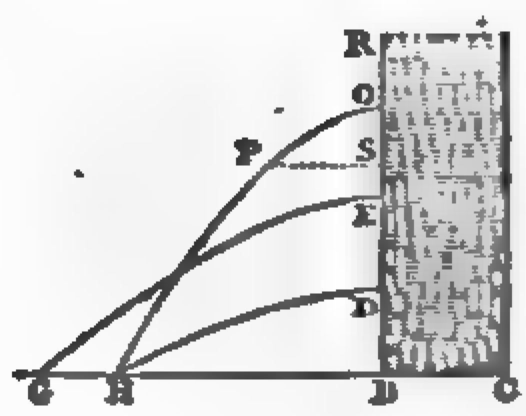
Corol. 3. Hence the exhaustion is performed in a less time through a pipe or tube inserted in the bottom; for its lower end becomes the aperture in that case; and since the distance from the surface of the water is thereby increased, the velocity will also be increased.

SCHOLIUM. The orifice is here supposed to be completely filled by the effluent water during the time of exhaustion; that however, is not always the case. For when the aperture is large

in proportion to the size of the vessel, the surface of the water, soon after the evacuation commences, is observed to move on all sides towards the column directly over the orifice, and this motion becomes more and more perceptible as the surface descends; the water therefore, meeting in different directions, forms an eddy by which means it acquires a spiral motion in descending to the orifice, and the circumjacent water joining in the whirl, produces a funnel-shaped vortex ABO that extends through the bottom at O, the inside ABO being totally void of water. But the difference in the pressure of the air above and below the stream may conduce to the formation of this vortex: for the force of the effluent water diminishes the pressure beneath, on which account the incumbent air following the stream, finds as it were an easier passage; and this appears the more probable, because the velocity of the water in the middle of the effluent column is always greater than that towards the sides, which is retarded by tenacity and friction.

We may therefore conclude that theory and experiment are most likely to agree when the orifice is small.

428. *Suppose* GC *the horizontal plane, and* RC *an upright vessel filled with water; to find the distance* BH *to which it will spout through an orifice* O *in the side of the vessel.*



We consider the effluent water as a projectile discharged horizontally at O with a velocity equal to that which a body would acquire in descending freely from rest through the height RO. And therefore (327, corol. 3, 4) if OS = OR, and SP (perpendicular to RB) = 2OS = 2OR, O will be the *vertex*, S the *focus*, and SP the *semi-parameter* of the *parabola* OPH described by the

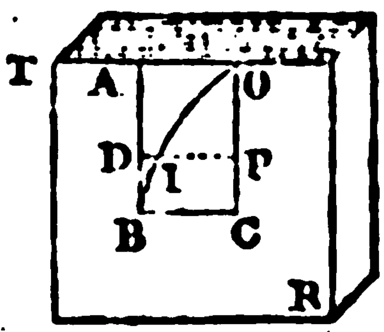
issuing stream. Hence by prop. of the parabola, OS (or OR) : SP^2 (or $4 \times OR^2$) :: OB : $4OR \times OB = BH^2$; consequently $2\sqrt{(OR.OB)} = BH$ the distance.

Corol. 1. Let E be any other aperture, then $BG = 2\sqrt{(ER.EB)}$; therefore the distances BH and BG are as $\sqrt{(OR.OB)}$ and $\sqrt{(ER.EB)}$. And if $BD = OR$, then $DB.DR = OR.OB$, hence it follows that BH is also the distance to which the water would reach through an aperture at D; that is, the horizontal distances are equal through apertures at equal distances from the top and bottom of the vessel.

Corol. 2. If $EB = ER$, the rectangle ER.EB is a maximum, and consequently $2\sqrt{(ER.EB)} = BG$ is also the greatest possible; but when $ER = EB$, BG is $= RB$, therefore the greatest horizontal distance to which the water can spout is equal to the height of the vessel.

If $RB = 13$ feet, OR and DB each $= 4$ feet, then $BH = 2\sqrt{(4 \times 9)} = 12$ feet, supposing the stream is not impeded by the resistance of the air. And when the orifice E is equally distant from the top and bottom, then $BG = BR = 13$ feet.

429. *Let* TR *be a vessel filled with water; to find the quantity that would flow out through the rectangular aperture* ABCO *in a given time, supposing the vessel to be kept constantly full.*



Make O the vertex, and CB the ordinate to the axis CO of the parabola OIB, and let PI be any other ordinate.

Since $\sqrt{OC} : \sqrt{OP}$ :: *veloc. of fluid at* BC : *veloc. at* DP,

And $\sqrt{OC} : \sqrt{OP}$ :: BC : IP, by prop. of the parabola;

Therefore the velocities of the effluent water at the depths OC and OP are as the ordinates BC and IP. Hence, if BC and DP are the perpendicular sections of two indefinitely thin horizontal

laminæ of the issuing water, the quantities of the laminæ flowing out in the same time, will be as those sections drawn into the respective velocities, or as $BC \times BC$ and $DP \times PI$; but it will amount to the same thing if the issuing section is denoted by the part PI and its constant velocity by DP or its equal BC, because $PI \times BC = DP \times PI$; consequently if the part PI (or ordinate) were to issue with the velocity of the fluid at the bottom PC, the same quantity would be discharged in the same time as when the whole DP runs out with its real velocity: but the sum of all the ordinates or PI's, &c. together, are equal to, or make up the surface of the parabola; hence if the water ran through the parabolic section OIBC only, but with a velocity every where equal to that at BC, the quantity issuing *would be* the same as that which flows through the whole aperture ABCO.

Put $s = 16\frac{1}{12}$ feet, and let the dimensions of the aperture be also in feet; then $2\sqrt{s\,OC}$ feet is the velocity per second which a body would acquire in descending freely from O to C, or the velocity of the water at BC; and $OC \times BC$ being the area of the aperture, that of the parabola will be $\frac{2}{3}OC \times BC$ (304); hence $\frac{2}{3}OC \times BC \times 2\sqrt{s\,OC} =$ the cubic feet per second, the quantity running out.

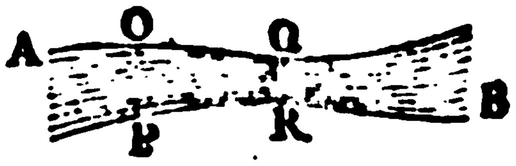
The same conclusion however, is obtained without the parabola by finding the sum of all the velocities from o at the top O, to $2\sqrt{s\,OC}$ or $2\sqrt{s} \times \sqrt{OC}$ at the bottom C, and dividing that sum by the number of velocities, for the mean velocity; thus, suppose OC to be divided into an infinite number of equal parts OP, &c. then the sum of the infinite progression of square roots $o^{\frac{1}{2}} \times OP^{\frac{1}{2}} + \&c.\ldots OC^{\frac{1}{2}}$, will be $\dfrac{OC^{\frac{1}{2}+1}}{\frac{1}{2}+1}$, or $\frac{2}{3}OC\sqrt{OC}$ (179), which divided by OC their number, gives $\frac{2}{3}\sqrt{OC}$, therefore $2\sqrt{s} \times \frac{2}{3}\sqrt{OC}$ is the mean velocity, this drawn into $OC \times BC$ the area of the aperture, gives $\frac{2}{3}OC \times BC \times 2\sqrt{s\,OC}$, as before.

Suppose the depth OC $= 3$ feet, breadth BC $= 2$ feet; then $\frac{2}{3}OC.BC.2\sqrt{sOC} = 55.3$ cubic feet, nearly, the quantity that runs out per second.

But if the velocity at BC is supposed to be equal to that which a body acquires in falling through $\frac{1}{2}OC$, then $\frac{2}{3}OC.BC.\sqrt{2sOC} = 39.3$ cubic feet, the quantity per second. This last result is probably less than would be found by experiment: and on account of the friction against the sides of the aperture, and the oblique motion of the particles in entering it, &c. we may conclude the former, or 55.3 feet, to be greater.

Corol. Hence for the quantity that runs out through an aperture DBCP not reaching to the top; find what would be discharged through ABCO, and ADPO separately, then the difference will be the answer.

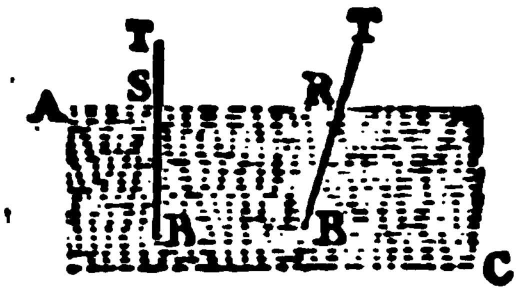
430. *Suppose* AB *to represent part of a river or canal whose breadth is unequal; then the velocity of the water at any two places, as* OP *and* QR, *will be reciprocally as the transverse sections of the stream at* OP *and* QR.



For let A and a represent the areas of the sections at OP and QR, and v and V the velocities of the water at those sections, respectively; then Av and aV will denote the quantities that run through OP and QR in the same time; but those quantities are equal, or $Av = aV$, that is $v : V :: a : A$. And the velocities in a pipe are found in the same manner, provided the water fills it at the different sections.

But could we obtain the velocity of the water at the surface, and also the exact dimensions of the section at any particular place (OP), that data would not be sufficient to compute the quantity of fluid that flows along in a given time: for by reason of the friction at the bottom and sides of the canal, the velocity

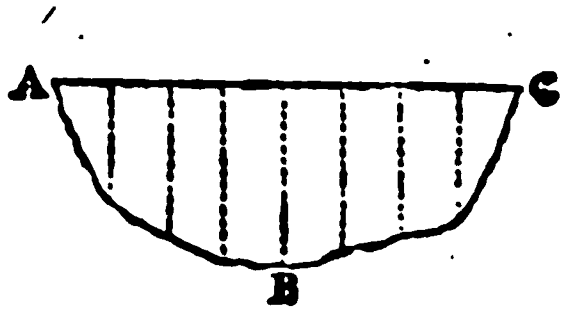
of the water is always greatest at the surface and middle of the stream. The following method however, has been proposed for determining a mean velocity.



Suppose AC is the section of a river or canal in which the water runs from S towards R. And let TB be a small uniform rod of wood loaded in the inside at one end B with a weight sufficient to make it swim with a part ST above the surface. Now this being put into running water, the greater velocity at the surface SR will make it incline towards the direction of the stream, and consequently when it has acquired an equilibrium, it must float along in an oblique position BRT, but the upper part will move slower than the water at the surface, and the lower end B faster than the stream at that depth; and hence we may conclude that the mean velocity of the water will be nearly the same as the velocity of the rod.

The experiment however, should be tried in different parts of the stream, and a mean taken of all the results.

The area of a transverse section may be nearly ascertained by taking a mean of several depths at equal distances across the stream:



Thus, let ABC represent the section; breadth at top AC $= 7\frac{1}{2}$ *feet*, and the depths at 7 equidistant places, 2, $2\frac{6}{12}$, $2\frac{8}{12}$, $2\frac{10}{12}$, $2\frac{4}{12}$, $2\frac{2}{12}$, and $2\frac{1}{12}$ *feet*, respectively; then

$$\frac{2+2\frac{6}{12}+2\frac{8}{12}+2\frac{10}{12}+2\frac{4}{12}+2\frac{2}{12}+2\frac{1}{12}}{8} = 2\frac{7}{96}$$

feet, the mean depth, which multiplied by $7\frac{1}{2}$ *feet* the breadth at top, gives $15\frac{35}{64}$ *feet*, the content of the section. (Mensur. *Art.* 271).

Now supposing the mean velocity of the water is found to be 78 *feet* per minute, we shall have $15\frac{35}{64} \times 78 = 1212\frac{21}{32}$ *cubic feet*, the quantity that flows along in one minute.

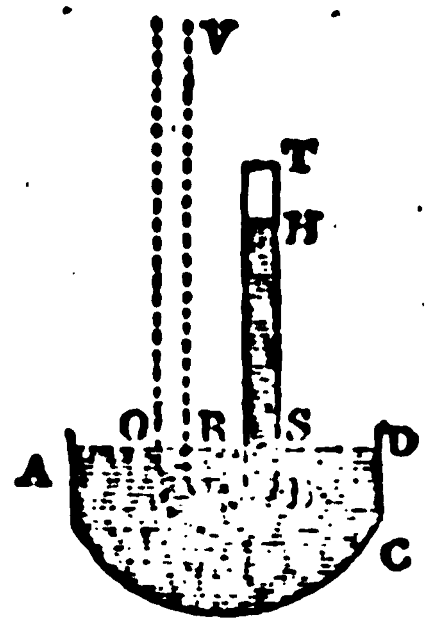
The result found in this manner will not, perhaps, be much

wide of the truth when the velocity of the water is uniform for a considerable distance. But in small irregular streams and rivulets, other methods must be adopted.

Of PNEUMATICS.

431. Pneumatics in the ancient schools, and what we understand by Metaphysics, were synonymous: but in the modern philosophy, pneumatics is the science which treats of the weight, pressure, elasticity, &c. of the air.

432. *The atmosphere or mass of air surrounding the globe, is a fluid that presses by its weight on all parts of the earth's surface.*



If a glass tube BT about a yard in length, close at one end T, be filled with mercury and the open end B stopped with the finger while it is immersed in a bason or vessel AC of the same fluid, then on removing the finger from the orifice B, the mercury in the tube will descend into the vessel till the weight of the column SH is equal to the weight of a column of the atmosphere of the same diameter but whose height is equal to the whole height of the atmosphere above the earth's surface. For the end T being close stopped or air-tight, the empty space TH is a vacuum, consequently there is no pressure on the surface of the mercury at H, and hence the column HS is a counterbalance to the weight of the atmosphere acting on the orifice B by its pressure against the surface of the fluid in the vessel: this also appears by letting in the air at the top T, for the mercury will all sink into the vessel; it is therefore kept suspended in the tube by the weight or pressure of the atmosphere. This is called the Torricellian experiment from Torri-

cellius the mathematician and pupil of Galileo, who made the discovery.

The length of the column SH varies from about 28 inches to 31. Suppose the height to be $29\frac{1}{2}$ inches, the tube uniform, and the area of the orifice B = 1 square inch, then the column of mercury SH will be $29\frac{1}{2}$ cubic inches, or 239 ounces; and therefore in a mean state of the air (or when the height of the column SH is $\frac{28+31}{2}$ inches) its pressure upon every square inch of the earth's surface is nearly 15 pounds. So if the orifice at T were stopped with the palm of the hand, a pressure equal to 15*lb.* would be felt on the back.

But the tubes commonly used are 33 or 34 inches long, and the internal diameter about $\frac{1}{4}$ of an inch; they are sealed hermetically at the end T; and to the upper part HT is attached a scale of 3 inches divided into 10ths, and subdivided to 100ths, by means of a sliding vernier; the lowest division is numbered 28, and the highest 31, these are inches, and the height of the mercury in the tube above the surface of the mercury in the vessel at bottom is shown at all times in inches and parts of an inch. The tube with its scale, &c. when fixed in a frame is called a BAROMETER.

In fair settled weather the mercury is up at 30 inches, and sometimes higher. When it falls to $29\frac{1}{2}$ a change is usually expected. But it seldom sinks so low as 28 inches, except in very stormy weather.

Mercury is about 14 times heavier than water, therefore if the fluid in the tube TB and vessel AC were water, the height of the column HS would be $29\frac{1}{2} \times 14 = 413$ inches, or about 34 feet, supposing the tube long enough, which is the reason that the piston of a common pump for raising water must work in the barrel or cylinder at a distance less than 34 feet from the surface of the water in the well.

But because the perpendicular pressure of the column of mercury BH and that of the column of air on the whole surface AD are in equilibrio, it may not be readily conceived why the weights of the two columns are unequal. Let us suppose the tube to be lengthened, but bent at the lower end so that its orifice OR is even with the surface of the mercury; also let RV be the column of air having OR for its base, and height RV equal to the height of the atmosphere; then if all the mercury were taken out of the vessel (that in the bent part of the tube excepted) it is evident the equilibrium would still remain, and therefore the *weights* of the columns SH and RV must be equal. For if the columns are cylinders of equal diameters, their weights will be as their pressures.

Since the atmosphere is a fluid variable in weight, it follows that bodies of unequal specific gravities will weigh differently in different states of the air. Thus if a piece of wood is an exact counterpoise to a pound of lead in a nice pair of scales when the mercury in the barometer is at 28 inches, it will not weigh a pound when the mercury stands at 30 inches. For the air is more dense in the latter case by about $\frac{1}{14}$ than in the former, therefore the lighter body or that whose bulk is greatest, will lose more weight or become more buoyant than the heavier, and consequently the lead must preponderate. Hence it has been suggested that advantage may be taken in buying and selling gold and jewels by weighing them in particular states of the weather.

It appears from many experiments that the weight of a cubic foot of air when in a mean state near the earth's surface, is $1\frac{1}{4}$ ounces avoirdupoise, very nearly. Hence the *absolute* weight of a body is what it would weigh in *vacuo*.

433. *If the air was of the same density at all altitudes as at the earth's surface, its height would be between 5 and 6 miles.*

Let RV represent the homogeneous co
weight to the column of mercury SH whi
inches. Then (422, corol.) the heights l
procally as the specific gravities of air an
$1\frac{1}{4}$: 14000 :: $29\frac{1}{2}$: 344166 *inches*, or 5.4
the column RV. The specific gravities o
ever, will vary if the temperature varies (
when the mercury in the barometer stan
the thermometer at 55 degrees, it has been
that the specific gravities of air, water, an
as $1\frac{1}{4}$, 1000, and 13600; hence we shall h
: 340000 inches, or 5.366 miles, the heigh
atmosphere whose density would be equal
the earth's surface, and weight the same a
mosphere.

Or taking $1\frac{1}{4}$ for the specific gravity of
or 5.269 miles.

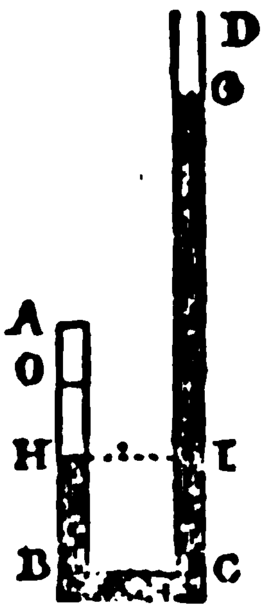
431. *The air is also an elastic fluid th*
or expanded.

One of the most simple instruments tha
elastic, and condensible, is a boy's *popg*
in one pellet with the handle or ramrod,
the whole extent of the barrel is conden
elastic force is sufficient to propel the othe
the *air-gun* is constructed, only the air
the chamber or barrel, and this being sudde
of a trigger, the air rushes out with great
bullet.

435. *When air is compressed, its densi*
are proportional to the weight or force th

Let ABCD be a long crooked glass tube
but open at D; the internal diameters of th

being equal. Fill the lower part BC carefully so as to leave the air in BA in its natural state but stop its communication with the external air. This done, pour in more quicksilver till it ascends in the leg BA to H the middle point between B and A; then IG the height of the mercury in the other leg above H will be found equal to the height of the barometer at that time.



The air in HA therefore is condensed into half its former volume by the weight or pressure of an additional atmosphere or the weight of the column of mercury IG which is equal to it; for the air in its natural state was pressed by the weight of the incumbent atmosphere only, but now the compressing force is the column IG together with that of the column of air directly over it; consequently the density is doubled by a double pressure. And since the pressure of the mercury and the action of the air are equal in opposite directions at H, the elastic force of the latter is increased in the same proportion as its density.

Again, if the pressure be doubled by pouring in a sufficient quantity of mercury at D, the mercury will rise to O the middle of HA, and condense the air into the space OA or $\frac{1}{4}$ of BA. In this manner it is found that the density is directly proportional to the compressing force.

The spaces BA, HA, OA, are reciprocally as the densities or as the compressing forces. It is said however, that after air is condensed into about $\frac{1}{4}$ or $\frac{1}{5}$ of the space it occupied in its natural state, the repulsive or elastic force increases in a greater proportion than the volume diminishes.

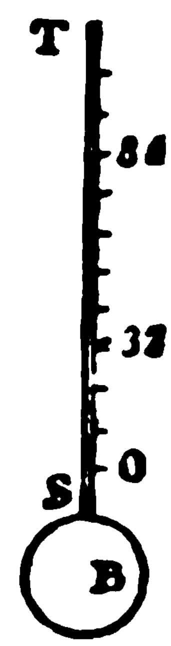
Corol. 1. Since the atmosphere at any height from the earth is pressed by the weight of the air above it, the greater that height, the less must be the air's density; thus it is more dense at the foot of a mountain than at the top; and still greater at the bottom of a deep pit than at the earth's surface.

Corol. 2. The elastic force of the air being equal to the force of compression, and (art. 433) 5.366 miles or 340000 inches = 28333 feet the height of an homogeneous atmosphere whose pressure would be equal to that of the real atmosphere, we shall have (art. 426), $\sqrt{(4 \times 16\frac{1}{12} \times 28333)} = 1350$ *feet*, the velocity per second with which common air at the earth's surface would rush into a vacuum.

436. *Heat expands, and cold condenses air.*

Thus, Let a bladder with a little air in it be tied very close, and laid before the fire; then as the inclosed air grows warm, the bladder will distend, and at last burst if the heat be continued. But when the bladder is removed to cool, it will contract again to its former size. The elastic force of the air is therefore increased by heat, and diminished by cold.

It is also found that all fluids, and most solids expand by heat, and contract by cold. And therefore in determining specific gravities, and making experiments with the barometer, it will always be necessary to mark the temperature as shown by the THERMOMETER. This instrument is well known. It consists of a glass bulb B with a small tube ST of the same material fixed in a frame; the bulb and lower part of the tube are usually filled with mercury, but the upper part of the bore should be a perfect vacuum, and the end T hermetically sealed. The degrees of temperature are shown on a graduated scale in the frame. Thus, when it just begins to freeze, the surface of the mercury in the tube will be at the division numbered 32, which therefore is called the freezing point. But when the weather changes to warm, the mercury in the bulb will expand, and consequently rise in the tube, so that in the summer it is sometimes as high or higher than the 80th. division. If the instrument be immersed in boiling water, the mercury will ascend to the 212th. division or degree. And if the heat be sufficient to boil mercury, it will



rise to 600 degrees, provided the tube be long enough. This is called *Fahrenheit*'s Scale, which was constructed thus: Having observed where the mercury stood in the tube in a very severe frost in Iceland, he marked that point with *o* for the lowest, or the beginning of his divisions; and then determined the other extreme or highest point of his scale by boiling the mercury; the distance between those extreme points he divided into 600 equal parts or degrees; and afterwards observed that the mercury in the tube stood at the 32*d*. division of his scale when water just began to freeze, or snow to melt; and that it rose to the 212th. by the heat of boiling water. It is found however, that the heat of boiling water encreases with the weight of the atmosphere as shewn by the barometer.

The Thermometers in common use contain about 100 or 120 of the 600 degrees. But for particular purposes the graduations are continued downward from *o*; thus mercury is congealed by the cold at 40° below that point on Fahrenheit's scale. The freezing point on Reaumur's Thermometer is o, and that of boiling water 80; sometimes however, this latter point is numbered 110°; hence, by proportion, the different thermometers are easily compared.

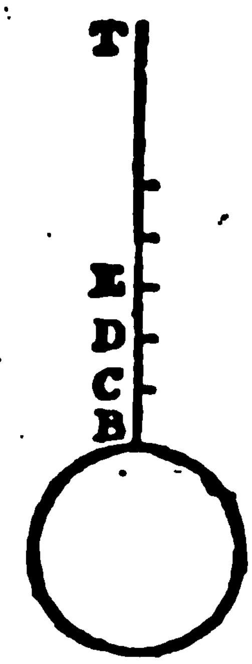
The temperate state of the air in England is reckoned at 55°, which is nearly a mean between freezing and summer heat. And to this degree of the thermometer, when the height of the barometer is 30 inches, the table of specific gravities (art. 419) is calculated.

437. The expansions of air, water, and mercury, answering to one degree on Fahrenheit's scale, are about $\frac{1}{435}$, [illegible], and [illegible] of their bulks, respectively.

Expansion of 1 foot of	Brass rod .... =	.0001263	of an inch, by one degree of Fahrenheit.
	Steel rod..... =	.000076	
	Cast iron .... =	.000074	
	Solid glass rod =	.0000536	

[redacted]

438. *If vertical distances be taken from the earth's surface in arithmetical progression, the corresponding densities of the atmosphere will decrease in geometrical progression; supposing the force of gravity constant.*



Let BT represent a cylindric column of the atmosphere perpendicularly incumbent upon the surface of the earth at B, and suppose this column to be divided into innumerable thin parts or laminæ of equal heights BC, CD, DE, &c. also let a, b, c, &c. denote their densities in succession from B upwards; then if the magnitude, or the height, of each lamina be represented by m, their weights will be proportional to $m \times a$, $m \times b$, $m \times c$, &c. or ma, mb, mc, &c. (considering the density of each as uniform); and since the densities are as the compressing forces or incumbent weights, the densities at B, C, D, &c. will be respectively

$$\begin{array}{rr} \text{as } ma+mb+mc+md, \text{ \&c.} & \text{or as } a+b+c+d, \text{ \&c.} \\ mb+mc+md, \text{ \&c.} & b+c+d, \text{ \&c.} \\ mc+md, \text{ \&c.} & c+d, \text{ \&c.} \end{array}$$

That is, $a : a+b+c+d$, &c. $::$ $b : b+c+d$, &c.
or, $a : b+c+d$, &c. $::$ $b : c+d$, &c. by division.

In like manner we get, $b : c+d$, &c. $::$ $c : d+e$, &c.

therefore by equality

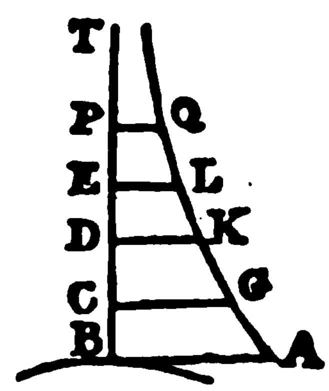
$a : b+c+d$, &c. $::$ $b : c+d$, &c. $::$ $c : d+e$, &c.

but $b+c+d$, &c. $c+d$, &c. are as b, c, &c.

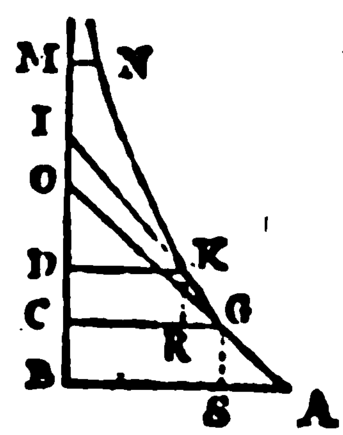
hence $a : b :: b : c :: c : d$, &c. That is, a, b, c, &c. are in geometrical progression.

Hence when the heights are in arithmetical progression (the common difference of the terms being m) the corresponding densities a, b, c, &c. will decrease in geometrical progression. Now when an arithmetical series of numbers is adapted to a geometrical one, the former will be analogous to the logarithms

of the latter (Arith. 155), it therefore follows that the altitudes increase as the logarithms of the corresponding densities decrease. Or if we make the heights beginning at the surface a descending series, the two progressions will decrease together. Hence any height (BE for example) is proportional to the difference of the logarithms of the densities at B and E, or to the difference of the logarithms of the heights of the mercury in the barometer at B and E, because the densities are measured by those heights. This however, is better explained by means of the *Logarithmic curve;*



439. Let BT be an indefinite right line perpendicular to the earth's surface at B, and suppose the densities at B, C, D, &c. to be represented by the lines BA, CG, DK, &c. (perpendicular to BT), then a line through the extremities A, G, K, &c. is the logarithmic curve. For if PE, ED, DC, &c. are equal, the abscissas PE, PD, PC, &c. are in arithmetical progression, while the corresponding ordinates (or densities) PQ, EL, DK, &c. are in geometrical progression, which is the property of a logarithmic curve. Thus in Briggs's scale, or the common logarithms, if PQ, EL, DK, &c. are 1, 10, 100, &c. then PE, PD, &c. are 1, 2, &c. (the logarithms of 10, 100, &c.) And since the series 100, 10, 1, $\frac{1}{10}$, &c. may be continued *in infin.* it is evident the curve can never meet BT, which therefore is an assymptote.



440. *In this curve all the subtangents are equal.* Let KI be a tangent at any point K, KD the ordinate at that point, then DI is the subtangent. Suppose DC, CB are indefinitely small but equal parts of the axis; draw the ordinates CG, BA, and let GO be a tangent at G, also make KR, GS perpendicular to CG, BA. Then since CG is a mean proportional between DK and BA, we have

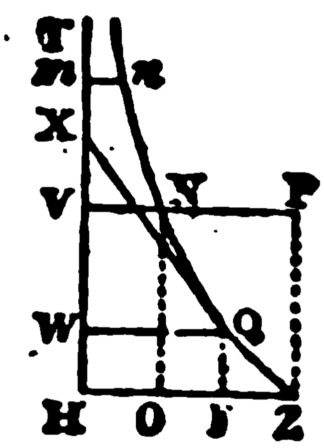
DK : CG :: CG : BA :: (by division) CG—DK (or RG) : BA—CG (or SA),

therefore DK : RG :: CG : SA. But the indefinitely small arcs KG, GA may be considered as right lines coinciding with IK, OG at the points K, G, respectively, and the triangles IKD, KGR; OGS, GAS will be respectively similar, whence

DI : RK :: DK : RG :: CG : SA (by the former proportion) :: CO : SG;

that is, DI : RK :: CO : SG; but RK = SG, therefore DI = CO. And since the same reasoning will hold good at any point in the curve, all the subtangents must be equal.

441. *Let* NGA (*preceding fig.*) *and* nQZ *be two logarithmic curves*, MN = mn = 1, MC, mH, *the logarithms of equal numbers or ordinates* CG, HZ; GO, ZX *tangents*; *Then subtang.* CO : *subtang.* HX :: MC : mH.



Suppose CD, HW are indefinitely small, but proportional to the logarithms MC, mH, or so that MC : CD :: mH : HW; and draw DK, WQ parallel to CG, HZ; and KR, QF perpendicular to CG, HZ; then CR and RG are equal to HF and FZ, and the triangles RGK, CGO; FZQ, HZX being respectively similar, we have

CO : RK :: CG : RG :: HZ : FZ :: HX : FQ.

And alternately, CO : HX :: RK (or CD) : FQ (or HW) :: MC : mH.

These constant subtangents of the logarithmic curves are called the *moduli* of the systems. In the hyperbolic logarithms the subtangent is 1, and the logarithm of 10 is 2.3025851, now 1 being the logarithm of 10 in Briggs's scale, we have by the last proportion, 2.3025851 : 1 :: 1 : .43429448 the modulus of the common or Briggs's logarithms. Hence if the common logarithm of a number be multiplied by 2.3025851, or divided by its reciprocal .43429448, the result is the hyperbolic logarithm of that number.

442. *If a number be very nearly equal to* 1, *its excess above* 1 *is to its logarithm, as* 1 *to the subtangent.* For let WQ = 1, QX a tangent at Q, HZ the number or ordinate at a small distance from WQ, and WH its logarithm; then the arc QZ and its chord will be nearly equal, and the triangles FQZ, WXQ similar; whence FZ : QF (or WH) :: WQ : WX.

443. *Let* HZ, VN *be any two ordinates,* NO *perpendicular to* HZ, *and* ZX *a tangent at* Z; *then the quadrilinear space* HZNV = OZ *multiplied by the subtangent* HX. For suppose WQ is parallel to, and indefinitely near HZ, and QF perpendicular to HZ; then the triangles HXZ, FQZ may be considered as similar; hence HZ : HX :: FZ : FQ, and HZ.FQ = HX.FZ; but WQ being indefinitely near HZ, the rectangle HZ.FQ may be taken for the area of the quadrilinear HZQW, consequently HX.FZ = the area HZQW; hence if we conceive the quadrilinear HZNV to be composed of an infinite number of elementary quadrilinear spaces HZQW, &c. their areas together will be HX × all the FZ, but all the FZ together = OZ, therefore the quadrilinear HZNV = HX.OZ.

Corol. Hence the infinitely long space contained by the ordinate HZ, assymptote HT, and curve ZNT is = the subtangent HX drawn into the ordinate HZ: For ultimately OZ = HZ.

444. Now let the densities of the air at the earth's surface H, and heights W, V, &c. be denoted by the corresponding ordinates HZ, WQ, VN, &c. respectively; also suppose HV = the height of an uniform atmosphere (Art. 433), and complete the parallelogram VHZP; then the pressure on the surface at H will be proportional to the sum of all the densities or (HZ × subtang. HX) the area of the infinitely long logarithmic space HTZ which is composed of the infinite progression of ordinates. But this is also represented by the parallelogram HP, or HV × density HZ; therefore HV the altitude of an uniform atmosphere is the subtangent of the atmospheric logarithmic; and if ZX be a tangent at Z, the points V and X will coincide.

If therefore HZ be the density at the surface H, and WQ the density at W (which suppose to be the top of a high mountain, for example) then the height (HW) will be proportional to HW the difference of the logarithms of HZ and WQ: But if the curve were actually constructed, its subtangent would be 27819 *feet* (433), and the difference of the logarithms adapted to that subtangent would be the height HW in feet; therefore to find that difference by means of the common logarithms, let D and d be the densities at H and W, then $log.\ D - log.\ d = log.$ of $\frac{D}{d}$; hence to find the $log.$ of $\frac{D}{d}$ when the subtang. is 27819, we have $.43429448 : 27819 :: log.$ of $\frac{D}{d} : \frac{27819}{.43429448} \times log.\ \frac{D}{d}$, or $64056 \times log.$ of $\frac{D}{d}$, the $log.$ required, or the height HW in *feet*, when the temperature shown by Fahrenheit's thermometer is 55°, and the height of an homogeneous atmosphere = 27819 *feet*, answering to 30 *inches* the height of the mercury in the barometer: But if 29½ *inches* be the mean height of the barometer, we shall get 27355 feet instead of

27819; and taking the mean or 27587 feet for the length of the subtangent, gives 63521 × *log.* $\frac{D}{d}$ the height in feet, or 10587 × *log.* $\frac{D}{d}$ the heights in fathoms: But the densities (D, d) are measured by the heights of the barometer at the places of observation H and W, therefore the difference of the common logarithms of those heights × 10587 is the height of one place above the other in fathoms when the thermometer is at 55°. But as the air expands about $\frac{1}{435}$ of its bulk by 1 degree of Fahrenheit, and the compressing and expanding forces equal, it follows that the result or height must be corrected by adding the 435th. part of itself for every degree which the temperature is above 55°, or subtracting, when the temperature is below. The factor 10587 however, is usually changed to 10000, and the expression for the height given thus, 10000 × *log.* $\frac{H}{h}$ (where H and h are the heights of the barometer) which is found by reducing the temperature from 55°: Thus, let $l = log.\ \frac{H}{h}$, then since the height $10587l = 10000l + 587l$, and $\frac{10587l}{435}$ answers to 1 degree, we have $\frac{10587l}{435} : 1 :: 587l : \frac{587 \times 435}{10587} = 24\frac{1257}{10587}$ degrees, this (neglecting the fraction) when taken from 55°, leaves 31° the temperature in which the expression 10000 × *log.* $\frac{H}{h}$ gives the height in fathoms; and the result is to be augmented by adding its 435th. part for every degree that the temperature is above 31°; but diminished by subtracting, when it is below.

Examp. 1. What is the air's density at 7 miles above the earth's surface?

7 miles = 6160 fathoms; therefore 10000 × *log.* $\frac{D}{d}$, or 10000 (*log.* D — *log.* d) = 6160, whence *log.* $d = \frac{10000\ log.\ D - 6160}{10000}$; and assuming D the density at the earth's surface = 10, its *log.* is 1, which gives *log.* d = .3840 the *log.* of 2.42, which is nearly ¼ of 10; therefore the air is about 4 times rarer at the height of 7 miles than at the surface of the earth.

Examp. 2. Suppose the mercury in the barometer is 29.74 inches high at the foot of a mountain, and 26.41 inches at the summit; what is its height, if the mean temperature be 50°?

$$29.74 \quad log. \quad 1.473341$$
$$26.41 \quad log. \quad 1.421768$$
$$\frac{H}{h} \quad log. \quad 0.051573 \qquad \text{and } 10000 \times .051573 = 515.73.$$

$50° - 31° = 19°$ temperature above $31°$; and $\frac{19}{435}$ of $515.73 = 22.53$ add
515.73
Height = 538.26 *fath.*

Computing by the formula $10587 \times log. \frac{H}{h}$, we have $10587 \times .051573 = 546.$

$55° - 50° = 5°$ temp. below $55°$; and $\frac{1}{435}$ of $546 = 6.28$ subtract
546.
Height = 539.72 *fath.*

The difference of the results arises in consequence of neglecting the fraction $\frac{587}{10587}$.

Examp. 3. If the heights of the barometer at the bottom, and top of a hill, are 29.37 and 26.59 inches, respectively, and the mean temperature 26°; what is the height?

$$29.37 \quad log. \quad 1.467904$$
$$26.59 \quad log. \quad 1.421718$$
$$\frac{H}{h} \quad log. \quad 0.043186 \qquad \text{and } 10000 \times .043186 = 431.86.$$

$31° - 26° = 5°$ temp. below $31°$; and $\frac{1}{435}$ of $431.86 =$ 4.96 subtract
431.86
Height 426.90 *fath.*

443. But on account of the great difference of temperature in low, and elevated situations, several corrections are necessary to make the results from barometrical observations agree with geometrical measurement. Before M. de Luc began his experiments with the barometer, a mean of the two temperatures shown by the thermometer attached to the barometer, and the heights of the mercury in the barometer, at the bottom and top of a hill, were thought sufficient to determine its height. M. de Luc however, found that an additional or detached ther-

mometer was also necessary, (see his *Recherches sur les Modifications de l'Atmosphere*), and this has been confirmed by the experiments of Gen. Roy, and Sir G. Shuckburgh. The formulæ for the height (in fathoms) according to the two latter observers are the following:

Gen. Roy.............. $(10000l \mp .468d) \times (1 + (f - 32^\circ) \times .00245)$.

Sir G. Shuckburgh... $(10000l \mp .440d) \times (1 + (f - 32^\circ) \times .00243)$.

Where l = the difference of the logarithms of the heights of the barometer at the two stations,

d = the difference of the degrees shown by Fahrenheit's thermometer attached to the barometer,

f = the mean of the two temperatures shown by the detached thermometer exposed for a few minutes to the open air in the shade at the two stations.

The sign — takes place when the attached thermometer is highest at the lower station, and the sign + when it is the lowest at that station.

Examp. To find the height of a mountain from the following observations taken at the foot, and summit:

	barom.	attached therm.	detached therm.
Lower station.......	29.862	68°	71°
Higher station......	26.137	63°	55°

inches.

Barom. 29.862...*log.* 1.475119 attached therm. 68° detac. ther. 71°
26.137...*log.* 1.417256 63° 55°
diff. 0.057863 = l. diff. 5 = d mean 63 = f.

By the first formula, $f - 32^\circ = 31^\circ$, and $1 + (31 \times .00245) = 1.07595$

$10000l = 10000 \times .057863 = 578.63$

$.468d = .468 \times 5 = 2.34$ subtr. 2.34

576.29, and $576.29 \times 1.07595 = 620$ *fathoms*, the height.

In computing the height by the formula $10000 \times log. \frac{H}{h}$ (in the preceding article) we take the mean temperature by the detached thermometer;

and correct the barometer for the difference of temperature shown by the attached thermometer: Thus, since mercury expands about $\frac{1}{9600}$ of its bulk by 1 degree of the thermometer, $\frac{5}{9600}$ of 26.137, or .014 of an inch will be the correction for 5°, this added to 26.137 gives 26.151 inches the corrected height where it was coldest.

29.862	*log.*	1.475119	63° — 31° = 32° mean temp. above 31°.
26.151	*log.*	1.417488	
	diff.	0.057631	

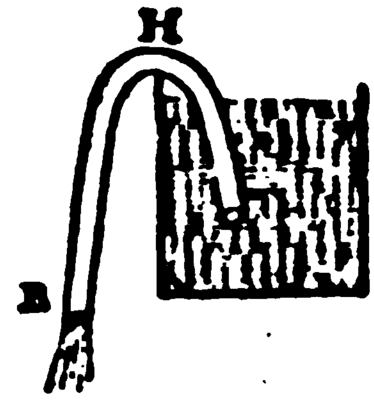
Then $10000 \times .057631 = 576.31$, and $576.31 + \frac{32}{435} \times 576.31 = 618.7$ *fath.* the height.

Ramsden's engraved Table gives the height = 3780 feet, or 621½ fathoms. This Table is on a slip of paper 1 foot long, and about 3½ inches wide; the logarithmic differences from 25 to 31 inches are given to 500ths. of an inch, and the corrections for the thermometers at both stations found by inspection.

Remark. In determining altitudes by the barometer, it is best to make the observations at the upper and lower stations at one and the same time as nearly as can be; but great care must be taken that the two barometers, and also the thermometers, are alike; that is, they should precisely agree when together in all states of the air. It is also necessary that the specific gravity of the mercury be well ascertained, because it is not equally pure in all barometers; which is the principal reason why different results are so frequently obtained from observations made with different barometers at the same stations. Other circumstances however, not generally known, may contribute to such disagreement: thus, Mr. Ramsden proved by experiment that the quicksilver in barometer tubes made of different sorts of glass will be suspended at different heights.

Of the SYPHON and PUMPS.

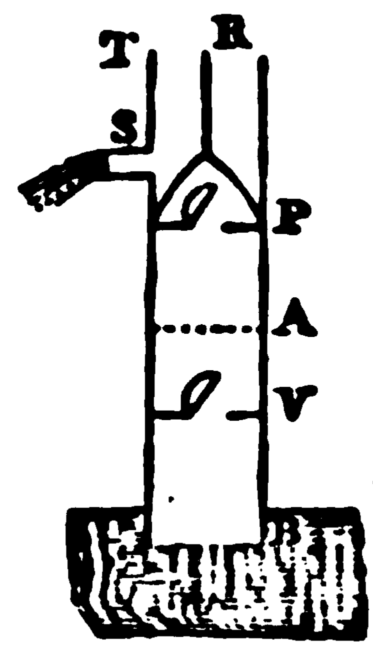
446. The Syphon is a bent tube RHO for drawing off liquors: one leg HR is usually made longer than the other, so that when it is resting on the side of a vessel, the outward end R falls lower than O the end immersed.



To use the syphon, fill it with the liquor, and stop both ends while the end O is immersed in the vessel, then if they are opened, the fluid will continue to run out at R as long as that end is lower than the surface of the liquor in the vessel, provided the end O be kept under that surface. For the weight of the column of fluid in HR is greater than that of the column in the other leg, therefore (considering the bend at H as the fulcrum) the former column must descend; and the effluent stream is continued by the constant pressure of the atmosphere on the surface of the liquor in the vessel, which makes it ascend in the leg OH.

If the vessel contain water, the bent part of the syphon (H) must be less than 33 or 34 feet from the water's surface, because that is the greatest height to which water will ascend by the pressure of the atmosphere.

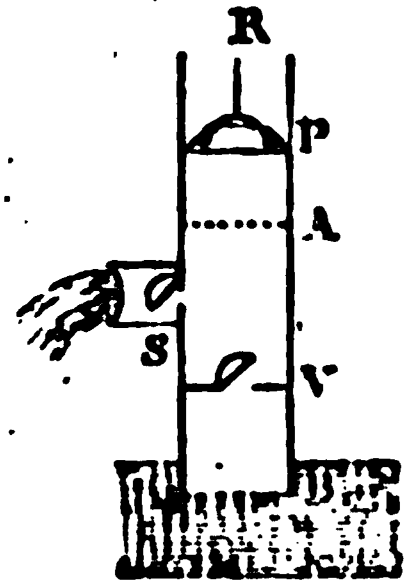
447. *The common* sucking Pump. This is a hollow cylinder or barrel TB containing a fixed valve V, and a piston P moveable up and down by means of a rod R fixed to a handle; in the piston is another valve, and both valves open upwards.



To work the pump. Let PA be that part of the barrel in which the piston moves, and suppose both valves to be shut, and the lower

end B immersed in water. Force down the piston, and the air beneath will open its valve, then draw it up, and the valve shuts by the pressure of the air above, by which means the column of air in AP is lifted up or drawn out of the barrel; now the quantity of air which occupied PB being diminished, the air in BV will expand and open the valve V; thus the internal air becomes rarefied, and therefore the external air by its pressure on the surface of the water at B will raise it a little in the barrel. Again, force down the piston, and the air in PV will shut the lower valve, but open the upper one, then by lifting the piston another quantity of air is expelled, and the water in consequence rises higher; thus by continuing the operation, all the air is drawn out of the pump, and the water will ascend above the valve V and be lifted by the piston till it runs out at the spout S.—Water poured into the top of the pump will exclude the external air, should the piston not fit the barrel quite close enough to be air-tight.

The pressure of the external atmosphere must raise the water above A, which therefore cannot be more than about 32 or 33 feet from the surface at B.



448. Forcing Pump. In this the piston P, which is without a valve, works above the spout S where a valve opens outwards. To expel the air (the valves being shut) force down the piston, suppose to A, and a quantity of air equal to that in PA will escape at S by forcing open the valve, then on drawing it up again that valve shuts by the pressure of the external air, and the air in PV being thus diminished, the valve V is opened by the expansion of the air beneath as in the other pump. Now if the piston be again forced down, the lower valve will close and another quantity of air be forced out at S; and since the water rises in the barrel every time the piston is drawn up, it will finally ascend to P (if PB is not more than

about 32 feet) and be forced through the descent of the piston. On this principle th guishing fires is constructed.

If the end B of the barrel be closed, and open in the contrary directions, or that at the other at S inwards, it becomes a *cond* the piston P is pressed down (suppose to V) be forced into VB; then on lifting the pis shut by the spring of the inclosed air, and rush in at S and fill the space VP; and by d again, another quantity of air is forced in, manner the air in VB will become more an

449. *The* AIR PUMP. This is a machine contrived for drawing the air out of a vesse which in experiments, is usually called the *receiver*. The principle is the same as in the common pump: Thus, TV is a barrel in which the piston P (with a valve opening upwards) works perfectly air tight, R is th this communicates internally with the barrel tube OV, and at V is a valve that opens common pump. Now when the piston is and then drawn up, it lifts out or expels the immediately filled again by the air in the ve through the tube OV; in like manner, by ing the piston, another quantity of air is dr operation be continued, the air in the recei ed till its elastic force is too weak to open th

Hence if the capacity of VP, and that tube are given, we can find how much the number of lifts or strokes of the piston thus

Let 1 denote the air in PV, VO, and that in PV,

Then $1-\frac{1}{p}$ or $\frac{p-1}{p}$ is the air left after the first stroke; and since the remainders are successively diminished by the pth. part, we have

$\frac{p-1}{p}-\frac{p-1}{p^2}=\left(\frac{p-1}{p}\right)^2$ the remainder after the 2d. stroke;

$\left(\frac{p-1}{p}\right)^2-\left(\frac{p-1}{p}\right)^2\times\frac{1}{p}=\left(\frac{p-1}{p}\right)^3$.. after the 3d. &c.

that is, the remainders form a descending geometrical progression, the first term being 1, and common ratio $\frac{p-1}{p}$ and therefore the remainder after n strokes will be $\left(\frac{p-1}{p}\right)^n$; but the remainders successively occupy the same space, and consequently the densities are denoted by the terms of the series.

Suppose the capacity of the tube VO and receiver R together is equal to 10 times that of PV the part of the barrel in which the piston works, then PV is $\frac{1}{11}$ of the whole, or $p=11$, and let $n=50$; then $\left(\frac{p-1}{p}\right)^n=\left(\frac{10}{11}\right)^{50}$ $=.008518$ the density of the inclosed air after 50 strokes or lifts of the piston, which is nearly $\frac{1}{117}$ of 1 the first density; so the air is rarefied about 117 times by 50 strokes.

450. Should it be required to determine how many strokes would be necessary to rarefy the air a proposed number of times, let $r=$ that number, then the density will be $\frac{1}{r}$, and we get $\left(\frac{p-1}{p}\right)^n=\frac{1}{r}$, whence $n\times\log.\frac{p-1}{p}=\log.\frac{1}{r}$. Suppose $r=60$, and let $p=11$ (as above), then n $=\frac{\log.\frac{1}{60}}{\log.\frac{10}{11}}$, or (taking the reciprocals) $n=\frac{\log.60}{\log.\frac{11}{10}}=43$ nearly, the number of strokes by which the air would be rarefied 60 times.

But a complete air pump is constructed with two barrels; and furnished with various apparatus for different experiments.

Of the RESISTANCE, and the FORCE of FLUIDS.

451. *When a body moves in a fluid at rest, the resistance is as the square of the velocity.*

For the resistance is evidently as the velocity of the body drawn into the number of particles it strikes, or if v = the velocity, and n = the number of particles, the resistance will be as nv; but the number of particles struck in any time is as the velocity, therefore substituting v for n gives v^2, that is, the resistance is as the square of the velocity. And since action and reaction are equal, the force of a fluid moving against a body at rest, is as the square of its velocity.

Corol. If the body be a plane moving perpendicularly to its surface p, the resistance will be as pv^2. For the resistance or reaction of the fluid against an indefinitely small part of the plane is as v^2 or $1 \times v^2$, against double that part as $2 \times v^2$, &c. and therefore against the whole plane p, as $p \times v^2$. Also because the number of particles struck in any time is proportional to the density (d) of the fluid, the resistance will be as $d \times p \times v^2$, or dpv^2.

452. *If a plane* PL *move in the direction* PR *or obliquely against a fluid, and s = sine of* LPR *the angle of inclination (radius being* 1), *then supposing the resistance in a perpendicular direction to be dpv^2)* (as in the last corol.), *the resistance in the oblique direction is dpv^2s^2.*

Make LR perpendicular to PR. Then the number of particles struck by the plane moving in the perpendicular, and oblique positions, will be as PL to LR, or as radius to the *sine* of the angle LPR; hence the

expression dpv^2 when reduced in that proportion becomes dpv^2s; for $1 : s :: dpv^2 : \frac{dpv^2s}{1}$ or dpv^2s; therefore supposing LR a plane moving perpendicular to its surface, the resistance would be as dpv^2s. Let this resistance be represented by PR, and draw RO and OB perpendicular to PL and PR, respectively; then since the resistance perpendicular to LR is to the resistance in the direction RO, as *radius* to the *sine* of the angle LRO (=LPR) the inclination of RO to RL (319, corol. 4) we have *rad.* : PR :: *sin.*OPR : RO, or $1 : s :: \frac{dpv^2s}{1} : \frac{dpv^2s^2}{1^2}$, consequently RO will represent the resistance or the reaction of the fluid in a direction perpendicular to the plane PL; and hence by the resolution of forces, RB and BO will respectively be the resistances in the direction of, and perpendicular to the plane's motion, whence $1 : \frac{dpv^2s^2}{1^2}$ (or OR) :: s (or *sin.* BOR) : $\frac{dpv^2s^3}{1^3}$ or dpv^2s^3, (or RB), the resistance to the plane in the direction of its motion.

Corol. 1. Hence the resistances to LR and LP in the direction RP, are as the square of *radius* to the square of the *sine* of LPR, for $1^2 : s^2 :: \frac{dpv^2s}{1} : \frac{dpv^2s^3}{1^3}$.

Corol. 2. Let c = *cosine* of s, or *sine* of the angle BRO; then $1 : dpv^2s^2 :: c : dpv^2s^2c$ or BO, the resistance to the plane in the direction BO or perpendicular to the direction of its motion; this resistance therefore varies as s^2c, or as $(1^2-c^2)c$, because $s^2 = 1^2-c^2$.

Hence we may determine what will be the most advantageous angle the rudder of a ship can make with her way to bring her round. Let LP represent the top of the rudder, and LA or PR the direction in which the vessel LA moves; then the resistance to the rudder in a direction at right angles to LA or PR, that is, in the direction BO, must be the greatest possible, or

$(1^2-c^2)c=1^2c-c^3$, a *maximum*, which (Art. 406) will be when $c^2=\frac{1^2}{3}$; whence $c=\sqrt{\tfrac{1}{3}}=.57735$ the natural cosine of $54^\circ\ 44'$ = LPR the angle which the rudder must make with the ship's way to produce the greatest effect in turning her.

And in the same manner it is found that the wind blowing in the direction of the axis of a windmill, will have the greatest effect to turn the sails at the beginning of the motion, when it strikes them in an angle of $54^\circ\ 44'$.

433. *If a fluid moving with a given velocity v, act against a plane in a perpendicular direction, the real or absolute force on the plane is equal to the weight of a column of the fluid whose base is the plane, and height equal to the height through which a heavy body must descend from rest by its own gravity to acquire the velocity v.*

This is manifest from *art.* 426: for the weight or pressure of such a column of the fluid will generate the velocity v; the fluid therefore moving with that velocity must act with a force equal to the weight or pressure which generates it. And if the plane be urged with the velocity v against the fluid at rest, the resistance will be equal to that force of the moving fluid, because action and reaction are equal.

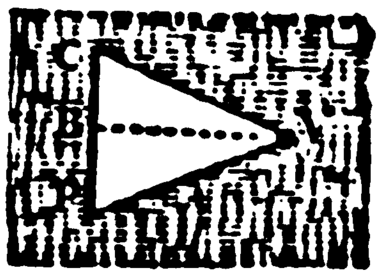
Thus suppose water to move at the rate of 10 feet $(=v)$ per second against a plane surface (p) 1 foot square, and let $d=62\frac{1}{2}lb.$ the weight of a cubic foot of water, and $s=16\frac{1}{12}$ feet; then $\frac{v^2}{4s}$ feet is the altitude due to the velocity v, or the height of the column, $\frac{pv^2}{4s}$ its cubic contents, and the weight or force $=\frac{dpv^2}{4s}=\frac{62\frac{1}{2}\times 1\times 100}{4\times 16\frac{1}{12}}=97lb.$ nearly, the force of the water against the plane, or the resistance to the plane if it moved perpendicular to its surface through the water at rest.

This force of resistance to the moving plane is not called the retarding force; for if the plane be the face of a body having

weight, its momentum, with the same velocity, will vary as its weight, hence the greater that weight, the less will be the retarding force, we therefore divide the resisting force by the weight of the body resisted, and the quotient is the retarding force. So if w = the weight of the body whose plane face is p, then $\frac{dpv^2}{4sw}$ will denote the retarding force, the motive or resisting force being $\frac{dpv^2}{4s}$.

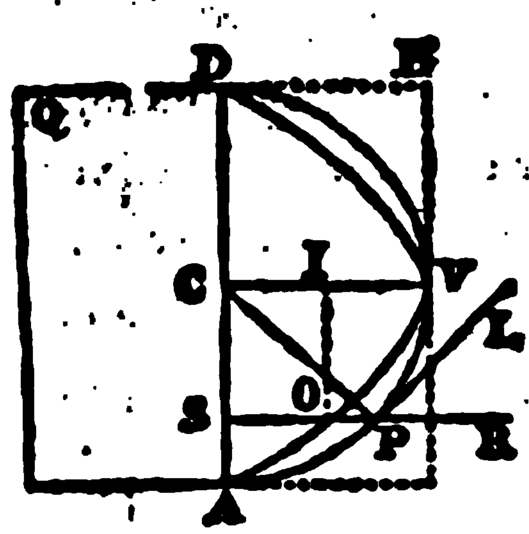
Corol. If the plane be inclined to the direction of its motion in an angle whose *sine* = s; then (452) by diminishing $\frac{dpv^2}{4s}$ in the triplicate ratio of *radius* to the *sine* of inclination s, we get $\frac{dpv^2s^3}{1^3\times4s}$ or $\frac{dpv^2s^3}{4s}$ the resistance to the plane in the direction of its motion.

454. *If a right cone* CVP *move against a fluid at rest with its vertex foremost in the direction of the axis* BV, *the resistance, to the resistance of a cylinder having an equal base* CP, *and moving also in the direction of its axis, will be as* PB^2 *to* PV^2.



For the same quantity of fluid is struck by the cone and cylinder, and every part of the cone's slant surface is inclined to BV, the direction of its motion, in the angle BVP; therefore (452, corol. 1.) the resistances will be as the square of the *sine* of BVP to the square of *radius*, that is, as PB^2 to PV^2.

455. *When a sphere and cylinder of equal diameters move in a fluid with the same velocity in the direction of the axes, the resistance to the sphere is but half the resistance to the cylinder.*



Let DA be the diameter of the sphere and the cylinder QA, and CV or SR at right angles to DA, the direction of motion. Make LP a tangent to the circle DVA at P, and from the centre C draw CP, and the angles LPR, PCS are equal. Then since the tangent LP and surface of the sphere are struck in the same direction RPS at the point P, the resistance to the sphere at that point, to the resistance at the point S on the face of the cylinder, will be as the square of the *sine* of the angle LPR (or SCP), to the square of *radius*, or (making CP *radius*) as SP^2 to CP^2 (459, corol. 1.). Let CV denote the resistance to any point S on the face of the cylinder, then CP^2 or $CV^2 : SP^2 :: CV : \frac{SP^2}{CV}$ the resistance to the corresponding point P on the sphere, being a third proportional to CV and the *sine* SP. On DA describe the parabola DVA about the axis CV, then $\frac{SP^2}{CV} = SO$:

For let OI be perpendicular to VC;

Then (268) $VC : VI :: CA^2 : IO^2$,

and $VC-VI(=SO):VC::CA^2-IO^2:CA^2(=VC^2)$ by division;

But $CA^2-IO^2=(CA+IO)(CA-IO)=DS\times SA=SP^2$, by prop. of the circle,

whence $SO : VC :: SP^2 : VC^2$, or $\frac{SP^2}{VC} = SO$: therefore the *locus* of the third proportionals SO, &c. or resistances to the semicircular arc DVA, is the parabola DVOA. Now if lines equal to CV are drawn parallel to CV from every point on the end or base of the cylinder, their sum together will denote the resistance to the cylinder, and the corresponding third proportionals on the same points, the resistance to the sphere; but the aggregate sum of the former lines constitute a cylinder BA whose length or height is CV, and all the latter a paraboloid of the same base and height, which in that case is equal to half the cylinder (305), consequently the resistance to the sphere is but half that to the cylinder.

Corol. Hence if p = the area of the great circle of the sphere or base of the cylinder, v = the velocity, $s = 16\frac{1}{12}$ feet; and d = the density or the specific gravity of the fluid; then (431) the resistance to the cylinder will be $\frac{dpv^2}{4s}$ and that to the sphere $\frac{dpv^2}{8s}$.

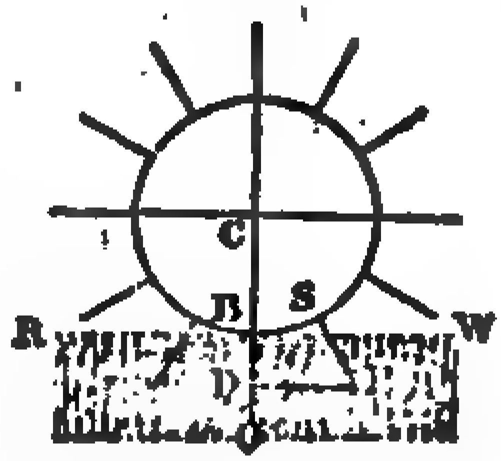
Thus suppose an 18 *lb.* iron-shot to be discharged with a velocity of 1500 feet per second, we have $p = .138326$ of a foot, nearly, $d = 1\frac{1}{4}$ ounces $= \frac{1}{12.8}$ *lb.* the weight of a cubic foot of air, and $v = 1500$; then $\frac{dpv^2}{8s} = 185$ *lb.* the resistance to the ball.

But the air rushes into empty space with a velocity not greater than between 1300 and 1400 feet per second (435, corol. 2.), the ball therefore moving at the rate of 1500 must leave a vacuum, or the air will cease to act by its pressure on the ball behind during a short space of time, and consequently, in addition to the above result, the ball will be resisted by the weight of a column of the air having a circular base whose diameter is that of the ball. The area of this circle is 19.948 inches, nearly, and allowing 15 *lb.* avoirdupois for the pressure of the atmosphere upon every square inch (432), we get $19.948 \times 15 = 299$ *lb.* which added to 185 *lb.* gives 484 *lb.* the resistance, exclusive of the resistance arising in consequence of the air's being condensed in front by the rapid motion of the ball. And hence it is found that the horizontal ranges are not increased by discharging balls with initial velocities beyond a certain limit.

The theory of Resistances however, and experiment give results considerably different. See Dr. Hutton's Mathematical and Philos. Dictionary, vol. 2, p. 363, &c.

436. *Let a stream of water moving in the direction* WR, *with a velocity* = V, *turn an undershot wheel whose centre*

is C; *then if* v = *the velocity of the pallets or floats* SW, BO, &c. *and* m = *the momentum of the wheel,* m *will vary as* $(V-v)^2 \times v$, *or* $m \propto (V-v)^2 \times v$.



Since the water and floats move with the respective velocities V and v, the former will strike the latter with the relative velocity $V-v$, or the impingent velocity of the water upon the floats is the same as it would be if they were at rest and the water moved with the velocity $V-v$, and hence if the water strike the floats perpendicular to their surfaces, its force (451) will be as $(V-v)^2$.

But the absolute force of the water moving with the velocity $V-v$ is equal to the weight of a column of the fluid having a base equal to the surface of a pallet or float, and whose height is equal to the height through which a heavy body must descend by gravity to acquire that velocity (453), this weight may therefore be represented by the force $(V-v)^2$, whence it follows that the momentum of the wheel will be directly proportional to that weight or force drawn into the velocity v, or to $(V-v)^2 \times v$.

In estimating the force of the stream upon the floats, we take a column of the fluid whose base is equal to the surface of one float only, because the section of the impinging stream which is perpendicular to the direction of its motion is equal to that surface.

Corol. 1. Hence if the velocity of the stream be given, we can determine that of the wheel when its effect is the greatest possible in a given time; for in that case $(V-v)^2 \times v$ must be a maximum; but (406, corol. 1), $(V-v)^2 \times v$ is a maximum

when $v = \frac{1}{2}(V - v)$, hence $v = \frac{1}{3}V$, that is, the velocity of the wheel $= \frac{1}{3}$ the velocity of the stream.

Corol. 2. And since the whole force of the impingent water, is to its force on the floats, as V^2 to $(V - v)^2$, or as V^2 to $(\frac{2}{3}V)^2$, that is, as 1 to $\frac{4}{9}$, therefore the resistance of the wheel, including friction, &c. when its effect is a maximum, will be $\frac{4}{9}$ of the resistance which would be just sufficient to counteract or balance the whole force of the water.

SCHOLIUM. The above conclusion is deduced by Maclaurin, Atwood, and other writers who have considered the subject. But it appears from experiments made by Mr. Smeaton, that the maximum effect of a water-wheel is when its velocity, instead of being $\frac{1}{3}$, is nearly equal to $\frac{1}{2}$ the velocity of the fluid. This disagreement it seems, induced a writer (Mr. W. Waring) in the 3d. vol. of the Transactions of the American Philosophical Society, to reject the preceding theory as fallacious, and adopt another founded on the following principle, namely, "that while the stream is invariable, whatever be the velocity of the wheel, the same number of particles or quantity of the fluid, must strike it some where or other in a given time"; and hence it is infered that the force of the stream upon the wheel is in the "simple direct proportion of the relative velocity"; hence (retaining the above notation), the momentum of the wheel will be as $(V-v)\,v$; now if V be a quantity or line divided into two parts $V - v$ and v so, that their rectangle $(V - v)\,v$ is a maximum, it follows from *art.* * 239, that the two parts are equal, or $v = \frac{1}{2}V$, that is, the velocity of the wheel $= \frac{1}{2}$ that of the fluid. Which is Mr. Waring's conclusion.

Let the circumference described by the floats be 30 feet, the number of floats $= 30$, and the surface of each $= 1$ foot square; also suppose the velocity of the stream $= 31$ feet per second, and that of the wheel $= 1$ foot; then the quantity of water that strikes a float or floats in one second of time will be 30 cubic feet.

Again, if we conceive the wheel to move round once in a second, the motion of the stream will be $1\frac{1}{10}$ feet, and that of a float 1 foot in $\frac{1}{10}$ of a second, and the quantity of fluid that strikes a float in that time is $1\frac{1}{10} - 1$ or $\frac{1}{10}$ of a cubic foot; now when the lowest float BO is perpendicular to the direction of the stream, four other floats are partly in the water; let us however, suppose that the whole surfaces of 4 floats are struck at the same time, then the quantity of fluid that strikes 1 float in every revolution will be $\frac{1}{10} \times 4$ of a cubic foot, which multiplied by 30 the number of floats gives only 4 cubic feet the quantity of fluid impinging on all the floats in one second of time or during a revolution of the wheel: the difference of the two results is 26 cubic feet: it therefore appears, that the number of particles or quantity of fluid which strikes in a given time will depend on the relative velocity; and consequently Mr. Waring's principle (quoted above) must be erroneous. This is also evident from the following consideration, that the velocity of a body, after being struck by running water, may become equal to that of the stream, in which case the body floats without being struck by the fluid.

By increasing the number of pallets on the wheel, the number constantly moving in the water will also be increased, but it does not follow that more surface would be struck, or the velocity of the wheel thereby accelerated; for the number of floats upon a wheel of any diameter may be augmented till its motion in consequence becomes actually diminished.

But in computing the velocity of the wheel according to the common theory, we estimate the force of the water too great by supposing the floats are constantly struck perpendicular to their surfaces, for the direct impact, which can only take place upon a float when in the position BO, is momentary. The particles of water are also conceived to act in succession without impediment, but it is not easy to comprehend how that can actually take place, because the particles in immediate contact with the

floats have not room to escape before they are struck by those which follow; the force of the stream therefore seems to be compounded of pressure and percussion. Now these circumstances all tend to sh.w (what Mr. Smeaton's experiments prove) that the actual force of a stream upon a water-wheel is less than that deduced from the common theory.—We have not considered the effect of friction, because no general rule has yet been devised for that purpose.

ADDITIONAL EXAMPLES

IN

THE APPLICATION OF ALGEBRA, CONIC SECTIONS, MECHANICS, HYDROSTATICS, &c.

1. GIVEN the area of a rectangle $= a$, and the ratio of the sides as m to n; to find the sides.

Ans. $\sqrt{\frac{ma}{n}}$, and $\sqrt{\frac{na}{m}}$.

2. If a rectangle be inscribed in a circle whose diameter $= d$, what are the sides when they have the ratio of m to n?

Ans. $d\sqrt{\frac{m^2}{m^2+n^2}}$, and $d\sqrt{\frac{n^2}{m^2+n^2}}$.

3. If the side of a square be denoted by s, what is the length of that line, drawn from the middle of one of its sides, which divides the area into two parts having the proportion of 2 to 1?

Ans. $\frac{1}{3}s\sqrt{10}$.

4. What is the length of a line drawn from an angle of a rectangle whose sides are S and s that divides the area into two parts having the ratio of 2 to 1?

Ans. $\frac{2}{3}\sqrt{(2\frac{1}{4}S^2+s^2)}$. Or $\frac{2}{3}\sqrt{(2\frac{1}{4}s^2+S^2)}$.

5. If the circumference of a circle and the perimeter of a square are equal, which contains the greatest area?

Ans. The circle is to the square as 16 to 12.56637, nearly.

6. If a sphere and cube have equal surfaces, which has the greatest cubic content?

Ans. The sphere to the cube as $\sqrt{6}$ to $\sqrt{3.14159}$ &c.

7. If the three perpendiculars let fall from a point within an equilateral triangle upon the sides, are denoted by a, b, c; what is the side of the triangle?

$$Ans. \ \frac{2a+2b+2c}{\sqrt{3}}.$$

8. What plane triangle is that, the natural tangents of whose angles are whole numbers?

Ans.

9. Given the base of a triangle $= b$, the angle opposite the base, and the right line drawn from that angle to bisect the base $= l$; to find the perpendicular.

$$Ans. \ \frac{l^2}{b} - \tfrac{1}{4}b.$$

10. If the base and perpendicular of a triangle are denoted by b and p; what is the side of that inscribed square, one side of which coincides with the base?

$$Ans. \ \frac{bp}{b+p}.$$

11. If the sides of a triangle are 28, 25, and 17; what is the side of its greatest inscribed square?

Ans. $10\frac{50}{709}$.

12. If the base and sides of a triangle are denoted by b, S, and s; then what are the expressions for the perpendicular, and segments of the base?

$$Ans. \text{ Perp.} = \sqrt{\left(\frac{S^2+s^2}{2} + \frac{S^2s^2}{2b^2} - \frac{S^4+s^4+b^4}{4b^2}\right)}.$$

$$\text{Greater segm.} = \frac{b^2-s^2+S^2}{2b}. \quad \text{Less} = \frac{b^2+s^2-S^2}{2b}.$$

13. Having observed the elevation of a distant object, I advanced 60 yards directly towards it on a level ground, and then observed the elevation to be the complement of the former to a right angle; advancing 20 yards still nearer, the elevation

now appeared to be just double the first. Hence the height of the object is required?

Ans. 74.16 *yards.*

14. If the radius of a circle be 10, what are the sides of an inscribed triangle when they have the proportion of 2, 3, and 4?

Ans. 9.6824
14.5237
19.3649

15. If r = the radius of a circle, what are the sides of the regular inscribed trigon, tetragon, pentagon, hexagon, octagon, and duodecagon?

Ans. Trigon $r\sqrt{3}$. Octagon $r\sqrt{(2-\sqrt{2})}$.
Tetragon $r\sqrt{2}$. Decagon $r(\frac{1}{2}\sqrt{5}-\frac{1}{2})$.
Pentagon $r\sqrt{(2\frac{1}{2}-\frac{1}{2}\sqrt{5})}$. Duodecagon $r\sqrt{(2-\sqrt{3})}$.
Hexagon r.

16. If the side of a regular trigon, tetragon, pentagon, hexagon, octagon, decagon, and duodecagon be denoted by s, the expressions for their areas are

Trigon $s^2\sqrt{\frac{3}{16}}$. Octagon $s^2(2+\sqrt{8})$.
Tetragon s^2. Decagon $s^2\sqrt{(\frac{125}{4}+\sqrt{\frac{3125}{4}})}$.
Pentagon $s^2\sqrt{(\frac{25}{16}+\sqrt{\frac{125}{64}})}$. Duodecagon $s^2(6+\sqrt{27})$.
Hexagon $s^2\sqrt{\frac{27}{4}}$.

Required the investigations?

17. Let the linear side or side of a face of a tetraedron, hexaedron, octaedron, dodecaedron, and icosaedron (the 5 regular bodies or solids), be denoted by s; then the expressions for their cubic contents will be

Tetraedron $s^3\sqrt{\frac{1}{72}}$. Dodecaedron $s^3\sqrt{\frac{235+\sqrt{55125}}{8}}$.
Hexaedron s^3.
Octaedron $s^3\sqrt{\frac{2}{9}}$. Icosaedron $s^3\sqrt{\frac{175+\sqrt{28125}}{72}}$.

N. B. The Tetraedron has 4 equilateral triangular faces.
Hexaedron or cube, 6 equal square faces.

Octaedron, 8 equal equilateral triangular faces.
Dodecaedron, 12 equal regular pentangular faces.
Icosaedron, 20 equal equilateral triangular faces.
These solids can be inscribed in a sphere.

18. If the length, breadth, and depth of a rectangular parallelopiped are denoted by l, b, and d; what is the diameter of its circumscribing sphere?

Ans. $\sqrt{(l^2 + b^2 + d^2)}$.

19. If the perimeter of a triangle be denoted by p, and the three perpendiculars let fall from the angles upon the opposite sides by a, b, and c; what are the expressions for the sides?

Ans. $\frac{pab}{ab+ac+bc}$. $\frac{pac}{ab+ac+bc}$. $\frac{pbc}{ab+ac+bc}$.

20. In any trapezium, the sum of the squares of the two diagonals is equal to twice the sum of the squares of the two lines joining the middle points of the opposite sides of the trapezium. Required the demonstration?

21. Having the base of a triangle $= b$, the perpendicular upon that side $= p$, and the rectangle of the other two sides $= r$; to find the angle opposite the base.

Ans. $\frac{bp}{r} =$ the *natural sine* of the required angle, (*radius* being 1).

22. Let the base of a triangle $= b$, the tangent of the opposite angle $= t$, and the perpendicular let fall from that angle upon the base $= p$; to find the segments of the base made by that perpendicular.

Ans. $\frac{1}{2}b + \sqrt{\left(\frac{bp}{t} - p^2 + \frac{1}{4}b^2\right)}$, and $\frac{1}{2}b - \sqrt{\left(\frac{bp}{t} - p^2 + \frac{1}{4}b^2\right)}$.

23. If a, b, and c denote the sides of a triangle, what is the radius of its inscribed circle?

Ans. $\sqrt{\frac{(h-a)(h-b)(h-c)}{h}}$, where $h = \frac{1}{2}$ the sum of the three sides.

24. Given the base and vertical angle of a triangle; to find the *locus* of the centre of the inscribed circle.

Ans. The arc of a circle.

25. If the hypothenuse of a right-angled triangle $= h$, and the radius of its inscribed circle $= r$; what are the sides?

Ans. $\frac{1}{2}h+r+\sqrt{(\frac{1}{4}h^2-hr-r^2)}$, and $\frac{1}{2}h+r-\sqrt{(\frac{1}{4}h^2-hr-r^2)}$.

26. If $r =$ the radius of three equal circles in contact with each other; what are the radii of the two circles described to touch them internally and externally?

Ans. $2r\sqrt{\frac{1}{3}} - r$, and $2r\sqrt{\frac{1}{3}} + r$.

27. If $r =$ the rectangle made by two lines, and $d =$ the difference of their squares: what are those lines?

Ans. $\sqrt{(\sqrt{(r^2+\frac{1}{4}d^2)} + \frac{1}{2}d)}$, and $\sqrt{(\sqrt{(r^2+\frac{1}{4}d^2)} - \frac{1}{2}d)}$.

28. Let the perimeter of a right angled triangle $= p$, and its area $= a$; to find the base and perpendicular.

$$Ans.\ \frac{p^2+4a}{4p} \pm \frac{1}{4p}\sqrt{(p^4 - 24p^2a + 16a^2)}.$$

29. If the perimeter of a rectangle $= p$, and its diagonal $= d$; what are the sides?

Ans. $\frac{1}{4}p \pm \frac{1}{4}\sqrt{(8d^2 - p^2)}$.

30. If S and s denote the segments of the base made by a perpendicular let fall from the vertical angle of a triangle, and $r =$ the rectangle under the two sides containing that angle; what is the perpendicular?

$$Ans.\ \left(\sqrt{r^2 + \left(\frac{S^2-s^2}{2}\right)^2} - \frac{S^2+s^2}{2}\right)^{\frac{1}{2}}.$$

31. If the three perpendiculars let fall from the angles of a plane triangle upon the opposite sides, are denoted by a, b, and c; what are the sides?

Ans. $\frac{1}{3}\sqrt{(8a^2+8b^2-4c^2)}$, $\frac{1}{3}\sqrt{(8a^2+8c^2-4b^2)}$, & $\frac{1}{3}\sqrt{(8b^2+8c^2-4a^2)}$.

32. If the three lines drawn from the angles of a plane triangle to bisect the opposite sides, be denoted by a, b, and c; then what are the expressions for the sides of the triangle?

Ans. $\frac{2}{3}\sqrt{(2b^2+2c^2-a^2)}$, $\frac{2}{3}\sqrt{(2a^2+2c^2-b^2)}$, & $\frac{2}{3}\sqrt{(2a^2+2b^2-c^2)}$.

33. To divide an angle whose sine and cosine are denoted by s and c (the radius being 1,) into two parts such, that their sines may have the given ratio of m to n.

Ans. $\frac{sn}{\sqrt{(s^2n^2+(m\pm cn)^2)}}$ = *sine* of less; and $\frac{sm}{\sqrt{(s^2n^2+(m\pm cn)^2)}}$ = *sine* of greater. Where the sign + takes place when the proposed angle is acute, but — when it is obtuse.

34. If the hypothenuse of a right angled triangle $= h$; what are the other sides when the area is the greatest possible?

Ans. Each side $= h\sqrt{\frac{1}{2}}$.

35. What are the sides of the greatest rectangle that can be inscribed in a semicircle, the radius of the circle being denoted by r?

Ans. $r\sqrt{\frac{1}{2}}$, and $r\sqrt{2}$.

36. What is the area of that right angled triangle whose base, perpendicular, and hypothenuse are denoted by x^r, x^{2r}, and x^{3r}, respectively?

Ans. 1.02908, nearly.

37. Given the area of a triangle $= 126$, the sum of the three sides $= 54$, and the sum of their squares $= 1010$. Required the sides?

Ans. 13, 20, and 21.

38. In 4 sides of a regular pentagon traced out for a fortification, stand 4 objects which are found to be at the angular points of a square; now if the side of the pentagon be 180 fathoms, what is the side of that square?

Ans. 190.89 *fath.*

39. A lead ball d inches in diameter is to be cast into two other balls whose diameters are in the given ratio of m to n. Required those diameters.

$$Ans. \ \frac{dm}{(m^3 + n^3)^{\frac{1}{3}}}, \text{ and } \frac{dn}{(m^3 + n^3)^{\frac{1}{3}}}.$$

40. A square piece of ground whose side = 30 yards is to be surrounded by a ditch dug 6 feet deep, and it is necessary that the earth thrown out should be sufficient to raise the interior surface 4 feet higher than the present level; now what must be the breadth of the ditch at bottom, supposing it the same all round, when the slope on each side is 45°, and the inner slope continued up to the new made surface?

Ans. 5.488 *feet.*

41. To determine the height of a hill we observed the elevations of an object on its summit at three stations A, B, and C, in the same horizontal right line, and found them to be 2° 45′¼, 3° 39′¼, and 7° 47′, respectively; the distance from A to B was 900 yards, and that from B to C 750 yards. Hence the height is required?

Ans. 171 *yards*, nearly.

42. Three detachments of foot having orders to occupy a certain post, begin their march from three towns, A, B, and C, at 6 o'clock in the morning; the detachment from A march 4 miles per hour, that from B march 3 miles per hour, and the other from C march 2 miles per hour, and they all arrive at the place of destination exactly at the same time, which was between 10 and 11 o'clock; now the distance from A to B was 15 miles, from B to C 8 miles, and from A to C 20 miles. Hence the distances from the post to the three towns are required.

Ans.	17.6858 *miles*	from A.
	13.2643	from B.
	8.8429	from C.

43. In the course of a survey, at a station on the top of a hill, we took the depressions of three objects, A, B, and C, which were nearly on the same horizontal level, and found them to be $4^\circ\ 59'$, $4^\circ\ 30'\frac{1}{2}$, $7^\circ\ 13'\frac{1}{4}$, respectively; now the distance from A to B was 4 miles, from B to C $3\frac{1}{4}$ miles, and from C to A 3 miles. Hence the perpendicular height of the hill is required?

Ans. 329 *yards*, nearly.

44. If the axes of an ellipse be 60 and 80; what are the lengths of two conjugate diameters, the longest of which makes an angle of 20° with the transverse axis?

Ans. 68.7861, and 79.1737.

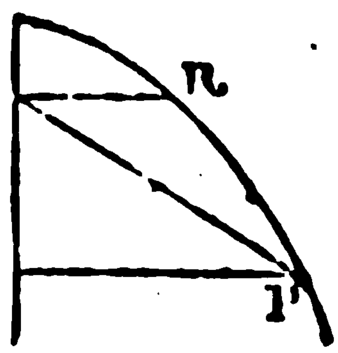
45. Let the axes of an ellipse be 60 and 100 inches; to find the radius of a circle described to touch the curve when its centre is in the transverse axis at the distance of 16 inches from that of the ellipse.

Ans. 27.49545 &c. *inches*.

46. Let VO be the axis of any conic section VRP, F the focus, and FR, OP, two ordinates at right angles to VO; then

$$FP = VF + \frac{(FR - FV)\,VO}{VF}.$$

Required the investigation?



47. If the axes of an ellipse be 80 and 60 yards; what are the areas of the two segments into which it is divided by a line drawn parallel to the conjugate axis at the distance of 10 yards from the centre?

Ans. 1291.27 and 2478.63 *yards*, nearly.

48. If the base of a triangle be given, and also the sum of the squares of the other two sides; what is the *locus* of the vertex of the triangle?

Ans. The arc of a circle.

49. In a triangle, if the base, and the difference of the other

two sides, are given; what is the *locus* of the vertex?

Ans. An hyperbola.

50. Let the base, and the difference of the two angles at the base of a triangle, be given; required the *locus* of the vertical angle in that case?

Ans An hyperbola.

51. Suppose a person, the height of whose eye is 5 feet 6 inches, while standing on a level floor, holds a 9 *lb.* iron shot in his hand so, that its centre is 15 inches from the eye, and 4 feet 9 inches from the floor; now how much of the floor's surface is hid by the shot from the eye; the shot's diameter being 4 inches?

Ans. 8.36344 *feet* square.

52. A heavy body was observed to descend freely from rest, by its own gravity, from the top of a tower to the bottom in $4\frac{1}{4}$ seconds of time; required the tower's height?

Ans. $72\frac{3}{4}$ *feet.*

53. From what height must a heavy body descend by its own weight to acquire a velocity of 1000 feet per second, supposing the air to be without resistance?

Ans. 15625 *feet.*

54. Suppose a heavy body to fall from the height of a mile above the earth's surface, with what velocity would it strike the ground, and what would be the time of descent?

Ans. Velocity = 582.8 *feet* per second.
Time = 18.12 *sec.* nearly.

55. If a heavy body descend $\frac{1}{3}$ of the whole distance fallen in the last second of time; at what height did it commence its motion?

Ans. 477.6 *feet,* nearly.

56. If a ball with a velocity of 1000 feet per second, enters a block of wood to the depth of 10 inches, what will be the velocity of the ball when the penetration is 16 inches, supposing the resistance of the wood to be uniform?

Ans. 1265 *feet* per second.

57. A quiescent body *C* is struck at the same instant of time by two other bodies *A* and *B* with forces that would separately carry it forward in the directions AC and BC at the rate of 15 and 10 feet per second, respectively; required the velocity and direction of *C* after the impact, if the directions of *A* and *B* form an angle at C of 70°?

Ans. Velocity $= 20.7$ *feet* per second.
Direction 27° 1¾' with that of *A*, or 42° 58'¼ with the direction of *B*.

58. Let the directions of *A* and *B*, and the force of *A* continue as before, and suppose after the impact that *C* moves with a velocity of 30 feet per second; required its direction, and the velocity which *B* would communicate alone?

Ans. Direction 28° 1'¾ with that of *B*, or 41° 58'¼ with *A*.
Velocity which *B* would produce alone $= 21.36$ feet per second.

59. If the velocities communicated to *C* by *A* and *B* when acting separately, and together, are respectively as 5, 4, and 3; what is the angle formed by the two directions in which *A* and *B* move when they act together?

Ans. 143° 7'⅘.

N. B. In this and the two preceding examples, the bodies are supposed to be globular, and the points of impact in the lines joining the centres.

60. Suppose the weight *C*, *art.* 320, corol. 2, to be 60 *lb.* what are the tensions of the cords AB, GB, and BC, or the forces with which they are stretched?

Ans. 48, 36 and 60 lbs.

61. Let a ring of metal weighing 8 *lb.* slide freely on a string 5 feet long whose ends are fastened to two tacks 2 feet asunder in a line making an angle with the horizon of 45°; to find the stress or force on each tack when the ring rests in equilibrio.

Ans. 4.415 *lb.* on each tack.

62. Suppose the ends of a thread 10 feet long be fastened to two tacks in the same horizontal line, at the distance of 6 feet; where must two weights, the one 3, and the other 5 ounces, be fixed to the thread, so as to hang at rest in the same horizontal line at the distance of three feet from the level of the tacks?

Ans. At 3.1479, and 2.395 *feet* from the ends of the thread.

63. Suppose a 12 *lb.* shot moving with a velocity of 1000 feet per second, to meet another of 9 *lb.* whose motion in an opposite direction is at the rate of 1200 feet per second; what is the velocity after congress, if the balls are non-elastic?

Ans. 57⅐ *feet* per second.

64. With what velocity must a 6 *lb.* shot meet another of 24 *lb.* that is moving at the rate of 400 feet per second, so as just to stop it; the balls being non-elastic, and the stroke in the direction of the centres?

Ans. 1600 *feet* per second.

65. A body at rest, but not fixed, when struck by a musket bullet weighing 1 ounce, moved with a velocity of 6 feet per second; now if the body weighed 10 *lb.* what was the velocity of the bullet?

Ans. 966 *feet* per second, if the bullet and body were non-elastic.

66. With what velocity will a 32 pounder recoil when discharged horizontally, if the gun and carriage together are 63 hundred weight, and the initial velocity of the ball = 1800 *feet* per second of time?

Ans. 65 *inches* per second, nearly.

67. Required the ratio of the masses of two elastic balls *A* and *B*, so that *A* striking *B* at rest shall lose ¼ of its velocity?

Ans. As 7 to 1.

68. Suppose two cannon shot, one 18 *lb.* the other 12 *lb.* when moving in the same plane, to strike one another in an angle of 80° with the respective velocities of 600 and 1000 feet per second; requuired the velocities and directions after the impact, if the balls are non-elastic.

Ans. Velocity of the greater shot 615 *feet* per second.
of the less 768.
Change in the direction of the greater shot 38° 18'½.
in the less 48° 45'⅔.

N. B. The tangent to the two balls at the point of impact is supposed to bisect the given angle 80°.

69. If a cannon ball be discharged from the top of a tower 80 feet high, with an initial velocity of 1500 feet per second, at what distance from the tower will it strike the ground, the elevation of the piece above the horizontal line being 20° 5', and the air supposed to be without resistance?

Ans. 45335 *feet.*

70. Required the elevation of a mortar to hit an object distant 7333 feet on a plane depressed 11°; the greatest horizontal range being 8190 feet?

Ans. 63° 23'½, or 15° 36'½.

71. If the horizontal range of a shell be 2000 feet when projected at an elevation of 30°; what is the time of flight, and the greatest height to which the shell ascends?

Ans. Height = 287½ *feet.* Time = 8½ *sec.*

72. If the impetus be 4000 feet, what must be the elevation of a mortar to hit an object whose distance on the horizontal plane is 5600 feet, and height above that plane 812 feet?

Ans. 63° 45', or 32° 30'.

73. A shell being thrown from a mortar at an elevation of 30°, the report of its fall on the horizontal plane was heard at the mortar just 20 seconds after the explosion. Hence the range is required?

Ans. 6033 *feet.*

74. The random of a piece on the plane of the horizon with a given charge of powder at an elevation of 30° being 1500 yards; to find the elevation when planted at 44 yards above the level of the horizon, so that the ball may fall at the greatest distance possible.

Ans. 44° 17'½.

75. A shell discharged at an elevation of 45° struck an object 30 yards above the horizon; required the distance of the object from the mortar, the horizontal range of the shell being 2000 yards.

Ans. 700, or 1300 *yards.*

76. A tower built on level ground is 65 feet high, now at what distance must I stand with a musket to hit an object on the top with the greatest force, the musket being held 5 feet above the ground?

Ans. At 60 *feet* from the tower.

77. In what time would a heavy body descending freely on a plane inclined to the horizon in an angle of 40° acquire a velocity of 100 feet per second?

Ans. 4.837 *seconds.*

78. A body descending freely by its own weight on an inclined plane whose length is 484 feet, descends 123 feet in the last second of time; required the plane's inclination to the horizon?

Ans. 34° 1'½.

79. A cylinder was observed to *roll* down a plane 400 feet long in 16 seconds of time; required the plane's inclination to the horizon, the cylinder having descended by its own gravity?

Ans. 8° 21'½.

80. If one end of a beam 20 feet long be 6 feet higher than the other end, what force acting in direction of the beam would keep a weight of one ton laid upon it from sliding down, supposing the friction between the weight and beam is equal to half the necessary force?

Ans. A force equal to 3 hundred weight.

81. If a man can draw a weight of 84 *lb.* up the side of a perpendicular wall 10 feet high, what weight will he be able to raise along a plank 20 feet long laid aslope from the top of the wall, the resistance from friction on the plank being equal to $\frac{1}{4}$ of the weight so raised?

Ans. 126 *lb.*

82. Two planes HV and OV whose lengths are 5 and 3 feet, respectively, meet at V above the horizontal line HO, and two weights *A* and *B* connected by a string passing over a pulley at V, are in equilibrio on the planes; now the pressure of *A* upon the plane HV is double that of *B* against OV. Hence the height of V above the horizon HO is required?

Ans. $\sqrt{\frac{11}{3}}$ *feet.*

83. Suppose two weights, one of 6, the other of 2 pounds, to be suspended upon a pin by means of a string, to determine how far the greater will descend, and the other ascend in 1 second of time, neglecting the friction on the pin.

Ans. $8\frac{1}{24}$ *feet.*

84. If a pendulum vibrating in an arc of 24° be 40 inches long, what is its velocity at the lowest point, supposing a body descends $16\frac{1}{12}$ feet in the first second of time?

Ans. 11.68 *inches* per second.

85. If the distance from the point of suspension to the centre of oscillation of a pendulum be 3 inches, how many vibrations will it perform in a minute, in the latitude of London?

Ans. 217 nearly.

86. What must be the length of a pendulum to vibrate only 40 times in a minute?

Ans. 88$\frac{1}{5}$ *inches.*

87. If a slender uniform rod 4 feet in length, be suspended at one end, and made to vibrate in small arcs, how many times will it oscillate in a minute?

Ans. 66.38 nearly.

88. What weight can a man raise with a handspike or iron crow 8 feet long, if the fulcrum or prop is 5 inches from one end, and he presses with a force equal to 150 *lb.* at the other?

Ans. 2730 *lb.*

89. If one arm of a steel-yard is 3 inches, what must be the length of the other that a counterpoise of 10 *lb.* may be sufficient to weigh a hundred weight, supposing the weight of the instrument itself is not considered in the account?

Ans. 33.6 *inches.*

90. The cylinder or axle over a common draw-well is 3 inches in diameter, the rope $\frac{3}{4}$ of an inch in diameter, and the handle describes a circle 30 inches in diameter; now what weight can a man draw up who acts with a force equal to 40 *lb*?

Ans. 320 *lb.*

91. Which is drawn with the least force on a rough uneven road, a carriage having wheels of 3 feet in diameter, or one with wheels that are 5 feet in diameter?

Ans. The advantage in favour of the greater wheels is as 5 to 3.

92. If the screw of a press be turned with a lever 7 feet long, and the threads of the screw are 1 inch asunder; what is the force of the press when the power at the end of the lever is = 100 *lb.* supposing the screw to act without friction?

Ans. 52779 *lb.*

93. A man with a combination of pullies raises a heavy body $1\frac{1}{4}$ inches at every pull which draws the rope 36 inches; now what is the weight of the body if he pulls with a force equal to 80 *lb*?

Ans. 1920 *lb.*

94. A barrel of gunpowder weighed 81 *lb.* in one scale, but when put into the opposite scale, it was found to weigh only 78 *lb.* $12\frac{1}{2}$ *oz.* Hence the true weight is required?

Ans. 79 *lb.* 14 *oz.*

95. Three inches from one end of a cylindrical pole is hung a weight of 30 *lb.* the pole is 8 feet long, and weighs 10 *lb.* now how far from that end must I place the pole on my shoulder to carry the weight with the most ease?

Ans. $14\frac{1}{4}$ *inches.*

96. What must be the length of a cylinder, the diameter of whose base is a yard, so that it may just stand by its own weight on sloping ground which rises 1 yard in 10?

Ans. $\sqrt{99}$ *yards.*

97. To find the centre of gravity of a quadrangular board of uniform thickness, two adjacent sides being 17 inches each, the other two 14 inches each, and the shortest diagonal = 16 inches.

Ans. 13.8297 *inches* from the sharpest corner.

98. A beam of timber 20 feet long is to be supported in an horizontal position by two props; the ends of the beam are squares whose sides are 2, and 3 feet, respectively; now if one prop stands 4 feet from the greatest end, at what distant from the less end must the other be to bear an equal weight?

Ans. $6\frac{1}{1}\frac{1}{9}$ *feet.*

99. To determine the weight of a tapering beam of timber 20 feet long, we found that it rested in an horizontal position on a prop or fulcrum 16 feet from the less end, but when the middle

of the beam was brought over the prop, it required the weight of a man, which was 200 *lb.* at the less end to keep it in equilibrio. Hence the weight is required?

Ans. 2000 *lb.*

100. The weight of a ladder 20 feet long is 70 *lb.* and its centre of gravity 11 feet from the less end; now what weight will a man sustain in raising this ladder when he pushes directly against it at the distance of 7 feet from the greater end, and his hands are 5 feet above the ground?

Ans. 63 *lb.* nearly.

101. If the quantity of matter in the moon, be to that of the earth, as 1 to 39, and the distance of their centres 240000 miles; where is their common centre of gravity?

Ans. 6000 *miles* from the earth's centre.

102. Supposing the *data* as in the last question, to find the distance from the moon in the line joining the centres, where a body would be equally attracted by the earth and moon; the force of attraction in bodies being directly as the quantities of matter, and inversely as the squares of the distances from the centres.

$$Ans.\ \frac{240000}{1+\sqrt{39}} = 33126\tfrac{1}{4}\ miles,\ \text{nearly}.$$

103. If two fires, one giving 4 times the heat of the other, are 6 yards asunder; where must I stand directly between them to be heated on both sides alike; the heat being inversely as the square of the distance?

Ans. 2 *yards* from the less fire, or 4 from the greater.

104. To what height above the earth's surface should a body be carried to lose $\frac{1}{10}$ of its weight; the earth's radius being 3970 miles, and the force of gravity inversely as the square of the distance from its centre?

. 214½ *miles.*

105. How far beneath the surface should the body be to lose $\frac{1}{10}$ of its weight, the force of gravity, in that case, being directly as the distance from the centre?

Ans. 397 *miles.*

106. If a line $= l$, be drawn from a point P to the centre of a circle whose diameter $= d$, and they revolve together about the point P, the circle, moving perpendicular to its plane, will generate a ring (like the ring of an anchor); required its solid content?

Ans. Let $c = 3.1416$, then $\frac{1}{2}lc^2d^2 =$ the content.

107. Suppose the point P to be at the circumference of the circle, or let the circle revolve about a tangent to its circumference as a fixed axis, then what is the content of the generated solid?

Ans. $\frac{1}{4}c^2d^3$.

108. Let a semicircle revolve about the tangent parallel to its diameter; required the content of the solid in that case.

Ans. $d^3\left(\frac{1}{8}c^2 - \frac{1}{6}c\right)$.

109. If a spar of wood 2 inches broad, and ½ an inch thick, will bear 50 *lb.* with its broadest side horizontal; what would it support when that side is vertical?

Ans. 200 *lb.*

110. A spar of oak when resting on its ends in an horizontal position will bear 200 *lb.* at a certain point; now what weight will it support (at the same point) when it is inclined to the horizon in an angle of 60°?

Ans. 400 *lb.*

111. Let $a =$ the magnitude of a mass or ingredient, and A = its weight.

$b =$ the magnitude of another ingredient, $B =$ its weight.

$m =$ the magnitude of any mixture or mass of both, and $M =$ its weight:

Then $\frac{Mba - Bma}{Ab - Ba}$, and $\frac{Abm - Mba}{Ab - Ba}$ will be the respective magnitudes of the ingredients in the compound. Required the investigation?

112. If n = the cubic feet in a mass of metal whose specific gravity or the number of ounces in a cubic foot $= m$, the specific gravity of wood $= d$, and the specific gravity of water $= w$; then $\frac{(m - w)\,n}{w - d}$ = the cubic feet of wood that will just float the metal. For example, 179.187 cubic feet of deal will float a cast-iron cannon of 52 hundred weight, in fresh water. Required the investigation?

113. How many empty 54 gallon casks (beer measure) when immersed in sea water, would float a brass cannon weighing 18 hundred weight, supposing the casks are water-tight, made of oak, and the weight of each = 50 *lb*?

Ans. 3.216, or 3 fifty-four gallon casks, and another that holds about 11 gallons.

114. If 4 *lb*. of fine silver, 6 *lb*. of copper, and 9 *lb*. of tin, are melted together, what is the specific gravity of the composion?

Ans. $8644\frac{2}{11}$.

115. If the weight of a shell 12¾ inches in diameter, be 198 *lb*. what is its thickness?

Ans. 2.02 *inches*.

116. If a sphere of wood 9 inches in diameter sinks, by its own gravity, 6 inches in fresh water; what is its weight, and specific gravity?

Ans. Weight 10 *lb*. 3.6 *ounces*.
Specific gravity 741.

117. To what depth would a globe of elm, whose diameter is 10 inches, sink by its own weight in fresh water?

Ans. 5.6707 *inches*.

118. Suppose the outward dimensions of a pontoon are

$$\left.\begin{array}{r}\text{Length at top} = 26\\ \text{at bottom} = 23\end{array}\right\}\text{feet.}\qquad \left.\begin{array}{r}\text{Breadth} = 2\frac{1}{4}\\ \text{Depth} = 2\frac{1}{3}\end{array}\right\}\text{feet.}$$

What weight will sink it 2 feet in fresh water?

Ans. 6160 *lb.* including the weight of the pontoon.

119. If a cube of wood floating in sea water, be $\frac{3}{4}$ wet, and it sinks $\frac{3}{10}$ of an inch deeper in fresh water; what is its magnitude, and specific gravity?

Ans. Side of the cube $= 13\frac{1}{3}$ *inches.*
Specific gravity $= 772\frac{1}{2}$.

120. It has been found by experiment that the mean specific gravity of human bodies when alive, is about 891 (that of fresh water being 1000); hence it is required to determine how many pounds of cork would be sufficient to float a person weighing 180 *lb.* with only $\frac{3}{4}$ of his body in water?

Ans. 9 *lb.*

121. Suppose a spherical balloon 27 feet in diameter can just raise 600 *lb.* including the balloon and its apparatus; now if that weight (600 *lb.*) be of the same specific gravity as water, and the specific gravity of common air $= 1\frac{2}{9}$; it is required to determine the specific gravity of the inclosed *gas* or inflammable air?

Ans. $\frac{291}{1000}$ nearly, or about $4\frac{1}{5}$ times lighter than common air.

122. If the diameter of a cylindrical vessel be 20 inches; required its depth, that when filled with a fluid, the pressure on the bottom and sides may be equal to each other?

Ans. 10 *inches.*

123. In what time would a ditch whose breadth at top $= 16$ feet, at bottom 8 feet, depth $= 6$ feet, and length $= 100$ yards, be filled with water through a rectangular opening 1 foot deep, and 2 feet wide, cut in the bank of a river; the top of the cut or

opening being on a level with the surface of the water in the river?

Ans. 47 *min.* 36 *sec.*

124. To find the whole force of water moving with a velocity $= \frac{1}{10}$ of a foot per second of time, against a rectangular flood-gate standing perpendicular to the horizon, whose breadth $= 12$, and depth $= 6$ feet.

Ans. $13500\frac{11}{16}$ *lb.*

125. Suppose a musket barrel $\frac{3}{4}$ of an inch in the bore to contain water, and let the water be forced down by means of a sponge at the end of the ramrod with a pressure $= 50$ *lb.* now (neglecting the resistance of the air) with what velocity will the water issue through the touch hole, if the sponge be air-tight, and the velocity of issuing water equal to that acquired by the free descent of a heavy body through the whole distance from the surface to the aperture.

Ans. 115 *feet* per second.

126. If an empty common glass bottle be corked and sunk in the sea 60 fathoms deep; with what force is the cork pressed by the water if the mouth of the bottle be $\frac{7}{8}$ of an inch in diameter?

Ans. $96\frac{3}{4}$ *lb.*

127. A glass cylindrical vessel whose depth $= 2$ feet, was sunk in the ocean with the open end downwards till the water rose 21 inches within the vessel; hence the depth to which it was sunk is required, supposing the pressure of the atmosphere to be $14\frac{3}{4}$ *lb.* on a square inch?

Ans. $38\frac{1}{2}$ *fathoms.*

128. If a man can push with a force $= 100$ *lb.* how far will he be able to introduce a sponge into a piece of ordnance whose length is 7 feet, and calibre 4 inches, when the barometer stands at 30 inches; the vent or touch-hole being stopped, and the sponge without windage?

Ans. $29\frac{1}{2}$ *inches,* nearly.

129. If a diving bell in the form of a cone, having the internal diameter of its base = 8 feet, and perpendicular height = 12 feet, be sunk in the sea to the depth of 13 fathoms; to what height will the water rise in the inside, and how much is the inclosed air condensed; the pressure of the atmosphere being $14\frac{3}{4}$ *lb.* on a square inch?

Ans. 4 *feet*, ascent of the water.
Density of the internal air, to that at the earth's surface as $3\frac{1}{2}$ to 1.

130. According to Humboldt, the height of the mountain Chimboraço one of the Cordilleras in South America, is 19600 French or 20889 English feet; now how much rarer is the air at the top of the mountain than at the bottom, supposing the height of the barometer at the latter situation to be 30 inches, and the specific gravities of air and quicksilver, $1\frac{1}{4}$ and 13600, respectively?

Ans. Nearly $2\frac{8}{11}$ times rarer.

131. To what height would the balloon in Examp. 121 ascend, if the attached weight and balloon together (exclusive of the inclosed *gas*) were only 500 *lb.* supposing the barometer stood at 30 inches, and the specific gravity of mercury 13600?

Ans. 3805 *feet.*

132. If a conical frustum, the diameter of whose base is 2 feet, and height $2\frac{1}{2}$ feet, move in a fluid in the direction of its axis with the least end foremost, to find the diameter of that end when the resistance is the least possible.

Ans. 6 *inches.*

THE END.

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Errata in Vol. II.

Page 10, *l.* 4, for *ba* r. *bx*.

79, examp. 3, for $2^{\frac{1}{3}}$ r. $2^{\frac{2}{3}}$.

91, —— 14, for *x*, *y*, and *x* r. *x*, *y*, and *z*.

110, —— 1, for 19 r. 15 the value of *x*.

124, *l.* 13, for -1 r. $+1$.

208, examp. 5, Simp. equat. for $\sqrt{\left(-\frac{a^2}{2}\right)}$ r. $\sqrt{(-2a^2)}$.

213, *l.* 3, for 2 r. $1\frac{1}{11}$.

217, examp. 1, for $(2100\frac{1}{4})^4$ r. $(2100\frac{1}{4})^{\frac{3}{4}}$.